# Generalized Symmetries and Gauging in $d = 2$ CFTs

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In some respects, this will be a continuum version of the type of questions Shu-Heng Shao will discuss in discrete lattice models.

# 1 Basics of 2d CFTs

References

- Well-known Di Francesco book
- Nice reading: Ginsparg `https://arxiv.org/abs/hep-th/9108028`
- Xi Yin 2017 TASI: 2d CFT from bootstrap philosophy
- Higher-dimensional CFT D. Simmons–Duffin 2016 TASI lecture notes; Rychkov 2016 EPFL notes.

## 1.1 CFT introduction

What are they? They are quantum field theories with conformal symmetry. That has a lot of consequences. For now we treat the general dimension case.

In $d$ spacetime dimensions with Lorentz signature, the conformal group is $SO(d,2)$ and contains the Poincaré group $ISO(d-1,1)$. In Euclidean signature it would be $SO(d+1,1)$ containing $ISO(d,1)$.

The Poincaré generators $M_{\mu\nu}$ and $P_\mu$ are completed by the dilation operator $D$ and by special conformal transformations $K_\mu = I \circ P_\mu \circ I$ where $I$ is the inversion transformation $I\colon x^\mu \mapsto x^\mu/|x|^2$. (Inversion is outside $SO$, because it reflects orientation.)

In these lectures we will focus on QFTs that are *local*, *unitary*, and *compact* in the sense that the theory on a compact spatial slice has a discrete energy spectrum. We assume translation invariance, which means that there exists a conserved stress tensor $T_{\mu\nu}$. We assume rotation invariance, which means $T_{\mu\nu}$ is symmetric. We assume conformal invariance, which means $T^\mu_\mu = 0$, so that $J^\mu_\epsilon = T^{\mu\nu}\epsilon_\nu$ for any vector $\epsilon$ obeying the conformal Killing equation

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu}.$$

Among such vectors we have those that obey the Killing equation $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 0$, which correspond to isometries of space, hence correspond to the Poincaré group inside the conformal group.

Once you have a symmetry you should organize everything according to it. Operators of the theory come into representations of the conformal group, called conformal families. In each family there is a lowest-dimension operator $\mathcal{O}_{\Delta,J}$, where the dimension $\Delta$ is the eigenvalue of the dilation operator $D$.

Correlators obey quite constraining conformal Ward identities.

**Physical contexts where CFTs arise.**

- Second-order phase transitions of various quantum models (lattice, continuous models), for instance boiling water.

- Fixed-points of renormalization group flows.[1]
- Model for quantum gravity through the AdS/CFT correspondence.

**Why?** CFTs carry the hallmarks of universality. They are characterized by a small set of physical data: critical exponents, OPE coefficients, etc. They give a powerful non-perturbative approach to QFT (hence also to quantum gravity). They are nevertheless much more interesting and generic than free theory (e.g., operator dimensions are generally not integers).

**How?** There is an operator algebra formulation allowing to describe the CFT by a spectrum of dimensions and spins, plus operator product expansions based on operator three-point functions. Then bootstrap axioms constrain that data. There is then an axiomatic approach to solve or constrain the CFTs, either analytically (in 2d) or numerically (in higher dimensions).

This is an ideal playground to investigate generalized symmetries (properties and dynamical consequences).

**Various questions and remarks.** *Question:* the whole structure is complicated. *Answer:* there is an infinite amount of data, and infinitely many constraints. It is not known whether this can be reduced in some way to finite data. In practice CFTs are often isolated, so that specifying some of the low-lying dimensions etc is often enough.

*Question:* can there be multiple operators with the same $\Delta$ and $J$? *Answer:* yes there can still be degeneracies, we are only writing $\Delta, J$ to keep notation simple, but there

*Question:* why the compact condition? *Answer:* there are other theories like Liouville CFT, but one ends up with complicated convergence questions. Also, the CFTs coming out of lattice models are naturally compact.

*Question:* what about extended operators? Why aren't we including the higher-dimensional operators? In 2d the local operators are actually enough to fix the line operators etc. It is related to how trivial TQFTs are in low dimensions.

## 1.2 CFT in $d=2$ (1+1d)

We now focus on 2d CFT. It is easier to make figures. But more importantly 2d CFTs have a very large symmetry algebra.

**Complex coordinates.** We introduce coordinates $z = x^1 + ix^2$ and $\overline{z} = x^1 - ix^2$ on Euclidean $\mathbb{R}^2$. The traceless condition sets $T_{z\overline{z}} = 0$. Conservation makes $T_{zz}$ holomorphic and $T_{\overline{z}\overline{z}}$ antiholomorphic, so instead of $T_{\mu\nu}$ we can

[1] Generally (but there are counterexamples) the low-energy limit of the RG flow is scale-invariant and this scale invariance generally enhances to full conformal invariance.

work with $T(z) = T_{zz}(z)$ and $\overline{T}(\overline{z}) = T_{\overline{z}\overline{z}}(\overline{z})$. In addition, this operator has $\Delta = J = 2$.

The fact that $T$ and $\overline{T}$ are separately conserved leads to a large symmetry algebra. There are many more conformal Killing vectors $\epsilon^\mu = (\epsilon(z), \overline{\epsilon}(\overline{z}))$ for any pair of holomorphic functions. This gives currents $j^\epsilon_\mu = (\epsilon T, \overline{\epsilon}\overline{T})$. Taking a basis of monomials $z^m$ we get charges

$$L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1}, \qquad \overline{L}_n = -\oint \frac{d\overline{z}}{2\pi i} \overline{T}(\overline{z}) \overline{z}^{n+1}.$$

The global conformal group $SO(3,1) \simeq PSL(2,\mathbb{C})$, or rather its Lie algebra, gets enhanced to an infinite-dimensional algebra $\mathrm{Vir}_c \times \mathrm{Vir}_{\overline{c}}$. The algebra is determined by the stress-tensor OPE, which is fixed by holomorphy and dimensional analysis to be

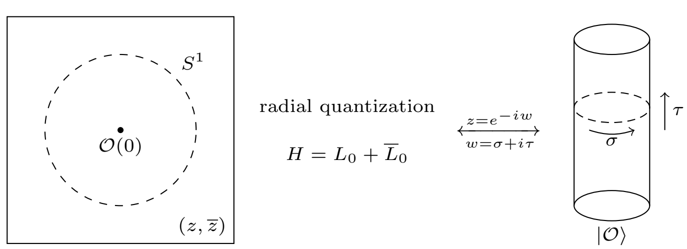
$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular}.$$

The second and third terms are necessarily there in order to reproduce the correct global conformal algebra. The first term involves a dimensionless number, called the central charge $c$, that depends on the theory. Then the charges $L_m$ obey the Virasoro algebra $\mathrm{Vir}_c$ with central charge $c$, namely

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

The global conformal algebra $\mathfrak{sl}(2,\mathbb{C})$ is generated by $\{L_0, L_{\pm 1}, \overline{L}_0, \overline{L}_{\pm 1}\}$. In particular $L_0 + \overline{L}_0 = \Delta$ and $|L_0 - \overline{L}_0| = J$.

