Nonrelativistic CFTs

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References:

- Nishida, Đàm Thanh Sơn `https://arxiv.org/abs/0706.3746`
- Hammer, Đàm Thanh Sơn `https://arxiv.org/abs/2103.12610`

1 Lecture 1

1.1 Non-relativistic QFTs

In these lectures d denotes the number of spatial dimensions. Morally, speed of light is infinite, so the light-cone is everything with $t > 0$.

1.2 A useful model

Free theory. Consider

$$S = \int dt d^d x \Big(i\psi^\dagger \partial_t \psi - \frac{|\nabla\psi|^2}{2m} \Big)$$

The equation of motion is the Schroedinger equation (not for the wavefunction, but for the field ψ).

$$i\partial_t\psi = -\frac{\nabla^2\psi}{2m}.$$

This shows up usually when discussing second-quantization. When quantizing the theory we impose commutation (or anti-commutation if we wanted to work with fermions)

$$[\psi_x, \psi_y^\dagger] = \delta(x-y).$$

We can expand into plane waves:

$$\psi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} a_k.$$

Contrarily to relativistic theory, ψ only contains annihilation operators, no creation operators, so $\psi(x)|0\rangle = 0$. *Question:* why? *Answer:* just see that this verifies the commutation relation, and contrarily to the relativistic case we do not need to ensure causality, so there is no need to add more stuff.

This is a realtively boring theory: the Hilbert space is the Fock space $a_{k_1}^\dagger \dots a_{k_n}^\dagger|0\rangle$, with energy $E = \sum_{i=1}^n k_i^2/(2m)$. It is a superposition of plane waves.

Turning on an interaction term. We want a non-relativistic version of the ϕ^4 theory:

$$S = \int dtd^dx\Big(i\psi^\dagger\partial_t\psi - \frac{|\nabla\psi|^2}{2m} - \frac{c}{2}\psi^\dagger\psi^\dagger\psi\psi\Big).$$

We want to do some power-counting. The mass m is just there to translate from momentum to energy (just like the speed of light in relativistic theories). So we set dimensions $[m] = 0$, and $[\nabla_i] = 1$, so $[\partial_t] = 2$, so $[dt] = -2$, $[d^dx] = -d$, and overall we want $[\psi] = d/2$ to get a dimensionless action. Finally, $[c] = 2-d$. The four-point interaction behaves differently in $d < 2$, $d = 2$ and $d > 2$ dimensions.

- If $d < 2$ the interaction term is relevant.
- If $d = 2$ the interaction term is marginal.
- If $d > 2$ the interaction term is irrelevant.

We will be mostly interested in the case $d = 3$, but for the moment let us concentrate on the case $d = 2$ which looks more interesting at first.

Many questions. *Question:* why is m dimensionless? *Answer:* there is a deeper reason which is that m is a parameter in the Galilean algebra so it is not renormalized. *Question:* what if there are multiple species? *Answer:* then a combination of the masses (times the number of each particles) is non-renormalized.

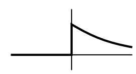
Question: can such a theory be obtained as a non-relativistic limit of a relativistic one? *Answer:* yes, take ϕ^4 theory and focus on the kinematic sector of the theory where we are just above the threshold of creating n particles, so that they all have very little energy. *Question:* but the relativistic kinetic term has two time derivatives $|\partial_t \phi|^2$; where did one time derivative disappear? *Answer:* in the suitable limit we have $\phi = e^{-imt}\psi/\sqrt{2m}$ where ψ is smooth while the prefactor is oscillatory. Then inserting in the usual ϕ^4 Lagrangian, and dropping the highly oscillating terms gives the non-relativistic Lagrangian.

Question: where did the dimension of time change in this process?

