Nothing of what I am going to say today is new. ⓪
In fact much of it is decades old, and covered by good reviews (and often the original papers are masterpieces of clarity!). So I'll be sometimes a bit quick, leaving details to these reviews. I can recommend the following:

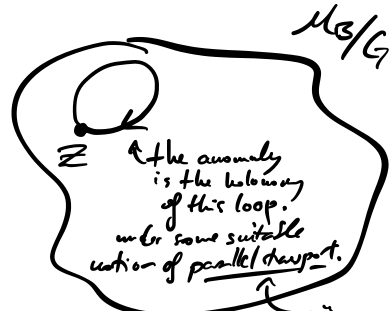
- Alvarez-Gaumé and Witten: "Gravitational anomalies"
- Alvarez-Gaumé: "An introduction to anomalies"
- Monnier: "A modern viewpoint on anomalies"
- Alvarez-Gaumé and Vázquez-Mozo: "Anomalies and the Green-Schwarz mechanism".
- Bilal: "Lectures on anomalies"
- Harvey: "TASI 2003 lectures on anomalies"

These lectures are about high energy physics/string theory, but my viewpoint on anomalies is very similar to the one in Max Metlitski's lectures:

$$\underset{\det(\not{D}_B)}{\overset{\shortparallel}{Z^{\text{Weyl}}_{\text{fermion}}(\mathcal{M}^d;B)}} = \underbrace{\left|Z^{\text{Weyl}}_{\text{fermion}}(\mathcal{M}^d;B)\right|}_{\text{well defined}} \; \underbrace{e^{i\varphi(\mathcal{M}^d;B)}}_{\text{Potentially not gauge invariant.}}$$

(B: background for some symmetry, can include metric.)

I like to think of this geometrically (the picture is most easily understood for continuous symmetries):



"Determinant line bundle"

Two cases:

- The determinant line bundle is not flat: small loops can generate changes in the phase: "local" or "perturbative" anomaly.
- The determinant line bundle is flat: holonomies come from non-trivial loops: "global" or "non-perturbative" anomalies.

In order to understand when holonomies arise we can use a prescription due to Dai-Freed and Yonekura-Witten:

$$Z_{\text{fermion}}(\mathcal{M}^d;B) := \left|Z_{\text{fermion}}(\mathcal{M}^d;B)\right| \, e^{2\pi i \eta(\not{D}_{\mathcal{N},\overline{B}})}$$

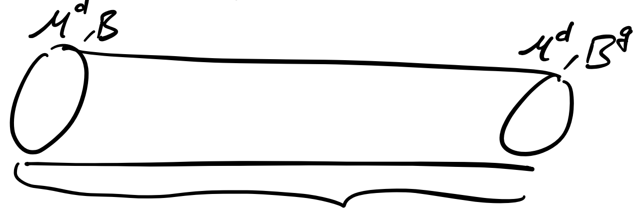
with $\partial\mathcal{N}^{d+1} = \mathcal{M}^d$, $\overline{B}$ an extension of B to $\mathcal{N}^{d+1}$, and

$$\eta(\not{D}_{\mathcal{N},\overline{B}}) := \frac{1}{2}\Big(\dim \operatorname{Ker} \not{D}_{\mathcal{N},\overline{B}} + \sum_{\lambda\neq 0} \operatorname{sign}(\lambda)\Big)$$

(λ: eigenvalues of $\not{D}_{\mathcal{N},\overline{B}}$)

(or simply $\sum_\lambda \operatorname{sign}(\lambda)$, if we take the convention $\operatorname{sign}(0)=+1$).