**State-operator correspondence.** There is a bijection between local operator $\mathcal{O}$ and states in the Hilbert space on a circle[2] The key is radial quantization, treating the radial direction as time. The corresponding Hamiltonian is $L_0 + \overline{L}_0$. To go from an operator $\mathcal{O}$ to a state, insert it at the origin, and evolve with that Hamiltonian until some (radial) "time slice". Alternatively, conformally map this to a cylinder.




[2]Some people in the audience are thinking about state-operator correspondence with non-local operators, but in 2d we can stick with local operators for our purposes.

**Hermitian structure.** This is inherited from the Lorentzian cylinder ($t = -i\tau$)

$$\mathcal{O}_L(t,\sigma)^\dagger = \mathcal{O}_L(t,\sigma), \qquad \mathcal{O}_L(t,\sigma) = e^{iHt-ip\sigma}\mathcal{O}_L(0,0)e^{-iHt+ip\sigma}.$$

Now Wick-rotate (the subscript $L$ stands for Lorentzian and $E$ for Euclidean)

$$\mathcal{O}_E(\tau,\sigma) \coloneqq \mathcal{O}_L(-i\tau,\sigma) \implies \mathcal{O}_E(\tau,\sigma)^\dagger \overset{H,P \text{ hermitian}}{=} \mathcal{O}_E(-\tau,\sigma).$$

Conclusion: Hermitian conjugation in Euclidean spacetime amounts to "time" reflection (and complex conjugation if the operator is not real). Note that this depends on what we use as Euclidean time.

On $\mathbb{R}^2$, hermitian conjugation amounts to BPZ conjugation: for a real scalar operator,

$$\mathcal{O}_\Delta(z,\bar{z})^\dagger = |z|^{-2\Delta}\mathcal{O}_\Delta\Big(\frac{1}{\bar{z}},\frac{1}{z}\Big)$$

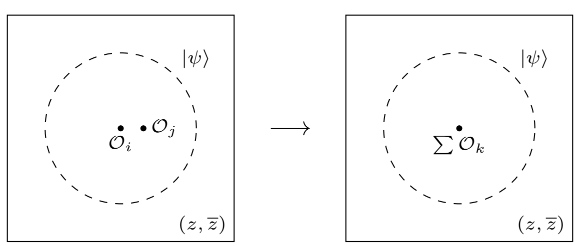
obtained using the relation between operators on the cylinder and on $\mathbb{R}^2$, namely $\mathcal{O}_{\text{cylinder}} = e^{\tau\Delta}\mathcal{O}_{\text{flat}}$. Similarly for the spinning case, for instance

$$T(z)^\dagger = \frac{1}{\bar{z}^4}T\big(1/\bar{z}\big) \qquad L_n^\dagger = L_{-n}, \qquad \overline{L}_n^\dagger = \overline{L}_{-n}.$$

Here, crucially, Hermitian conjugation has mapped $T$ to itself at a different point: on the right-hand side we do not have $\overline{T}$, but really $T$. It depends holomorphically on its argument, which happens to be the conjugate of $1/z$. Upon integrating, we get Virasoro modes of $T$, with no mixing between $T$ and $\overline{T}$ under the Hermitian conjugation.

You are free to work with a different Hermitian structure, but this would not correspond to a Wick-rotated Lorentzian theory. So to extract physical results you don't have a choice.

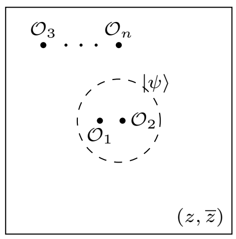
**Operator product expansion** Insert a pair of operators near the origin, consider the state that they produce upon the state-operator correspondence, then use the state-operator correspondence backwards to write everything in terms of operators at the origin:




$$\mathcal{H}_{S^1} \ni |\psi\rangle = \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)|0\rangle = \sum_k C_{ij}^k(z_1,z_2)\mathcal{O}_k(0)|0\rangle,$$

which is a convergent sum as a consequence of $\langle\psi|\psi\rangle = \langle \mathcal{O}_j(z_2)^\dagger \mathcal{O}_i(z_1)^\dagger \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle < +\infty$.

Then in any correlator we can replace a pair of operators that are close enough (that can be wrapped alone in a circle with no other insertion) by a sum of single operators with suitable coefficients




$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3 \dots \mathcal{O}_n\rangle = \sum_k C_{12}^k \langle \mathcal{O}_k(z_k)\mathcal{O}_3 \dots \mathcal{O}_n\rangle.$$

## 1.3 Virasoro representations

Local operators, or equivalently states, fall into unitary representations of $\mathrm{Vir}_c \times \mathrm{Vir}_c$ (we will assume that there is no gravitational anomaly, namely $\overline{c} = c$). We consider lowest-weight states $|h, \hbar\rangle$ of each irreducible representation, where $h, \hbar$ are eigenvalues of $L_0, \overline{L}_0$, namely

$$L_n|h,\hbar\rangle = \overline{L}_n|h,\hbar\rangle = 0, \qquad \text{for all } n > 0.$$

Under the state-operator correspondence, they correspond to primary operators $\mathcal{O}_{h,\hbar}$, which obey the following transformations under conformal transformation $z \to z'$ and $\overline{z} \to \overline{z}'$,

$$\mathcal{O}'_{h,\hbar}(z', \overline{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{-\hbar} \mathcal{O}_{h,\hbar}(z, \overline{z}).$$

Conformal invariance of correlators:

$$\langle \mathcal{O}'_1(z_1, \overline{z}_1) \dots \mathcal{O}'_n(z_n, \overline{z}_n)\rangle = \langle \mathcal{O}_1(z_1, \overline{z}_1) \dots \mathcal{O}_n(z_n, \overline{z}_n)\rangle.$$

Consequences

- Two-point and three-point functions of primary operators are completely fixed up to coefficients. Denoting $z_{ij} = z_i - z_j$,

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle = \frac{\delta_{h_1,h_2}\delta_{\hbar_1,\hbar_2}}{z_{12}^{2h_1}\overline{z}_{12}^{2\hbar_1}}$$
$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{c_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2} \overline{z}_{12}^{\hbar_1+\hbar_2-\hbar_3} \overline{z}_{23}^{\hbar_2+\hbar_3-\hbar_1} \overline{z}_{31}^{\hbar_3+\hbar_1-\hbar_2}}.$$

- Correlators of descendants are fixed by those of the primaries.

