# Invertible topological phases and anomalies

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## Contents

# 1 Lecture 1

Why study gapped phases? They exist (Quantum Hall Effect, topological insulators, spin liquids). They are simpler than gapless states of matter. They are windows to more general phases of matter.

References:

- A. Kapustin `https://arxiv.org/abs/1403.1467`
- A. Kapustin et al `https://arxiv.org/abs/1406.7329`
- E. Witten `https://arxiv.org/abs/1508.04715`

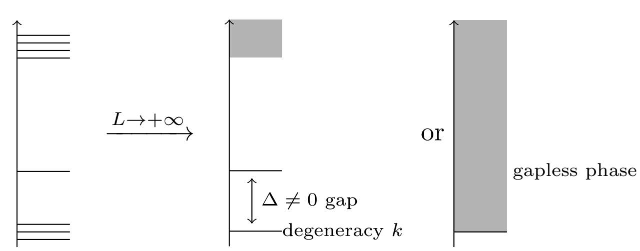
## 1.1 Phases of matter

Consider a system where the system's Hilbert space is $V = \bigotimes_i V_i$ with $V_i$ finite-dimensional Hilbert space of a local site. The Hamiltonian is also written as

$$H = \sum_i H_i$$

where $H_i$ is a Hamiltonian involving sites near $i$.

Consider a big periodic lattice with $L^d$ sites. Its spectrum can look like



with a gap between ground state and first excited level, or no gap. We will be interested in the first case (gapped phase).

### 1.1.1 Equivalence relation

We say that $H_1$ and $H_2$ belong to the same gapped phase if there is a (continuous) family[1] $H(s)$, $s \in [0,1]$ with $H(0) = H_1$, $H(1) = H_2$ and the gap $\Delta$ remains non-zero for all $s$.

### 1.1.2 Trivial phase

A **trivial phase** is one where the Hamiltonian can be brought to the free form

$$H_0 = -\sum_i |\phi\rangle_i \langle\phi|_i \,.$$

### 1.1.3 TQFT

Surprisingly a large set of gapped phases of lattice systems is described by a TQFT (topological quantum field theory) instead of being trivial.

## 1.2 Adding symmetries

In condensed matter systems we have many symmetries (translation, discrete rotation, glide, internal symmetries). Consider an internal unitary symmetry, characterized by unitary operators $U(g)$ with $U(g)U(h) = U(gh)$. It is called an on-site symmetry if it takes the form

$$U(g) = \prod_i U_i(g)$$

where $U_i(g)U_i(h) = U_i(gh)$ satisfies a usual group multiplication law.

A Hamiltonian $H = \sum_i H_i$ is symmetric under this symmetry provided $U(g)H_iU(g)^\dagger = H_i$.

## 1.3 Invertible phases

### 1.3.1 Stacking

Let us define *stacking* (in the sense of physically putting one on top of the other) two phases of matter $A$ and $B$ to obtain a system $A \oplus B$. Define

$$V = V_A \otimes V_B, \qquad H_{A\oplus B} = H_A \otimes 1 + 1 \otimes H_B.$$

We want to keep track of locality in the sense that we consider the $i$-th site of $A \oplus B$ to come from the $i$-th site of $A$ and that of $B$.[2]

[1] In fact, one must allow for stabilisation, namely adding a trivial part to the Hamiltonian to make the Hilbert space dimensions match at least.

[2] To be precise we don't really need the lattices to have exactly the same shape.

### 1.3.2 Invertibility

A phase is invertible if exists a phase $-A$ such that $A \oplus (-A) = 0$ is the trivial phase (continuously deformable to $H_0$). Gapped phases with an inverse are called "invertible gapped phases". Invertible phases form an abelian group under the stacking operation. All of these definition make sense with a particular symmetry group.

### 1.3.3 Example of non-invertible phase

Suppose a phase $A$ has (as a representative) a Hamiltonian with $k$-fold degenerate ground state, and likewise $B$ has a $m$-fold degenerate ground state. Then $A \oplus B$ has $km$ ground state degeneracy. This can only be trivial if $km = 1$. Thus, any phase with $k > 1$ ground state degeneracy on any manifold (e.g., torus) is non-invertible. Concrete examples are given by theories of anyons for instance.