A loop in $\mathcal{M}_{B/G}$ can be understood, in terms of the spacetime, as a cylinder (the "mapping torus"):



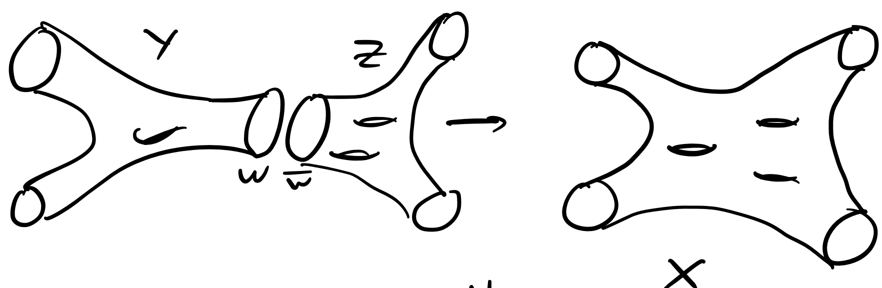
$T \leftarrow$ the mapping torus

Picking some N^{d+1}, we have

$$Z(\mathcal{M}^d;B) = |Z(\mathcal{M}^d;B)|\, e^{2\pi i \eta(\not{D}_{N,\bar{B}})}$$

$$Z(\mathcal{M}^d;B^g) = |Z(\mathcal{M}^d;B^g)|\, e^{2\pi i \eta(\not{D}_{N+T,\bar{B}})}$$

To make progress, we need to know that, if $X = Y \sqcup_W Z$ (X arises from gluing Y and Z along a common boundary W):



then $\eta(\not{D}_X) = \eta(\not{D}_Y) + \eta(\not{D}_Z)$ mod 1. (We are choosing some specific boundary conditions here, known as the APS boundary conditions.)
We also have $\eta(\not{D}_X) = -\eta(\not{D}_{\bar{X}})$ mod 1.
We find:

$$\frac{Z(\mathcal{M}^d;B^g)}{Z(\mathcal{M}^d;B)} = e^{2\pi i \eta(\not{D}_{T,\bar{B}})}$$

Now we use the Atiyah-Patodi-Singer (APS) index theorem: glue the two ends of T, so that we get an actual mapping torus $\tilde{T}$. Assuming that $\tilde{T} = \partial S$, we have

$$\mathrm{index}(\not{D}_S) - \int_S \hat{A}(TS)\, ch_R(E) = \eta(\not{D}_{\tilde{T}})$$

We have defined $\eta(\not{D}_{\tilde{T}})$ above. We have

$$\mathrm{index}(\not{D}_S) = \underset{\substack{\uparrow \\ \text{positive} \\ \text{chirality zero modes}}}{\dim \ker D_+} - \underset{\substack{\uparrow \\ \text{negative chirality} \\ \text{zero modes.}}}{\dim \ker D_-} \in \mathbb{Z}$$

The quantity $\hat{A}(TS)\, ch_R(E)$
is known as the "index density" (for the Dirac operator).

We have $\hat{A}(TS) = 1 - \frac{1}{24} p_1(TS) + \frac{1}{16}\left(\frac{7}{360} p_1^2 - \frac{1}{90} p_2\right) + \cdots$

where $p_i(TS) \in H^{4i}(S;\mathbb{Z})$ is the i-th Pontryagin class

$(p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}))$. In terms of the curvature

$$\underset{\substack{\uparrow \\ SO(n)\text{ bundle}}}{p_{\mathbb{Q}}(E)} := \det\Big(1 + i\underset{\substack{\uparrow \\ \text{curvature}}}{F}/_{2\pi}\Big) = 1 + p_1 + p_2 + \cdots$$

Finally $ch_R(E) := tr_R\left(e^{iF/2\pi}\right) = \dim(R) + ch_1(F) + ch_2(F) + \cdots$

For completeness, we can also define, for a complex bundle,

$$c(E) = 1 + c_1 + c_2 + \cdots \qquad [\text{Chern classes}]$$

such that $c(E) \otimes \mathbb{Q} = \det(1 + iF/_{2\pi})$

<u>Example 1</u> $U(1)$ bundle on S^4:

$\partial S^4 = 0$, so we have $\dim\ker D_+ - \dim\ker D_- = \int_{S^4} \hat{A}\, ch_R(E)$.