- Correlators of more than three local operators are fixed by two- and three-point functions.

Unitarity constraints imply that states in $\mathcal{H}_{S^1}$ should have positive-definite norms. Easy exercise: show that $c > 0$ and that $h, \hbar \geq 0$ (unitarity bound).

Systematic version: positivity properties of the Gram matrix of lowest-weight states $|h, \hbar\rangle$. We learn

- Determine the unitary irreps $M_{c,h}$ of $\mathrm{Vir}_c$.
- Unitary theories with $c < 1$ are minimal models, have $c = 1 - \frac{6}{m(m+1)}$ with $m = \underset{\text{Ising}}{3}, \underset{\text{tricritical}}{4}, 5, \ldots$ and dimensions are (with a further two-fold identification)

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad 1 \leq r \leq m-1,\ 1 \leq s \leq r.$$

Once we understand the representation theory we are not done. Coming from further constraints on CFT we will see that there are further constraints on $h$ versus $\hbar$.

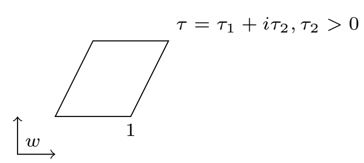
**Ising CFT.** Minimal model with $m = 3$, central charge $c = 1/2$, three primary operators, of dimensions $h = 0, 1/16, 1/2$. These are the critical exponents of the Ising CFT. They can be measured, for instance in the transverse-field Ising model.

## 1.4 Torus

One way to discuss the relation between $h$ and $\hbar$ is to study locality constraints coming from placing the CFT on the torus.

*Question:* why do you say locality? What it means is that we are allowed to place the theory on any (Euclidean) manifold, then select whatever spatial slice I want, and the correlators will give the same result regardless of what direction I choose as Euclidean time.

Consider a torus $T^2 = \mathbb{C}/2\pi(\mathbb{Z} + \tau\mathbb{Z})$ described as a quotient of the complex plane by a lattice.




Then

$$Z_{T^2}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_{S^1}}\left(e^{-2\pi\tau_2 H_{\mathrm{cyl}} + 2\pi i \tau_1 P_{\mathrm{cyl}}}\right)$$

where $H_{\mathrm{cyl}} = L_0 + \overline{L}_0 - c/24$ and $P_{\mathrm{cyl}} = L_0 - \overline{L}_0$, where the $c/24$ comes from a Schwartzian term, a non-trivial transformation of $T$ under Weyl transformations because $T$ is not a Virasoro primary.

This trace can also be written as

$$Z_{T^2}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_{S^1}}\left(q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24}\right), \qquad q = e^{2\pi i \tau}, \quad \overline{q} = e^{-2\pi i \overline{\tau}}.$$

**Positivity and integrality in the torus partition function.** The Hilbert space on a circle decomposes into a direct sum of Virasoro representations, with integer multiplicities $n_{h,\hbar}$ (finite because the theory is compact) so

$$Z_{T^2}(\tau,\overline{\tau}) = \sum_{h,\hbar} n_{h,\hbar}\chi_h(\tau)\chi_{\hbar}(\overline{\tau}),$$

where $\chi_h(\tau) = \mathrm{Tr}_{M_{c,h}}\, q^{L_0 - c/24}$ is a trace over the irreducible representation of $\mathrm{Vir}_c$ with lowest-weight state of dimension $h$.

For $c > 1$ and $h > 0$ we have

$$\chi_h(\tau) = \frac{q^{n-\frac{c-1}{24}}}{\eta(\tau)}, \qquad \eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n).$$

This has an expansion in positive powers of $q$, which counts descendants $L_{-n_1}L_{-n_2}\dots L_{-n_m}|h,\hbar\rangle$ of higher and higher dimensions. In special cases (for $c \leq 1$) there are null states, and the character changes (multiplicities decrease).

**Modularity of the torus partition function.** We only explain the case of a bosonic CFT. *Question:* what does bosonic/fermionic CFT mean? *Answer:* Bosonic means the CFT is completely defined by only specifying the metric of the manifold. Fermionic means that we need the additional data of a spin structure.

The torus $T^2 = \mathbb{C}/2\pi(\mathbb{Z} + \tau\mathbb{Z})$ is the same as that with $\tau \to \tau + 1$, and the same as the one with $\tau \to -1/\tau$ (modulo a conformal transformation). Together these transformations generate the group $PSL(2,\mathbb{Z})$ mapping $\tau$ to $(a\tau + b)/(c\tau + d)$ with $ad - bc = 1$.

More generally, the partition function on a higher-genus Riemann surface is invariant under the mapping class group. As it turns out, they are redundant: the constraints of the torus one-point functions, and sphere four-point functions are enough (work by Moore and Seiberg).

## 1.5 Consequences of modular invariance

The invariance under $\tau \to \tau + 1$ leads to a $e^{2\pi i(h-\hbar)}$ factor in front of each character. Invariance requires the spin $h-\hbar$ to be an integer: we get quantization of spins (for bosonic CFTs).

The invariance under $\tau \to -1/\tau$ leads to an identity

$$\chi_h(-1/\tau) = \int_{h'} S_{hh'}\chi_{h'}(\tau)$$

of characters where the integral may have a discrete $\sum$ part in the rational case. The kernel $S_{hh'}$ is known, everything converges nicely. Then the invariance of the partition function requires the multiplicities to obey

$$S \cdot n \cdot S^T = n.$$

### 1.5.1 Example of the Ising CFT.

Representation theory tells you $h, \hbar = 0, 1/16, 12$. Spin quantization $h - \hbar \in \mathbb{Z}$ forces $h = \hbar$.