1.3 Beta function

Feynman rules. The Green function in non-relativistic theories is retarded:

$$G(t,x)\begin{cases} = 0 & t<0, \\ \neq 0, & t>0. \end{cases}$$

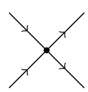


Indeed

$$\langle 0|T\psi(t,x)\psi^\dagger(0,0)|0\rangle = \langle 0|\psi^\dagger(0,0)\psi(t,x)|0\rangle = 0 \qquad t<0$$

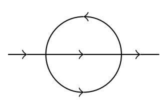
since $\psi|0\rangle = 0$. After Fourier transform one finds a propagator with the following $i\epsilon$ prescription, and a four-point vertex:

$$\longrightarrow \qquad G(\omega,p) = \frac{i}{\omega - \frac{p^2}{2m} + i\epsilon},$$



$$\text{vertex} = -2ic.$$

Non-renormalization of the mass. Then we can check that the mass is not renormalized (at least at first order) by drawing the leading correction to $\longrightarrow$:

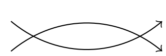


$$= (\text{coef})\big(G(t_x - t_y, x-y)\big)^2 G(t_y - t_x, y-x)$$

there is always one of the propagators going in the wrong time direction.

Renormalization of the four-point vertex. Corrections to the four-point vertex:

$= 0,$



$\neq 0.$

In $d=2$ we eventually get the beta function

$$\beta(c) = \frac{c^2}{2\pi} \qquad \text{for } d=2.$$

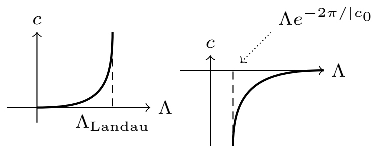
Then the flow can be integrated explicitly by solving

$$\frac{\partial c(\Lambda)}{\partial \log \Lambda} = \beta(c), \qquad c(\Lambda_0) = c_0.$$

This gives

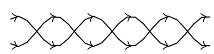
$$c(\Lambda) = \frac{c_0}{1 + \frac{c_0}{2\pi} \log \frac{\Lambda_0}{\Lambda}}$$

If the coupling constant starts positive, $c_0 > 0$, then in the IR, $c \to 0$, but at finite $\Lambda_{\text{Landau}} = \Lambda_0 e^{2\pi/c_0}$ we get a Landau pole. Starting instead from $c_0 < 0$ we get an IR divergence



Question: isn't $c < 0$ sick because the potential is unbounded? *Answer:* no vacuum instability because in this theory the number of particles is fixed.

Two-particles potential. The two-particles potential can be computed by Feynman diagrams of the form



because the particle number $N = \int dx \psi^\dagger \psi$ is conserved by the evolution. The Hilbert space splits into a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ where $\mathcal{H}_n$ has n particles.

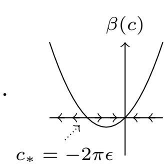
For some critical value of c_0 one finds a confining potential, leading to a bound state.

1.4 Non-relativistic conformal theories

1.4.1 Epsilon expansion

We consider $d = 2 + \epsilon$ dimensions. The beta function is

$$\beta(c) = \epsilon c + \frac{c^2}{2\pi}.$$



For positive c, or small enough negative c the RG flow make $c \to 0$. For negative enough c there is a particular value of c_0 that gives a fixed point.

1.4.2 Schrödinger (non-relativistic conformal) symmetry

What are the invariances of the Schrödinger equation $i\partial\psi/\partial_t = -\nabla^2\psi/(2m)$?