A useful relation is $\frac{1}{3}\int_{M^4} p_1(TM^4) = \text{signature}$. For S^4,

signature$(S^4) = 0$ since $H^2(S^4;\mathbb{R}) = 0$. So:

$$\dim\ker D_+ - \dim\ker D_- = \int_{S^4}\left(1 - \tfrac{1}{24}p_1\right)\left(1 + \overset{\text{Some irrep of } U(1)}{\frac{iF}{2\pi}} - \frac{1}{2}\frac{F^2}{4\pi^2}\right)$$

$= \int_{S^4}\left(-\tfrac{1}{2}\right)\frac{F^2}{4\pi^2} = 0$ since $[F] \in H^2(S^4;\mathbb{R}) = 0$ and the integral is in cohomology.

So $\underbrace{\dim\ker D_+}_{n_+} - \underbrace{\dim\ker D_-}_{n_-} = 0$. In fact, generically $n_+ = n_- = 0$. [Lichnerowicz formula]

<u>Example 2</u> Fermion in the fundamental of $SU(N)$ on S^4:

As before, $p_1(TS^4) = 0$, so the only contribution is from $ch(E)$. We have

$$n_+ - n_- = -\frac{1}{8\pi^2}\int \mathrm{tr}_{\square} F^2 = \int c_2 = n_{inst}$$

(In fact $n_+ = 0$ if $n_{inst} < 0$, and $n_- = 0$ if $n_{inst} > 0$,

Jackiw and Rebbi, 1977, Spinor analysis of Yang-Mills theory)

<u>Example 3</u> Neutral fermion on K3.

In this case the gauge bundle is trivial, so $ch(E) = 1$. For K3, the signature is -16, so $\frac{1}{3}\int_{K3} p_1 = -16 \Rightarrow \int_{K3} p_1 = -48$. We get:

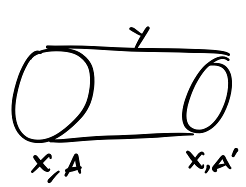
$$n_+ - n_- = -\frac{1}{24}\int_{K3} p_1 = -\frac{1}{8}\,\text{signature} = 2$$

(This has to be even, due to Kramers doubling. Rokhlin's theorem then follows: for a spin manifold signature $\in 16\mathbb{Z}$.)

Relation to the Chern-Simons invariant

η and the Chern-Simons invariant are closely related. Consider a connection A (including metric data) and a small deformation A'. Then, by the APS index theorem:

$$\eta_X(A') - \eta_X(A) = \mathrm{index}(\not{D}_Y) - \int_Y \hat{A}(TZ)\, ch(E)$$

(Here 

)

Generically $\text{index}(\not{D}_Y) = 0$ if $\text{index}(\not{D}_X) = 0$, so

$$\eta_X(A') - \eta_X(A) = \int_Y \hat{A}(TZ)\, ch(E)$$

Writing $\hat{A}(TZ)\, ch(E) = d\Omega_{CS}$, this gives

$$\eta_X(A') - \eta_X(A) = \Omega_{CS}(A') - \Omega_{CS}(A)$$

They are not necessarily the same, though. Consider a $SU(2)$ bundle in 5d. We have $\Omega_{CS} = Tr_R(AF^2 + \dots)$ which vanishes because in $SU(2)$ $tr_R(t^a t^b t^c) = 0$. But we will see later that η is nonvanishing. The relation to Ω_{CS} shows that in this case η is a deformation invariant. Also note: CS is not gauge invariant, η is gauge invariant but not continuous (jumps by 1).

Relation to bordism

In fact the right notion here is bordism. (Max explained this last week.)

We just saw that the index density encodes (via Ω_{CS}) the variation of the phase of Z under small deformations of the background:

$\Rightarrow$ Local anomalies will cancel if $\hat{A}(TX)\, ch(E) = 0$.