Uniqueness of the conformal vacuum $|0, 0\rangle$, namely $n_{0,0} = 1$ means we only need to fix $n_{1/2,1/2}$ and $n_{1/16,1/16}$. In the exercises we see that restricted to the representations of interest we have

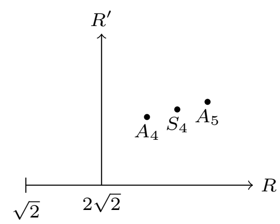
$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}.$$

Then we find $n_{1/2,1/2} = 1$ and $n_{1/16,1/16} = 1$. Altogether, we have three conformal primaries,

$$\mathcal{H}_{S^1} = \{1, \epsilon_{1/2,1/2}, \sigma_{1/16,1/16}\}.$$

### 1.5.2 A second application: the $c = 1$ CFT

See Ginsparg's review. For $c = 1$, representation theory allows for infinitely many possible values of $h$ for the $M_{c,h}$ unitary representations.




The line labeled by $R$ is the compact boson of radius $R$, with Lagrangian $\frac{R^2}{4\pi}\int \partial\phi\bar{\partial}\phi d^2z$ where $\phi$ is $2\pi$-periodic. Varying $R$ amounts to an exactly marginal deformation by the operator $\partial\phi\bar{\partial}\phi$ of dimensions $(h, \hbar) = (1, 1)$. It is invariant under T-duality $R \to 2/R$. Then $R = \sqrt{2}$ is the self-dual point.

At $R = 2\sqrt{2}$ there is the BKT point with additional symmetries and an additional exactly marginal deformation leading to the orbifold branch (orbifold under $\phi \to -\phi$).

There are also three discrete points obtained by orbifolding the self-dual boson by subgroups of $SO(3)$. To be precise, orbifolding by an $A$-type subgroup of $SO(3)$ gives points on the compact-boson branch, orbifolding by a $D$-type subgroup gives points on the orbifold branch, and finally we have $E_6$, $E_7$, $E_8$ subgroups.

### 1.5.3 Cardy formula

The Cardy formula is a statement about universality of heavy states. The central charge $c$ is supposed to count degrees of freedom.

One can take $\tau = i\beta/(2\pi)$ purely imaginary, and interpret $\beta$ as the inverse temperature. Then, denoting $\Delta = h + \bar{h}$, we have the large-$\beta$ and small-$\beta$ limits

$$Z(\beta) = \int d\Delta\, \rho(\Delta) e^{-\beta(\Delta - c/12)} \xrightarrow{\beta\to+\infty} e^{(c/12)\beta}$$
$$\xrightarrow{\beta\to 0} \int d\Delta\, \rho(\Delta) \text{ formally.}$$

Then the S transformation $\tau \to -1/\tau$ relates these two limits as $Z(\beta) = Z(4\pi^2/\beta)$. By playing around with these formulas and with inverse Laplace transform, we find that the large $\Delta$ density of states is

$$\rho(\Delta) \overset{\Delta\gg c}{\approx} \rho_0(\Delta) = \frac{e^{2\pi\sqrt{c\Delta/3}}}{\Delta^{3/4}}\Big(1 + O(\Delta^{-1/2})\Big) + \text{exponentially suppressed.}$$

The first term (with all power-law corrections in $\Delta$) has a closed form in terms of a Bessel function. This is very schematic but there are rigorous versions from the Tauberian theorems.

#### 1.5.4 Application 4

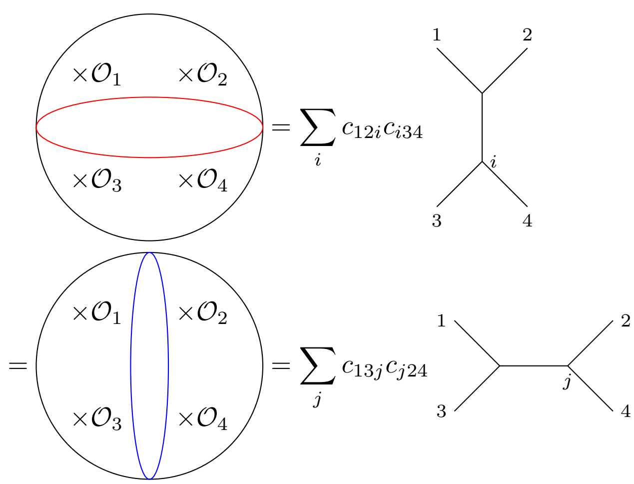
Discover features of familiar symmetries and new generalized symmetries.

### 1.6 General axiomatic constraints

Let us study constraints coming from locality. No matter how we build our Riemann surface out of Hilbert spaces on circles, we will always get the same answer for CFT observables. It all boils down to consistency of cutting and gluing CFT observables on Riemann surfaces.

Moore–Seiberg gave necessary and sufficient conditions for that:

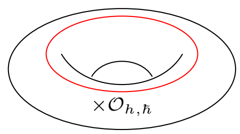
1. Quantization of spin.
2. Sphere 4-point crossing



3. Modular covariance of the torus one-point function[3] $\langle \mathcal{O}_{h,\hbar} \rangle_\tau$, namely

$$\langle \mathcal{O} \rangle_{-1/\tau} = \tau^h \bar{\tau}^{\hbar} \langle \mathcal{O} \rangle_\tau .$$

where the factor is a Weyl factor coming from the $w \to w'/\tau$ coordinate change.




*Question:* if you write the modular invariance on arbitrary Riemann surfaces, is that enough to recover the sphere four-point function condition? *Answer:* that's a very good question; in some sense the genus 2 Riemann surface can be cut open into a four-point function.

*Question:* how trivial is the Moore–Seiberg result; is it deep? *Answer:* it depends on your particular taste. It is just about cutting and gluing.

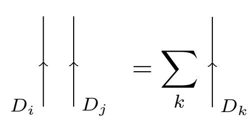
*Question:* are there cases where the sphere condition is satisfied but not torus modular invariance. *Answer:* have to think.

## 1.7 Generalized global symmetries in $d = 2$ CFT

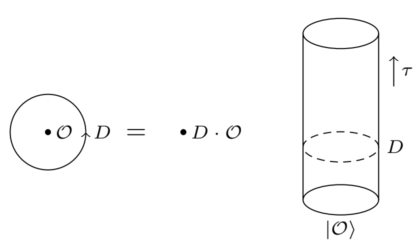
From our point of view, symmetries are the same as topological defect lines (TDL). Because we work in Euclidean signature with unitary Lorentz-invariant theories (actually CFT), so lines can be oriented in any direction.