- Spatial translations $\psi(t,x) \to \psi'(t,x) = \psi(t, x+a)$, also spatial rotation, reflection.
- Phase rotation $\psi = e^{i\alpha}\psi$.
- Galilean boost[1] K_i defined by $v_i K_i \colon \psi \to \psi' = e^{im\,v\cdot x - imv^2 t/2}\psi(t, x - vt)$ for a vector v. The phase factor cancels the effect of how time derivative now acts on the (boosted) spatial coordinate.
- Dilatation $\psi \to \psi' = \frac{1}{\lambda^{d/2}}\psi(\lambda^2 t, \lambda x)$.
- Proper conformal transformation[2]

$$C\colon \psi \to \psi'(t,x) = \frac{1}{(1+\alpha t)^{d/2}} e^{\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}}\psi\Big(\frac{t}{1+\alpha t}, \frac{x}{1+\alpha t}\Big).$$

1.4.3 Non-relativistic theory from light-cone restriction of a relativistic CFT

In principle, it would be good to directly take a non-relativistic limit of a relativistic theory. But this is tricky to do generally because the relativistic theory wants to generate particles whereas we would want a limit with constant particle number.

Consider a $(d+1)+1$ dimensional Minkowski space. The Klein–Gordon equation reads (with $i = 1, \ldots, d$)

$$\Big(-\partial_t^2 + \partial_i\partial_i + \frac{\partial^2}{\partial y^2}\Big)\phi = 0.$$

Switch to light-cone coordinates $x^{\pm} = (t \pm y)/\sqrt{2}$. Then the equation becomes

$$(-2\partial_+\partial_- + \partial_i\partial_i)\phi = 0.$$

Then require the field to take the form $\phi = e^{imx^-}\phi(x^+, x^i)$ then the equation becomes

$$\Big(-2im\frac{\partial}{\partial x^+} + \partial_i\partial_i\Big)\phi(x^+, x^i) = 0,$$

which is the Schrödinger equation. The Schrödinger algebra should arise by taking a similar operation on the $(d+1)+1$ dimensional conformal algebra $\mathfrak{so}(d+2,2)$: select generators that commute with one light-cone momentum P^+ (this forms a Lie algebra, which incidentally includes P_+ itself). Conversion from the relativistic conformal symmetry to the non-relativistic (here M is the total mass, N is the particle number)

[1] The precise expression needs to be checked.

[2] It looks like a special conformal transformation, a combination of inversion, translation, inversion.

relativistic	Schrödinger
P^+	$M = mN$
P^-	H
$D + M^{+-}$	D
M^{i+}	K^i
$K^+/2$	C

1.4.4 A few facts

In the Schrödinger algebra,

- N is central, namely $[N, \text{anything}] = 0$;
- $[K_i, P_j] = i\delta_{ij}M$ where M is the total mass $M = mN$;
- $[D, P_i] = iP_i$, $[D, K_i] = -iK_i$, $[D, H] = 2iH$, $[D, C] = -2iC$ scaling dimensions, consistent with the earlier naive dimension assignment;
- $[C, H] = iD$ so C, D, H form a $SO(2, 1)$ algebra.

Question: can a theory be invariant under P_i, K_i, H, D (and M), but not C? *Answer:* ?

Introduce some operators. From $\psi(x)$ and $\psi^\dagger(x)$ we build

$$n(x) = \psi_x^\dagger \psi_x$$
$$j(x) = \frac{-i}{2}\psi^\dagger \overleftrightarrow{\nabla}\psi.$$

Then

$$[n(x), n(y)] = 0, \qquad [n(x), j^i(y)] = -in(y)\nabla^i \delta(x-y),$$
$$[j^i(x), j^j(y)] = -i(j^j(x)\partial_i + j^i(y)\partial_j)\delta(x-y).$$

These are related to diffeomorphism invariance. (Bruno is lost.) But these are just operators, they typically don't commute with the Hamiltonian since we are doing quantum mechanics.

Then we can express many symmetry generators (but not the Hamiltonian for instance) in terms of these currents as

$$N = \int dx\, n(x), \qquad K_i = \int dx\, x_i n(x), \qquad C = \int dx\, x^2 n(x),$$
$$P = \int dx\, j(x), \qquad D = \int dx\, x \cdot j.$$

We have $\partial_t n + \nabla \cdot j = 0$, which lets us compute the time derivative of the moments N, K_i and C. We have $[H, K_i] \sim \int dx x \nabla j \sim P_i$. We can compute all the commutators between these operators. The most non-trivial aspect is how

H commutes with D. This is what distinguishes theories that are scale-invariant from those who are not.