More formally, assume that $\hat{A}(TX)\, ch(E) = 0$, and that $\partial X = Y - Y'$. We have:

$$\frac{e^{2\pi i \eta(\not{D}_Y)}}{e^{2\pi i \eta(\not{D}_{Y'})}} = e^{2\pi i \eta(\not{D}_{Y-Y'})} = e^{2\pi i\, \text{index}(\not{D}_X)} = 1.$$

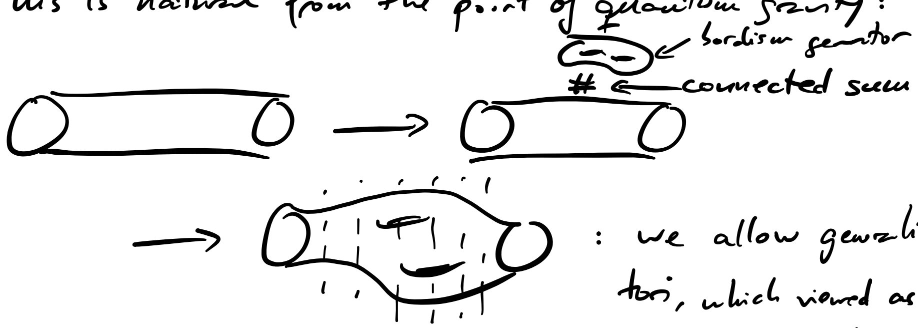
So any two bordant manifolds lead to the same phase if there are no local anomalies!

So the procedure for understanding anomalies splits into two steps:

1. Check vanishing of the local anomaly ≡ index density. This typically reduces to a group theory exercise.
2. See if there is some bordism class $\Omega_{d+1}(BG)$ where $e^{2\pi i \eta} \neq 1$.

The second step requires some explanation: from field theory the standard condition is about <u>mapping tori</u>: we want to require $e^{2\pi i \eta(\not{D}_{\tilde{T}})} = 1$ for all mapping tori $\tilde{T}$. But now we are imposing $e^{2\pi i \eta(\not{D}_{\mathcal{M}})} = 1$ for <u>all</u> $\mathcal{M}$, and not all $\mathcal{M}$ are bordant to mapping tori. (Examples where this is actually stronger in Witten, 1508.04715, for instance.) In 1808.00009 with M. Montero we called this stronger condition "Dai-Freed" anomaly cancellation, and studied some implications (see also Juven Wang's papers.)

This is natural from the point of quantum gravity:



: we allow generalized mapping tori, which viewed as a path in the space of configurations can be induced from topology change.

Example of a global anomaly: Witten's SU(2) anomaly (See also 1810.00844 by Wang, Wen, Witten)

Consider a Weyl fermion ψ in 4d in the fundamental of SU(2). It's not possible to write a mass term: $\psi_{\alpha i} \psi_{\beta j} \varepsilon^{\alpha\beta} \varepsilon^{ij}$ vanishes identically. So there could be an anomaly.

1 Is there a local anomaly? <u>No</u>: $\mathrm{tr}_{\square}(t^a t^b t^c) = 0$

and $\mathrm{tr}_{\square}(t^a) = 0$, so the degree $4+2=6$ part of $\hat{A}\,\mathrm{ch}(E)$

$= (1 - \frac{1}{24}\underbrace{p_1}_{\deg 4} + \underbrace{\cdots}_{\deg \geq 8})\, \mathrm{tr}_{\square}(e^{iF/2\pi}) = 0.$

2 Is there a global anomaly?

2.1 Let's compute $\Omega_5^{Spin}(BSU(2))$.

We will use a way of computing bordism known as the Atiyah-Hirzebruch spectral sequence. [An alternative, very powerful method, is the Adams spectral sequence, I recommend

Beaudry, Campbell 1801.07530

Debray, Dierigl, Heckman, Montero 2302.00007]

The basic idea of spectral sequences is that one can build increasingly accurate approximations to some algebraic object by starting from simple pieces. For bordism, assume that X is a fibration $F \to X \to B$. AHSS then gives:

$$H_p(B; \Omega_q^{Spin}(F)) \Rightarrow \Omega_{p+q}^{Spin}(X)$$

In our case we will use the universal fibration

$$pt \longrightarrow BSU(2) \xrightarrow{id} BSU(2)$$

to write $H_p(BSU(2); \Omega_q^{Spin}(pt)) \Rightarrow \Omega_{p+q}^{Spin}(BSU(2))$.