By definition, topological defects obey

- Topological invariance
- Fusion with integer coefficients (see later for a reason based on locality)




The action on charged operators preserves $h, \hbar$ since the operator commutes with the stress-tensor. It can be depicted in radial quantization or on a cylinder,




[3]By translation invariance there is no dependence on where the operator is inserted on the torus. By spin quantization there is no need to write $\tau \to \tau + 1$ invariance. In fact, one can show that only operators of even spins can have non-zero one-point functions.

**Quantum dimension of a defect.** Because the operator does not change the dimensions, and we have assumed we have a unique vacuum, the defect must simply rescale the vacuum by some number $\langle D\rangle = \langle 0|D|0\rangle$ called the quantum dimension,

$$D|0\rangle = \langle D\rangle|0\rangle.$$

In a unitary theory, $\langle D\rangle \geq 1$. If $\langle D\rangle = 1$ then $D$ is invertible. If $\langle D\rangle > 1$ then $D$ is not invertible.

### 1.7.1 Example: Ising CFT

What are the symmetries? In fact, symmetries are equivalent to Ward identities, so whenever you find Ward identities you should find the corresponding symmetry.

There is the obvious $\eta$ symmetry, acting as $1 \to 1$, $\sigma \to -\sigma$, $\epsilon \to \epsilon$. Shows that $\langle\sigma\sigma\sigma\rangle = \langle\sigma\epsilon\epsilon\rangle = 0$ etc.

Surprisingly (at first) one has $\langle\epsilon\dots\epsilon\rangle = 0$ whenever there is an odd number of $\epsilon$, so there should be some symmetry guaranteeing that. But it cannot be just a $\mathbb{Z}_2$ charge of $\epsilon$ since $\langle\sigma\sigma\epsilon\rangle \neq 0$. It will be a non-invertible symmetry $\mathcal{N}$ sending $\epsilon \to -\epsilon\langle\mathcal{N}\rangle$ and $\sigma \to 0$ so as to allow $\langle\sigma\sigma\epsilon\rangle \neq 0$.

This leads us to go beyond groups and to discuss fusion categories.

# 2 Topological Defects and Fusion Category

References for today:

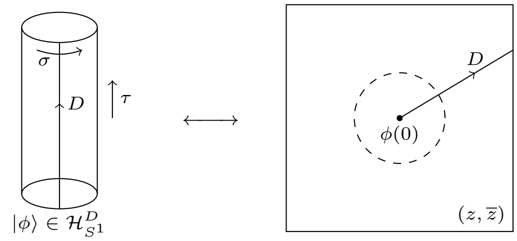
- `https://arxiv.org/abs/1704.02330` Bhardwaj, Tachikawa,
- `https://arxiv.org/abs/1802.04445` Chang, Lin, Shao, Wang, Yin.

The goal is to understand in what sense fusion categories are the natural object that comes up when studying generalized symmetries in 2d, just like groups arise when studying invertible symmetries.

## 2.1 Axiomatic approach to symmetries in 2d CFT

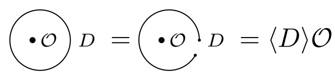
Here we focus on compact, unitary CFTs with a single ground state. Without symmetries the relevant axioms are Moore–Seiberg axioms (on fusion and braiding and torus S-move). We now want to refine these axioms by decorating them by topological defect lines.

We work in Euclidean signature. A given symmetry defect can be taken as wrapping the $S^1$ spatial direction, in which case it is simply a symmetry operator on the Hilbert space, or can be placed in the time direction at a point in space,

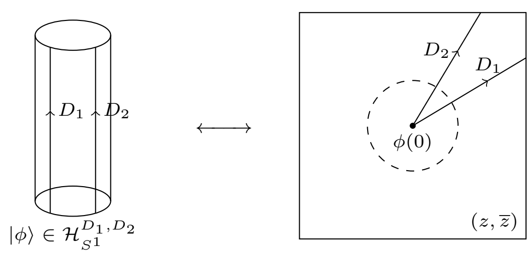



The Hilbert space in the presence of a defect $D$ is denoted as $\mathcal{H}^D_{S^1}$ and called the $D$-twisted sector. Under the state-operator map it corresponds to an operator in the $D$-twisted sector.

**Faithfulness condition.** We assume the faithfulness condition which states that the only defect that acts trivially on all local operators is the identity defect. Equivalently, defects $D \neq 1$ cannot end topologically: otherwise you could cut it open and see that it acts trivially:




**Multi-defect Hilbert space.**




The Virasoro action allows you to move the defects around. It introduces some factors so strictly speaking the Hilbert space $\mathcal{H}^{D_1,D_2,\cdots}_{S^1}$ depends on the separations and Hilbert spaces for different separations are easily isomorphic. The Hilbert space is also invariant (up to an important isomorphism) under cyclic permutations of the defect (to do things properly we need to include a marked point etc).

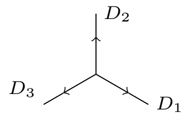
Besides the Hilbert space $\mathcal{H}^{D_1,D_2,\cdots}_{S^1}$ we also define

$$\mathcal{H}^{D_1+D_2}_{S^1} = \mathcal{H}^{D_1}_{S^1} \oplus \mathcal{H}^{D_2}_{S^1},$$

corresponding to the insertion of a direct sum of defects at the same place. This will be useful when discussing the fusion of defect.

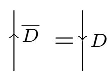
*Question:* if you insert a non-topological defect do you break Virasoro? *Answer:* you break Vir × Vir to the diagonal subalgebra.

**Topological junction.** A topological junction is an operator of dimension $h = \bar{h} = 0$ inside $\mathcal{H}_{S^1}^{D_1 \ldots D_n}$. The space of such junctions is denoted $V_{D_1 \ldots D_n}$. An element $v \in V_{D_1 \ldots D_n}$ is visualized on the plane as




In the invertible case defects are labeled by group elements $g_i$ and $\dim V_{g_1 \ldots g_n}$ is 1 if $g_1 g_2 \ldots g_n = 1$ and is otherwise zero.

**Dual defect.** The dual defect $\overline{D}$ may have $D\overline{D}$ different from 1. The dual defect is simply defined as the orientation-reversed defect




A defect is simple if $\dim V_{D\overline{D}} = 1$. As a consequence we can show that $D \neq D_1 + D_2$. Conversely when $\dim V_{D\overline{D}} \geq 2$ then we can always split $D$ into pieces.