Contrapositively, *invertible phases have a unique ground state on any spatial manifold.* [3]

Question (converse): if a phase has a unique ground state on every manifold, is it automatically invertible? (The consensus in the room seems to be yes provided we talk only about invertible symmetries, and not SPT based on non-invertible symmetries.)

Question: Why not track degeneracies of excited states too? Because even if these degeneracies change, that does not lead to non-analyticity in the partition functions on compact manifolds, and other similar observables. Thus, no phase transition.

### 1.3.4 Classification of invertible phases with no symmetry

Purely bosonic theories:

| dimension | group | generator |
|---|---|---|
| $1+1$ | – | – |
| $2+1$ | $\mathbb{Z}$ | $E_8$, $c_- = 8$ |
| $3+1$ | – | – |

Theories with fermions as well (means we need to respect $(-1)^F$ fermion number, but this requires having a spin structure — this notion is subtle on general lattices):

| dimension | group | generator |
|---|---|---|
| $1+1$ | $\mathbb{Z}_2$ | Kitaev wire |
| $2+1$ | $\mathbb{Z}$ | $p+ip$ superconductor, $c_- = 1/2$ |
| $3+1$ | – | – |

[3]Here there is a subtlety in the vocabulary: are we talking about properties

(Bruno vaguely remembers:) Stacking 16 copies of the “$p + ip$ superconductor” Hamiltonian gives the generator corresponding to purely bosonic theories.

It is believed that the mathematical interpretation of these types of classifications are **(spin) bordisms groups** enriched by suitable symmetries.

#### 1.3.5 Non-liquid phases

Technically here we cheated, we have classified topological quantum field theories. But some gapped phases are not described by a TQFT. Consider a infinite stack[4] of (2 + 1 dimensional) $p + ip$ superconductors to make a 3 + 1 dimensional lattice, with a clear preferred direction. This has no TQFT description. Related to fractons.

# 2 Lecture 2

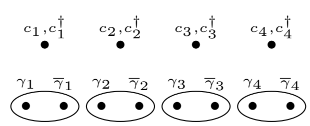
## 2.1 Kitaev chain

### 2.1.1 Trivial spin chain

Consider a spin chain with a fermion (with a complex two-dimensional Hilbert space $V_i$) at each site, with creation/annihilation operators with $\{c_i, c_j^\dagger\} = \delta_{ij}$. Split each site into Majoranas, $c_j = \frac{1}{2}(\gamma_j + i\overline{\gamma}_j)$, with[5]

$$\begin{aligned} \gamma_j^\dagger = \gamma_j, \qquad \overline{\gamma}_j^\dagger = \overline{\gamma}_j, \\ \{\gamma_i, \gamma_j\} = \{\overline{\gamma}_i, \overline{\gamma}_j\} = 2\delta_{ij}, \qquad \{\gamma_i, \overline{\gamma}_j\} = 0. \end{aligned}$$

Draw these as two sites in a picture:




Then the trivial Hamiltonian can be written as

$$H_0 = \sum_{i=1}^{N} i\gamma_i\overline{\gamma}_i = \sum_{i=1}^{N} (2c_i^\dagger c_i - 1)$$

The unique ground state has $c_i^\dagger c_i \left|\text{g.s.}\right\rangle_0 = 0$; $i\gamma_i\overline{\gamma}_i \left|\text{g.s.}\right\rangle_0 = -\left|\text{g.s.}\right\rangle_0$, gap $\Delta = 2$.

[4] Previously we only stacked a few systems so the dimension remained the same. Here we can either think of it as keeping the same dimension of treating the stacking label as an extra dimension.

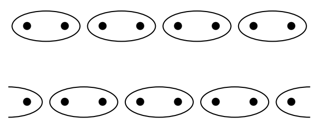
[5] Concretely, one can take $c_j = \cdots \otimes 1 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots$ where the non-trivial matrix acts on the $V_j$ factor of the Hilbert space. In shortened notation, $c_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j$. Then $c_j^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j$, $\gamma_j = (\sigma_1)_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j$, $\overline{\gamma}_j = (\sigma_2)_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_j$.