(a) $BSU(2) \cong \mathbb{HP}^\infty$, so in particular

$$H_n(\mathbb{HP}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{when } n \in 4\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}$$

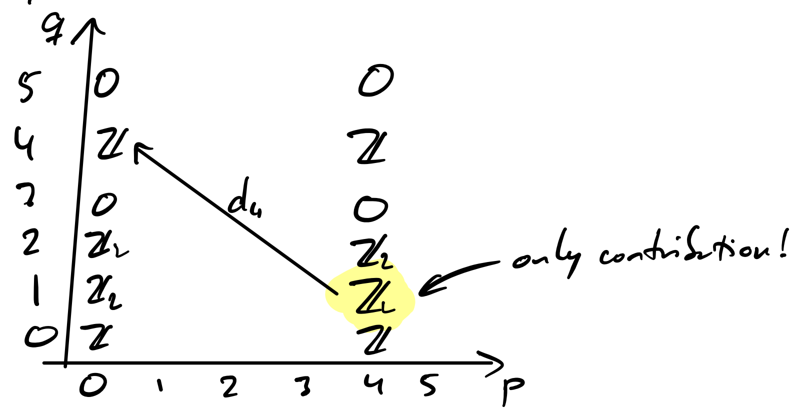
and

$$\Omega_\bullet^{spin}(pt) = \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}\oplus\mathbb{Z}, \ldots\}$$

(originally worked out in Anderson, Brown, Peterson, tables and many more calculations in 1808.00009)

So we find: $H_p(BSU(2); \Omega_q^{Spin}(pt)) = \begin{cases} \Omega_q^{Spin}(pt) & \text{if } p \in 4\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}$

Graphically:



There are some potential subtleties that in this case can be easily seen not to matter. So we find:

$$\Omega_5^{Spin}(BSU(2)) \cong H_4(BSU(2); \Omega_1^{Spin}(pt))$$
$$\cong \underbrace{H_4(BSU(2); \mathbb{Z})}_{\cong H^4(BSU(2);\, \mathbb{Z}) \,\cong\, c_2 \text{ of field configurations}} \otimes \Omega_1^{Spin}(pt)$$
$$\cong \mathbb{Z}_2$$

The $H_4(BSU(2); \mathbb{Z}) \otimes \Omega_1^{Spin}(pt)$ presentation suggests looking into a 4d single instanton configuration times an S^1 with periodic boundary conditions (so it generates $\Omega_1^{Spin}(pt)$). This is indeed a good space where to compute η, since whenever $\dim(A) \in 2\mathbb{Z}+1$ and $\dim(B) \in 2\mathbb{Z}$: [Gilkey, "The Geometry of Spherical Space Form Groups"]

$$\eta(\not{D}_{A\times B}) = \eta(\not{D}_A)\, \mathrm{index}(\not{D}_B)$$

The idea here is dimensional reduction: reducing on B gives a theory on A with $\mathrm{index}(\not{D}_B)$ massless fermions (+ gapped/gappable

modes) and $\eta(\not{D}_A)$ encodes the anomaly for these.
In detail, choosing $M^5 = S^4 \times S^1$, and putting a single instanton bundle on S^4 and trivial $SU(2)$ bundle on S^1 (but periodic boundary conditions ~ insertion of $(-1)^F$ in the path integral) we find

$$\eta(\not{D}_{S^4\times S^1}) = \mathrm{index}(\not{D}_{S^4})\ \eta(\not{D}_{S^1}) = 1 \cdot \tfrac{1}{2} = \tfrac{1}{2} \neq 0.$$

So there is indeed an anomaly.
We can make the dimensional reduction argument more precise: after reducing the 4d theory on the S^4 with the instanton, we have a theory in 0d (matrix model) for 1 massless fermion. The unbroken gauge group is $Z(SU(2)) = \mathbb{Z}_2$ $\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$, which acts on the single fermion measure by $D\psi \to -D\psi$. So there is an anomaly: the measure picks up a phase under gauge transformations.
Note: $Z(SU(2)) \sim (-1)^F$ when acting on the fermions, so this agrees with the fact that we want periodic boundary conditions when computing $\eta(\not{D}_{S^1})$.