**Fusion.** When fusing defects we get a new defect, which can be decomposed into simple defects, so we can introduce notation in the case of simple defects:

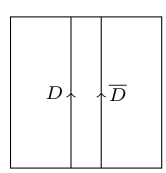
$$D_i D_j = \sum_k N_{ijk} D_k.$$

One can check that $N_{ijk} = \dim V_{D_i D_j \overline{D}_k}$. This differs from the group-like multiplication law $D_g D_{g'} = D_{gg'}$ for invertible symmetries. Special case

$$D_i \overline{D}_i = 1 + \ldots$$

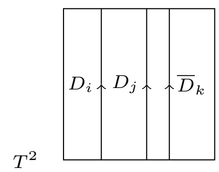
Let us check that the leading term is 1.

**Using the thermal partition function.** ..... missing discussion of how the thermal partition function allows one to show that the leading term in $D_i\overline{D}_i$ is $\dim V_{D\overline{D}}$ .....



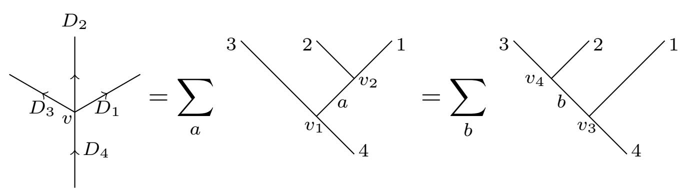

$T^2$

Likewise in an exercise we should prove $N_{ijk} = \dim V_{D_i D_j \overline{D}_k}$ using the low-temperature limit of the torus partition function with three defect insertions



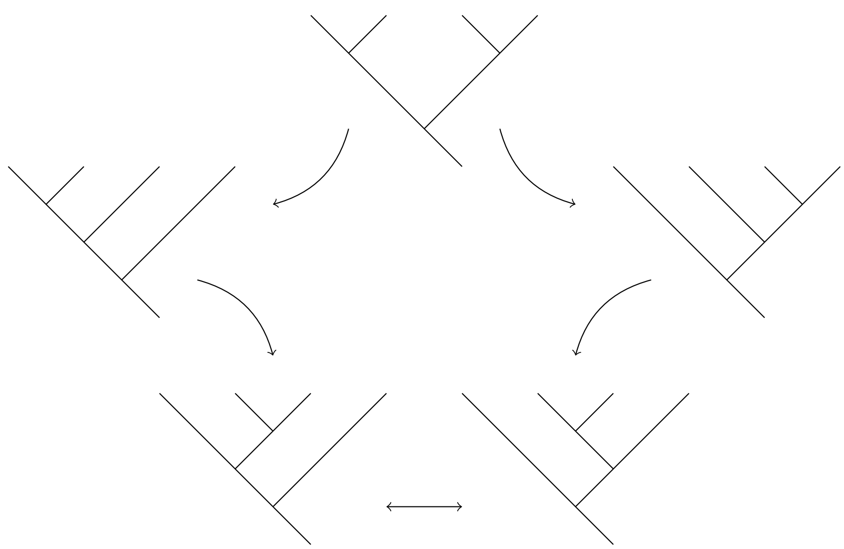

**Topological junctions as morphisms.** Topological junctions $v \in V_{D_1 D_2 D_3}$ are morphisms between $D_1 D_2$ and $\overline{D}_3$.

**F-symbols (associators).** In a general junction vector space there is no preferred basis. A several bases of $V_{D_1 D_2 D_3 \overline{D}_4}$ can be constructed from bases of three-fold junctions:




We have a unitary change of basis $(F_4^{321})_{ab}$ mapping the basis of the form $v_1 \otimes v_2$ to that of the form $v_3 \otimes v_4$.

**Pentagon identity.** The fusion coefficients $F$ have to obey an equation of the form $FF = \sum FFF$, obtained by performing the following fusion steps:

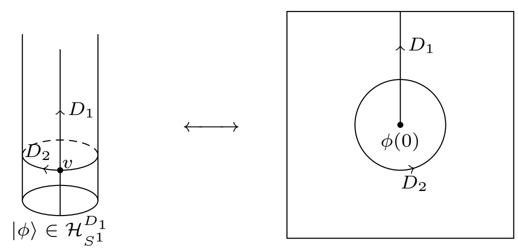


This is an analogue of the cocycle condition for $H^3(G,U(1))$ which classifies anomalies for the group $G$. In a sense, F-symbols capture the anomalies.

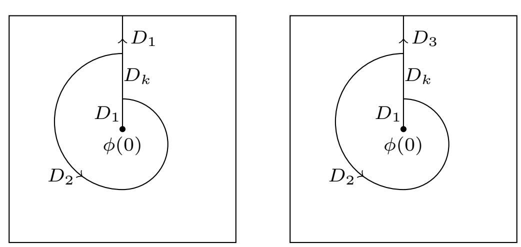
Should we continue with six-fold junctions etc? MacLane coherence theorem: the full set of consistency conditions is automatically satisfied once the pentagon identity is obeyed.

## 2.2 Symmetry action in defect Hilbert space

How can a symmetry labeled by a defect $D_2$ act on a twisted Hilbert space $\mathcal{H}_{S^1}^{D_1}$, namely on the $D_1$-twisted sector? We need a topological junction $v$ between the operators.



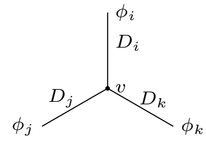

This can be resolved into an intermediate defect $D_k \in D_1 D_2$. There are normally multiple choices of this intermediate defect, hence $V_{D_1 D_2 \overline{D}_1 \overline{D}_2}$ is typically of dimension larger than 1 (in contrast to invertible symmetries). This means that there are multiple possible actions of $D_2$ on $\mathcal{H}_{S^1}^{D_1}$. This is called the Lasso action. A generalization is that the action can change $\mathcal{H}_{S^1}^{D_1}$ to $\mathcal{H}_{S^1}^{D_3}$ as in the second picture below. This generates the Tube algebra.




## 2.3 Definition of symmetry-enriched CFT

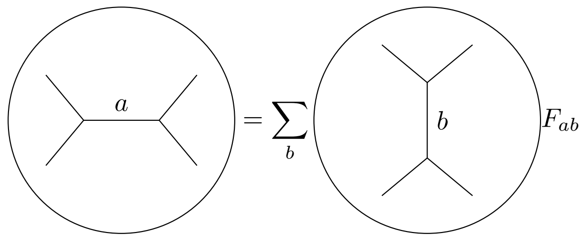
A CFT enriched by a collection of topological defect lines $\{D_i\}$ is given by

- **Data:** $\mathcal{H}_{S^1}^{D_i \ldots D_j}$ and three-point functions of operators attached to defects,




  This includes the usual $\mathcal{H}_{S^1}$ and three point functions.