Claim: at the critical point (at the fixed point) the commutator of the Hamiltonian and dilation is the one we expect (the scale-invariant one).

Claim: if the theory is constructed from ψ and $\psi^\dagger$, then scale-invariance implies Schrödinger symmetry.

2 Lecture 2

First we will describe local operators in a quantum mechanics language. Then we will study a concrete application of the formalism.

Each operator in the Schrödinger algebra has a certain dimension under D namely $[D, A] = i\Delta_A A$, with

A	Δ_A
P_j	1
K	-1
H	2
C	-2

$$[P_i, K_j] \sim i\delta_{ij} N,$$
$$[H, C] \sim iD.$$

Consider now a local operator $\mathcal{O}(x)$ (with $x = (t, \vec{x})$ such as $\psi(x)$. Then

$$[N, \mathcal{O}(x)] = iN_{\mathcal{O}}\mathcal{O}(x), \qquad [D, \mathcal{O}(0)] = i\Delta_{\mathcal{O}}\mathcal{O}(0)$$

where $N_{\mathcal{O}}$ is the charge of the operator and $\Delta_{\mathcal{O}}$ the dimension.

For instance $N_\psi = -1$, $\Delta_\psi = 3/2$ in the free theory.

Starting from an operator $\mathcal{O}$ with definite charge and dimension, $[P_i, \mathcal{O}]$ have dimension $\Delta_{\mathcal{O}} + 1$ while $[H, \mathcal{O}]$ has dimension $\Delta_{\mathcal{O}} + 2$. Similarly $[K_i, \mathcal{O}]$ has dimension $\Delta_{\mathcal{O}} - 1$ and $[C, \mathcal{O}]$ has dimension $\Delta_{\mathcal{O}} - 2$.

Primary operator. An operator is a primary operator if it commutes with K and C, namely $[K_i, \mathcal{O}(0)] = [C, \mathcal{O}(0)] = 0$. Acting with P_j and H produces a tower of descendants of the original primary operator, like in relativistic CFT.

Example: in the free theory,

$$[C, \psi(0)] = \Big[\int dx\, x^2 \psi^\dagger(x)\psi(x), \psi(0)\Big] \sim x^2\delta(x = 0)\psi = 0.$$

Comment: the interpretation of P_j and K_j as raising and lowering operators is not clear if N acts trivially, because then they just commute.

Two-point functions of charged operators. Let us assume $N_{\mathcal{O}} \neq 0$. Then Schrödinger invariance implies the form

$$\langle \mathcal{O}(t, x)\mathcal{O}^\dagger(0, 0)\rangle = \frac{c}{t^{\Delta_{\mathcal{O}}}} \exp\Big(i\frac{N_{\mathcal{O}} x^2}{2t}\Big).$$

In a reflection-positive theory the constant c has to be positive (e.g., take the spatial position $x = 0$). For $N_{\mathcal{O}} = 0$ it is less clear.

Three-point functions. Three-point functions are only partially constrained. For instance

$$\langle\psi(y)(\psi^\dagger\psi)(x)\psi^\dagger(0)\rangle \sim f\Big(\frac{(y-x)^2}{t_y-t_x}\Big)$$

where the dependence on this ratio cannot be fixed by Schrödinger invariance. This means in particular that any theory with a non-trivial f cannot derive from a light-cone reduction of a relativistic theory.

See [Golkar–Sσn 2014].

A modified Hamitonian. Define

$$H_{\text{osc}} = H + C.$$

In the free theory since $C = \int dx \frac{x^2}{2}\psi^\dagger\psi$ we can think of this as adding a harmonic trap keeping things close to $x = 0$.

In a general theory we cannot have a similar interpretation of C.