Other kinds of matter with anomalies

Spinors are not the only representation of Spin(d) which leads to gravitational anomalies: in $d \in 4\mathbb{Z}+2$ we can also have:

- Gravitinos (spin $3/2$)
- Self-dual forms (bosonic!)

Gravitinos

Gravitinos are spin-$\frac{3}{2}$ particles, of the form $\psi_{\alpha\mu}$. We can view them as spinors charged in the vector representation of the tangent bundle. Naively:

$$I_{d+2}^{\text{gravitino}} \overset{\text{wrong!}}{=} \hat{A}(TM)\, \underbrace{ch(TM)}_{tr(\exp i R/_{2\pi})}$$

But we should be careful: a spinor-vector is not just spin $\frac{3}{2}$, but it includes an extra spinor.

(recall, for example, the Clebsch–Gordan rule for $SU(2)$:

$$\underbrace{R_1 \otimes R_{1/2}}_{\text{spin}} = R_{3/2} \oplus R_{1/2})$$

For $Spin(6) = SU(4)$: spinor is 4, and vector is 6. We have

$$\bar{4} \times 6 = \underbrace{20}_{\text{gravitino}} + 4 \leftarrow \text{extra spinor}$$

so we need to substract this contribution (Note: in sugra this is automatic, it gets removed by fermionic gauge transformations), to get:

$$I_{d+2}^{\text{gravitino}} = \hat{A}(TM)\left(tr\left(e^{iR/_{2\pi}}\right) - 1\right)$$

[Note: This was a little quick. A better way of obtaining the -1 comes from noticing that the quantisation of a gravitino requires ghost spinors, and including these, see van Nieuwenhuizen '81]

Self-dual forms

In $d = 4n+2$ dimensions, we can have $2n$-forms B such that $dB = * dB$.

These self-dual forms can also have anomalies, despite being bosonic, because they live in a complex representation of the Lorentz group. A difficulty is that it's not straightforward to write a covariant Lagrangian for the self-dual forms, from which to derive Feynman rules. (But it can be done, see for instance Lechner, hep-th/9808025, using the PST action)

At any rate, it is the case that there's an anomaly: the self-dual tensor arises in the product of a spinor with itself (plus anomaly free representations), so we can write the index theorem for a fermion

$$I_A = -\frac{1}{8}\hat{A}(TM)\,\mathrm{tr}\left(\exp\left(\frac{i}{2\pi}\frac{1}{4}R_{ab}\gamma^{ab}\right)\right)$$
$$= \underset{\text{bosons}}{-}\frac{1}{8}\underbrace{L(TM)}_{\text{Hirzebruch polynomial}} \quad \left(\text{note: } \int L(TM) = \text{signature}\right)$$

Anomaly cancellation in IIB

These considerations are nicely illustrated in IIB string theory. This is a $d=10,\ \mathcal{N}=(2,0)$ supergravity with the following fields:

$g_{\mu\nu},\ B_{\mu\nu},\ \phi$

$C_0,\ C_2,\ C_4$ with $dC_4 = * dC_4$

$\underset{\text{gravitino (Weyl)}}{\psi_{\mu\alpha}},\ \overset{\text{dilatino (Weyl)}}{\chi_{\dot\alpha}}$