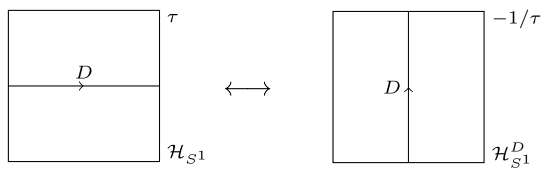
- **Bootstrap conditions (locality).** Sphere four-point crossing in the presence of topological defects,




Modular covariance of the torus one-point function with defects.

## 2.4 Modular invariance of the symmetry enriched CFT

Recall that for a symmetry topological defect line $D$ we can write




This leads to the following relation

$$\mathrm{Tr}_{\mathcal{H}_{S^1}}\left(\widehat{D}q^{L_0-c/24}\overline{q}^{\overline{L}_0-c/24}\right)=\mathrm{Tr}_{\mathcal{H}^D_{S^1}}\left(\widetilde{q}^{L_0-c/24}\overline{\widetilde{q}}^{\overline{L}_0-c/24}\right)$$

where $\widetilde{q}=e^{2\pi i(-1/\tau)}$.

**Ising model.** See homework. The Hilbert space on the LHS here is a direct sum of representations of Virasoro so

$$\begin{aligned}\mathrm{LHS}&=A_1|\chi_0|^2+A_{1/2}|\chi_{1/2}|^2+A_{1/16}|\chi_{1/16}|^2\\ \mathrm{RHS}&=\sum_{i,j}n_{ij}\chi_i(\widetilde{q})\chi_j(\overline{\widetilde{q}}),\end{aligned}$$

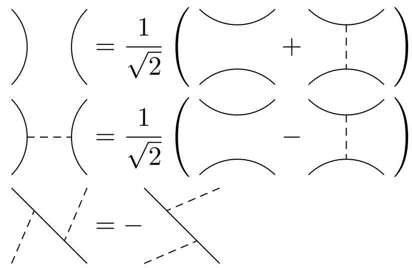
where the $A_i$ are not yet quantized and the $n_{ij}$ are non-negative integers. Using the modular transformations of Virasoro characters we find

- $(A_i)=(1,1,1)$ corresponds to the identity defect $D=1$;
- $(A_i)=(1,1,-1)$ corresponds to the $D=\eta$ defect;
- $(A_i)=(\sqrt{2},-\sqrt{2},0)$ corresponds to the $D=\mathcal{N}$ defect.

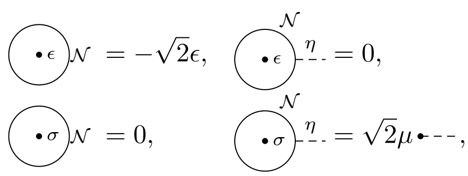
The fusion rule for $\mathcal{N}^2$ can be found by squaring these eigenvalues $A_i$ and reexpressing them in the basis of other solutions $(A_i)$. The same can be done for all fusion rules and we find

$$\mathcal{N}^2=1+\eta,\qquad \eta^2=1,\quad \mathcal{N}\eta=\eta\mathcal{N}=\mathcal{N}.$$

**Ising F-symbols.**



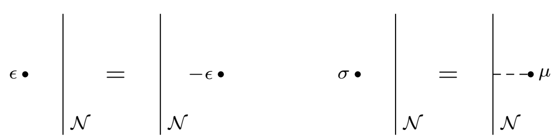

**Action on local operators.** Then we will find




where $\sqrt{2}$ is the quantum dimension $\langle \mathcal{N} \rangle = \sqrt{2}$, and $\mu$ is a primary operator in the twisted sector:

$$\mathcal{H}^{\eta}_{S^1} = \{\psi_{1/2,0}, \widetilde{\psi}_{0,1/2}, \mu_{1/16,1/16}\}.$$

**Passing TDL through local operators.**



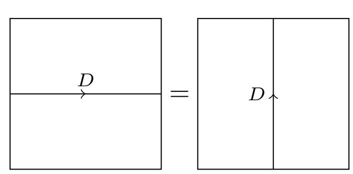

(Unrelated?) claim: the presence of $\mathcal{N}$ means that the CFT is self-dual under the $\mathbb{Z}_2$ orbifold.

## 2.5 Dynamic consequences of non-invertible symmetry

Assume that you have a UV theory $T_{\text{UV}}$ with $D$ symmetry and you deform it by a (marginally relevant or) relevant operator $\mathcal{O}_{h,\bar{h}}$ with $\bar{h} = h$ and $\Delta \leq 2$ then perform the RG flow to the IR theory $T_{\text{IR}}$. Assume also that the operator $\mathcal{O}$ commutes with the defect $D$ (namely the deformation preserves the symmetry).

**Claim 1** (Theorem)**.** *If $\langle D \rangle \notin \mathbb{Z}$ then $T_{\text{IR}}$ cannot be trivially gapped: we either get a CFT or spontaneous symmetry breaking.*

*Proof.* We have



If the IR is trivially gapped then there is a unique ground state so the left-hand side is $\langle 0|D|0\rangle = \langle D\rangle$. The right-hand side is a trace of 1 over the defect Hilbert space, which has to be an integer. Contradiction. If there were multiple ground states then $D$ can act differently on different ground states and somehow this resolves the problem. $\square$

**Example 1.** Tricritical Ising model $c = 7/10$. The symmetry is Ising $\boxtimes$ Fib where the "Ising" symmetry is the usual $\{1, \eta, \mathcal{N}\}$ and the "Fib" symmetry is generated by $W$ with $W^2 = 1 + W$, of quantum dimension $\langle W\rangle = (1+\sqrt{5})/2$.

Deforming this CFT by $\epsilon'_{3/5,3/5}$, which commutes with $\mathcal{N}$, gives an RG flow whose low-energy limit is either gapless (necessarily Ising by $c$ monotonicity) or gapped with at least three vacua. Both cases arise depending on the sign of the deformation, as can be shown using integrability.