Consider a state $\mathcal{O}^\dagger(0)|0\rangle$ with all particles concentrated at the origin (which is not a very physical state). Then act with e^{-H}:

$$|\psi_{\mathcal{O}}\rangle \coloneqq e^{-H}\mathcal{O}^\dagger(0)|0\rangle.$$

Then

$$H_{\text{osc}}|\psi_{\mathcal{O}}\rangle = e^{-H}(C-iD)\mathcal{O}^\dagger(0)|0\rangle = \Delta_{\mathcal{O}}|\psi_{\mathcal{O}}\rangle.$$

I don't understand the last equality Thus the dimension $\Delta_{\mathcal{O}}$ of the operator is the eigenvalue of the state under H_{osc}.

Non-QFT interpretation of local operators. All the examples we are studying are really quantum mechanics with short-range interaction.

Let us consider a $3+1$ dimensional system, treated as quantum mechanics of a collection of spin-1/2 fermions. Then the wavefunction is

$$\Psi = \Psi(x_1,\ldots,x_n,y_1,;y_m)$$

where x_i are coordinates of the spin up particles and y_i of the spin down ones.

For $n > 1$ or $m > 1$ we would have to impose that Ψ would be antisymmetric in the x variables, and antisymmetric in y. In particular Ψ would vanish as $x_i - x_j \to 0$ and likewise as $y_i - y_j \to 0$. What about when $x_i - y_j \to 0$? Focus on the case $n = m = 1$ to avoid worrying about the statistics of the fermions, and drop the i index.

Usually in quantum mechanics we would assume that the function Ψ is smooth as $x \to y$ but let us add a $1/|x-y|$ term to make the situation more interesting. Namely let us require that

$$\Psi(x,y) = \frac{C(\frac{x+y}{2})}{|x-y|} + \text{lower order terms}$$

where CC is a function of the center of mass. This is because we want the Hamiltonian to include $H = -\frac{1}{2}\left(\nabla_x^2 + \nabla_y^2\right)$. Naively second derivatives of $1/|x-y|$ can be of order $1/|x-y|^3$, but this specific differential operator acting on $1/|x-y|$ just gives a delta function, which can be ignored. We find that $H\Psi$ does not have any $1/|x-y|^3$ term and is simply of the same form $O(1/|x-y|)$ as Ψ itself, so this is a good candidate wave-function.

To be precise, the class of wave-functions and Hamiltonians is

$$\begin{aligned}
\Psi(x,y) &= \frac{C(\frac{x+y}{2})}{|x-y|} + \text{odd powers of } |x-y|, \\
H &= -\frac{1}{2}\left(\nabla_x^2 + \nabla_y^2\right) + \text{higher partial waves}
\end{aligned}$$

Basically instead of interactions we have put boundary conditions at $x \to y$. (This is a bit like anyons for which we impose a monodromy for x going around y.) The boundary condition can be obtained as an effective description of how wave-functions behave for a system with a potential allowing for a zero-energy bound state (particles interacting with resonant interactions). We get a (non-relativistically) conformally-invariant theory.

Local operators. The one-point function of the fermion operator ψ is not very interesting, it is just the wave function,

$$\langle 0|\psi_\uparrow(x)|\text{1-particle}\rangle = \Psi_{\text{1-particle}}(x).$$

Things start to be interesting when looking at two-point functions,

$$\langle 0|\psi_\uparrow(x)\psi_\downarrow(y)|\text{2-particle}\rangle = \Psi_{\text{2-particle}}(x,y).$$

How do we extract a local operator from that? We want to take $y \to x$ but the wavefunction blows up (due to our boundary condition). Just rescale:

$$\mathcal{O}_2(x) \coloneqq \lim_{y\to x} |x-y| \psi_\uparrow(x)\psi_\downarrow(y), \qquad \langle 0|\mathcal{O}_2(x)|\text{2-particle}\rangle = \lim_{y\to x} |x-y| \Psi_{\text{2-particle}}(x,y),$$

so this particular matrix element of $\mathcal{O}_2(x)$ is finite. In fact, all its matrix elements between multi-particle states can be shown to be finite. We actually have the OPE

$$\psi_\uparrow(t,x)\psi_\downarrow(t,0) \sim \frac{\mathcal{O}_2(t,0)}{|x|} + \text{regular}.$$