It's illuminating (as in the paper by Alvarez-Gaumé and Witten) to assume we have $n_{1/2}$ positive chirality Weyl fermions, $n_{3/2}$ positive chirality gravitinos, and n_A self-dual forms. We have

$$I_{1/2}\Big|_{d=12} = \frac{1}{967680}\left(-31p_1^3 + 44p_1p_2 - 16p_3\right)$$

$$I_{3/2}\Big|_{d=12} = \frac{1}{967680}\left(225\,p_1^3 - 1620\,p_1 p_2 + 7920\,p_3\right)$$

$$I_A\Big|_{d=12} = \frac{1}{967680}\left(-256\,p_1^3 + 1664\,p_1 p_2 - 7936\,p_3\right)$$

It is an easy calculation that $n_{1/2} = -1$, $n_{3/2} = 1$, $n_A = 1$ is a solution. [Exercise: show that it is the <u>only</u> solution if we assume $n_{3/2} = 1$, i.e. $\mathcal{N} = (2,0)$ sugra]

Anomaly cancellation in type I/heterotic

There are other versions of supergravity, coming from string theory, which are also potentially anomalous. Consider Type I/heterotic string theory: at low energies it is described by a $\mathcal{N} = (1,0)$ 10d supergravity coupled to super Yang-Mills with gauge group G.

We have:

$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ ← all bosonic

$\psi_{\mu\alpha}$ ← gravitino (Majorana-Weyl), χ_α ← dilatino (M-W)

A_μ ← gauge boson
λ_α ← gaugino (M-W)
} in the adjoint representation of G

The anomaly polynomial for this theory is (expanding the characteristic classes above, and omitting some constants): [Note: The following is sketchy, I encourage you to work through the details whenever you have a free afternoon!]

$$I_{12} \sim (\dim G - 496)\,p_3 + \#\,\mathrm{tr}_{adj}(F^6) + \ldots$$

For local anomalies to vanish, we therefore need $\dim G = 496$.

In fact, there are versions of string theory that satisfy this: type I string theory ($G = SO(32)$) and heterotic ($G = SO(32)$ or

$\mathcal{G} = E_8 \times E_8$). In both of these cases $\dim(\mathcal{G}) = 496$, so the purely gravitational anomaly cancels!

The good news don't end here: for these choices of $\mathcal{G}$ $c_3(\mathcal{G})$ factors: $\mathrm{tr}_{adj}(F^6) = \mathrm{tr}_{adj} F^2 (\ldots)$ and in fact the whole of I_{12} factorises:

$$I_{12} \propto (\mathrm{tr} R^2 - \mathrm{tr}_{adj} F^2)\, \underbrace{X_8}_{\mathrm{tr} R^4 + \ldots + \#\mathrm{tr}_{adj} F^4}$$

Whenever this happens, there is a way to cancel the anomaly, the Green–Schwarz mechanism:

- We postulate that under gauge/Lorentz transformations, B changes in such a way that $H = dB + (\Omega^R_{CS} - \Omega^{\mathcal{G}}_{CS})$ is gauge invariant (so we change the geometric nature of B! See for example hep-th/0011220 for more on the definition of the B field in these theories.)

  [Note: This modification is actually required by consistency of the supergravity action at one loop.]

  Note that $dH = \mathrm{tr} R^2 - \mathrm{tr} F^2$.

- Additionally, we add a coupling to the $10d$ action of the form

  $$B_2 \wedge X_8 .$$

  This coupling is not gauge invariant, due to the modified transformation laws of B, but its gauge transformation precisely cancels the anomaly from the fermionic sector.

- Toy example: $U(1)_A \times U(1)_B$. Under transformations of $U(1)_A$ we pick up a term $\sim \alpha(x) F_1^2$, which we can cancel by introducing an axion.

Loose ends:

· Other anomaly free 10d $\mathcal{N}=(1,0)$ sugras:

Adams, DeWolfe, Taylor hep-th/1006.1352

Montero, Vafa 2008.11729

Cvetič, Dierigl, Lin, Zhang 2203.03644

· Anomaly theories for all this

Monnier hep-th/1903.02828

· Dai-Freed anomalies for string theory: status

Monnier hep-th/1903.02828

Freed, Hopkins 1908.09916

Debray, Dierigl, Heckman, Montero 2107.14227

Basile, Debray, Delgado, Montero 2310.06895