*Question:* what operator tracks the RG flow arriving into the Ising model? *Answer:* it is an irrelevant operator, which turns out to be the $T\overline{T}$ operator constructed from the stress-tensor, which thus automatically commutes with all of the symmetries, including $\mathcal{N}$.

**Example 2.** The $1+1$ dimensional $SU(N)$ massless adjoint QCD also has a huge amount of non-invertible symmetries. This leads to a gapped phase with $\sim 2^N$ vacuum degeneracies. Heuristic explanation: the adjoint fermions are described by the WZW model $\mathrm{Spin}(N^2-1)_1$; gauging $SU(N)$ roughly amounts to taking a coset, which suggests the TQFT $\mathrm{Spin}(N^2-1)_1/SU(N)_N$, which has a ton of topological defects.

# 3 Topological Interfaces and Generalized Gauging

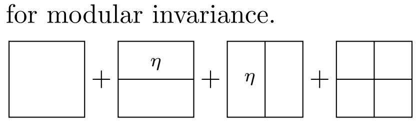
See Fuchs–Runkel–Schwaigert https://arxiv.org/abs/hep-th/0204148, ..., Diatlyk–Luo–Weller–Wang https://arxiv.org/abs/2311.17044

## 3.1 Gauging procedure

**Gauging a usual abelian group symmetry.** To gauge a $\mathbb{Z}_2$ symmetry of a theory $T$ (which is like an orbifold in string theory), two steps:

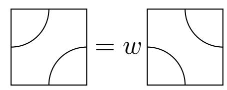
- project to the $\mathbb{Z}_2$-even sector;
- include $\mathbb{Z}_2$-twisted sectors.

For instance, the torus partition function is a sum of four terms: the first two terms are a trace in the usual Hilbert space but with a projection $(1+\eta)/2$ onto the $\mathbb{Z}_2$-even sector; the second two are the projection but in the twisted sector. All terms have to be there for modular invariance.

$$Z_{T/\mathbb{Z}_2} = \frac{1}{2}\Bigg($$


$$\Bigg)$$

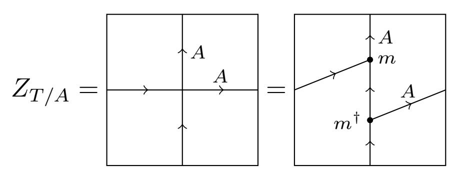
The last diagram is subtle and has to be resolved to be defined. We have no topological junction for a group symmetry so there are two resolutions simply related by a phase $w$ (equal to $\pm 1$ in the $\mathbb{Z}_2$ case)




If we pick one of these resolutions, then we have to worry about modular invariance of $Z$. If $w \neq 1$ we will find that the partition function is not $S$-invariant regardless of what we do. There is an anomaly preventing you from gauging.

Example: the Ising model has $w = +1$ while $SU(2)_1$ has $w = -1$ for the $\mathbb{Z}_2$ center symmetry.

**Gauging topological defect lines.** Pick a general topological defect line $A$. (For instance, to reproduce the previous gauging we would take $A = 1 + \eta$, or more generally for a group we would take the projector $A = \sum_{g \in G} g$.) Then the sum of partition functions we had before is reproduced by inserting a complete network of $A$ defects,

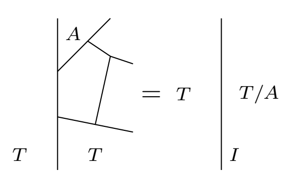



Data for generalized gauging: $(A, m, m^\dagger, u, u^\dagger)$ with $u, u^\dagger$ end-points of $A$ and $m, m^\dagger$ three-fold junctions. To avoid gauge anomaly, data has to form a symmetric special Frobenius algebra object. These latter data $m, m^\dagger$ capture "discrete torsion", 1+1 dimensional SPT phases. (We also assume $A$ is self-dual but it is not clear how much work this assumption is making.)

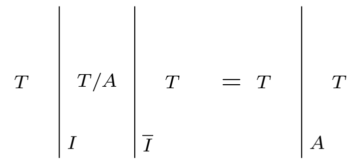
Abstractly: gauging is decorating the observables in $T$ with a network of $(A, m, m^\dagger)$ with a mesh that is fine enough.

## 3.2 Half-gauging and topological interfaces

Suppose you have a symmetry and a choice of $(A, m, m^\dagger)$. Then by gauging over a half-space you can make an interface between the theory $T$ and $T/A$:




Gauging just in a small slab (such as a time interval) defines a topological line in $T$, and furthermore this turns out to be $A$ itself. In other words $I\bar{I} = A$.




Special case: if $T \simeq T/A$ then the interface gives a topological defect line in $T$, so that actually $A$ factorizes as $I\bar{I}$ with $I$ being a topological defect line of $T$ itself.

Example: in the tricritical Ising model the Fibonacci line $W$ acts on $\phi$ (of dimensions $h = \bar{h} = 3/80$) by multiplying it by $(1-\sqrt{5})/2$. The gaugeable algebras are $A = 1 + \eta$ and $A = 1 + W$. It turns out that gauging gives the same theory, meaning that there are interfaces such that $A$ is the square of these interfaces. This is consistent with $\mathcal{N} = 1 + \eta$ and with $W^2 = 1 + W$.

Example: $\mathrm{Rep}(D_8)$ symmetry has $\nu^2 = 1 + \eta + \eta' + \eta\eta'$ (see Shu–Heng Shao's lectures). There turns out to be 12 gaugeable algebras. In particular consider the maximal gauging, gauging the whole category $A = A_{\max} = 1 + \eta + \eta' + \eta\eta' + 2\nu$. There are actually three possible choices of $m, m^\dagger$ obeying the conditions. They correspond to the three SPTs for $\mathrm{Rep}(D_8)$ that we saw in Shu–Heng Shao's lectures, namely choices of discrete torsion.

Interestingly, $\mathrm{Rep}(D_8)$ is the symmetry subcategory of the orbifold branch of the $c = 1$ CFT. Gauging $T_{\mathrm{orb}}(R)$ with $(A_{\max})_3$ gives the compact scalar theory $T_{\mathrm{circ}}(R)$. Gauging $T_{\mathrm{orb}}(R)$ with $(A_{\max})_1$ or $(A_{\max})_2$ gives the same orbifold theory but with $R \to 4/R$. In particular at $R = 2$ we have self-duality hence extra symmetries. This is the point where the theory is two copies of the Ising model, hence indeed has extra symmetry. See `https://arxiv.org/abs/2310.19897` and `https://arxiv.org/abs/2311.16230`.