If the times were t_1, t_2 then the function $1/|x|$ would be more complicated. The dimension of the operator $\mathcal{O}_2$ are deduced from this OPE and from $[\psi] = 3/2$, so

$$[\mathcal{O}_2] = 2.$$

So we have found the exact dimension of the charge-2 operator in this quantum mechanics.

Question: why did we need two species of fermions $\psi_\uparrow$ and $\psi_\downarrow$? *Answer:* because with a single fermion the lowest charge 2 operator would be $\psi\nabla\psi$, which is of quite high dimension.

One can check that

$$\Psi(x,y) = \frac{e^{-(|x|^2+|y|^2)/2}}{|x-y|} \quad \text{obeys } H\Psi = 2\Psi.$$

(The higher excited states can be obtained by applying suitable combinations of C, D, H, and take the form of the same Gaussian with Laguerre polynomials.)

Charge 3 operator. We seek the dimension of the charge 3 operator. For this we need to find the wavefunction $\Psi(x_1, x_2, y)$. Roughly-speaking the operator will be

$$\mathcal{O}_3(x) = \lim_{R\to 0} R^{?}\psi_\uparrow(x_1)\psi_\uparrow(x_2)\psi_\downarrow(y)$$

where $x_i = y + Rv_i$ for some fixed v_1, v_2. Problem solved by Efimov in the 70's. Outline of his strategy:

$$-\frac{1}{2}\Big(\nabla^2_{x_1} + \nabla^2_{x_2} + \nabla^2_y\Big)\Psi = \delta(x_1 - y)F(x_1, x_2) - \delta(x_2 - y)F(x_2, x_1)$$

due to the $1/|x_i - y|$ blow-up. This can be solved as

$$\Psi = \widetilde{\Psi}(x_1, x_2, y) - \widetilde{\Psi}(x_2, x_1, y), \quad \widetilde{\Psi}(x_1, x_2, y) = \text{S-wave wavefunction as } x_1 \to y.$$

To find $\widetilde{\Psi}$, Efimov used a suitable coordinate system described in terms of the overall separation $R^2 = |x_1 - x_2|^2 + |x_1 - y|^2 + |x_2 - y|^2$, and rescaled variables $v_i = (x_i - y)/R$ expressed in suitable polar coordinates. Then the Hamiltonian takes the form $\partial_R^2 + \frac{5}{R}\partial_R + \frac{1}{R^2}\Delta_5$ where Δ_5 is a Laplacian on a compact 5 manifold of these polar coordinates. Efimov found the spectrum. See details in `https://arxiv.org/abs/2309.15177`. The lowest-dimension operator in the $\ell = 0$ sector has dimension $\Delta_{\mathcal{O}_3} \simeq 4.666$, but the lowest-dimension operator is in the $\ell = 1$ sector and has dimension $\Delta \simeq 4.273$. That fact can be anticipated from free-field theory: the two $\psi_\uparrow$ fermions give spin 1 because of the Pauli exclusion principle.

For four particles the Schrödinger equation cannot be solved exactly so people can only do numerics, maybe bootstrap bounds could be interesting.

Question: intuition for why two-body operators have protected dimension? *Answer:* actually the anomalous dimension is -1, but no clear answer about what protects it.

For large n number of particles people can solve by using effective field theory at finite density.

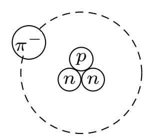
2.1 Application to nuclear physics

Let us discuss a case where we have an approximate non-relativistic conformal field theory, in nuclear physics.

Now ψ are neutrons, can have spin up or spin down. The interaction between these neutrons are almost fine-tuned to fit this picture. The neutron-neutron scattering length is equal to $a = -19\,\mathrm{fm}$. In nuclear physics the typical distance where particles interact is roughly $1\,\mathrm{fm}$ (roughly speaking the inverse mass of the pion). So $-19\,\mathrm{fm}$ is very large. The minus sign means that neutron pairs do not quite make a bound state.

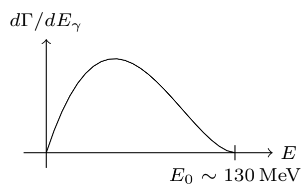
A neutron can be thought of as describing a flow from a non-relativistic CFT at high energies to a free theory at low energy, and studying it at an energy scale $E \sim 1/(ma^2) \sim 0.1\,\mathrm{MeV}$ (formula?). If we had $a = \infty$ we would sit at the NRCFT point and $a < \infty$ means we have a sort of relevant deformation thereof.

Triton (tritium nucleus). Among many nuclear reactions, one that was studied experimentally was the following. Consider a pionic atom, meaning instead of an electron you have a π^-, rotating around a nucleus. Specifically the nucleus we take is triton (the nucleus of tritium), consisting of a proton and two neutrons:



This thing decays by a process that suddenly the pion is absorbed by the nucleus and the whole thing becomes three neutrons that fly apart. The energy produced by this reaction is approximately the mass of the pion; a bit less, roughly $130\,\mathrm{MeV}$ because some energy is spent to break apart the bound state.

In a small portion of the decays (suppressed by the electromagnetic $\alpha \sim 1/137$) there is also a photon γ. Then γ carries away some of the energy. The differential decay rate as a function of energy looks as follows,



The end-point E_0 is an energy for which the photon has taken away all of the energy. Close to E_0, the neutrons that remain share a small amount of energy $E_0 - E$ (up to their center of mass motion). We can think of this process as creating two objects:

- a photon, which moves away very early on and decouples from the rest of the evolution;
- three neutrons, created by an operator $\mathcal{O}_3^\dagger$.

Then NRCFT should be a good approximation for energies very close to E_0, with corrections coming from the fact that the original nucleus has finite size (so the three neutrons are not quite at the same point, contrarily to $\mathcal{O}_3^\dagger$. We can then relate the decay rate with an NRCFT observable:

$$\frac{d\Gamma}{dE_\gamma} = \Im\langle \mathcal{O}_3 \mathcal{O}_3^\dagger \rangle.$$

The operator that should dominate is the lowest-dimension operator, so $\Delta \simeq 4.273$. Then we compute the Fourier transform etc and we predict that the differential decay rate should behave as $(E_0 - E_\gamma)^{\Delta - 5/2} = (E_0 - E_\gamma)^{1.77}$

Experimental data is not precise enough to confirm this. But people had set up some very elaborate nuclear physics models with 40 parameters to model the various interactions, and they could fit parameters (using the whole curve and other data sets). Using this model as the ground truth, one finds that indeed the power law behaviour matches the NRCFT as $E_\gamma \to E_0$. But the approximation is only valid for a small window of energies, $E_0 - E_\gamma < 2.5\,\mathrm{MeV}$ or so.

Question: why is the validity so limited? *Answer:* nature is really described by QCD. The flow from QCD down to the IR free theory (in this setup) runs very close to the NRCFT, but only for some range of energies. Estimating this energy range gives roughtly a $5\,\mathrm{MeV}$ range of energy where the approximation should be valid. Another source of error terms is the finite-size of the nucleus. To settle this question one could do experiments on Helium-4, as that nucleus is very tightly bound.

Question: what about deuterium? *Answer:* then the differential cross-section is completely known and has an $\sqrt{E_0 - E}$ behaviour as $E \to E_0$. This matches the expectation coming from the dimension of $\mathcal{O}_2$, but then the NRCFT does not bring anything new.