### 2.1.2 The same trivial spin chain

On a closed chain with $N$ complex fermions, the $2N$ Majoranas can be relabeled slightly to write an equivalent trivial Hamiltonian (with the understanding $\gamma_{N+1} = \gamma_1$)

$$H_1 = \sum_{j=1}^{N} i\overline{\gamma}_j \gamma_{j+1} = \sum_{j=1}^{N} \Big[ -(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1} \Big]$$

We are really solving the same problem but pairing up the Majorana differently:



Again we get a unique ground state $|\text{g.s.}\rangle_1$ which obeys $i\overline{\gamma}_i \gamma_{i+1} |\text{g.s.}\rangle_1 = -|\text{g.s.}\rangle_1$, gap $\Delta = 2$. This is what happens on a closed chain.

### 2.1.3 Kitaev Hamiltonian

On an open spin chain the second Hamiltonian $H_1$ does not have the $j = N$ term. The Majoranas $\gamma_1$ and $\overline{\gamma}_N$ are unpaired. Then we get a two-fold degeneracy of the ground state, labeled by the eigenvalue of $i\gamma_1\overline{\gamma}_N$ which now commutes with the Hamiltonian. Nota: acting with $\gamma_1$ maps one ground state to the other; same for $\overline{\gamma}_N$ but with sign differences.

This two-fold ground state degeneracy is delocalized in a sense: it is half due to the left boundary and half to the right boundary, so in some sense each boundary contributes a factor of $\sqrt{2}$ to the Hilbert space dimension. Of course this simply reflects the fact that a single Majorana does not exist on its own.

This is the simplest example of anomaly, it is like a gravitational anomaly (because there is no global symmetry involved) of the $1 + 1$ dimensional theory.

Consider now fermion parity.

$$(-1)^F = (-1)^{\sum_{j=1}^{N} c_j^\dagger c_j} = (-i\gamma_1\overline{\gamma}_1)\dots(-i\gamma_N\overline{\gamma}_N) = (-i\gamma_1\overline{\gamma}_N)\prod_{j=1}^{N-1}(-i\overline{\gamma}_j\gamma_{j+1})$$

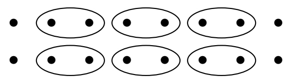
the $N - 1$ last factors act trivially on the ground state, so fermion parity acts as $(-i\gamma_1\overline{\gamma}_N)$ on the ground states.

## 2.2 Stacking Kitaev chains

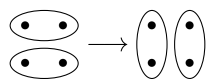
Let us explain that Kitaev + Kitaev = 0. Stack two copies of the Kitaev chain on top of each other. Instead of having a Majorana mode at each end, we now have two Majorana modes, and we will trivialize the Hamiltonian by deforming it.

Here Bruno is a bit confused. The open Kitaev chain has two vacua, so stacking with itself gives $2 \times 2$ vacua. To deform to the trivial Hamiltonian, we must allow to lift these ground states somehow, or to work enriched by $(-1)^F$ or similar, but Max Metlitski did not mention this. Alternatively work on the closed chain.

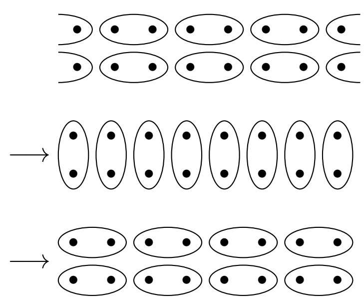
The starting point is



For every two pairs of Majoranas on top of each other (so four Majoranas) we deform the Hamiltonian using the basic building block



This is apparently easy because it is a finite-dimensional Hamiltonian and the number of ground states with each charge under the symmetries (only $(-1)^F$) are the same in both systems.



Consider the stacking of 2 Kitaev chains and look to the boundary. We can gap out the boundary modes by adding:

$$\Delta H = i\gamma_1\eta_1 + i\overline{\gamma}_n\overline{\eta}_n \tag{1}$$

This is a diagnostic that the stacking of 2 Kitaev chains is equivalent to the trivial phase. This can be shown explicitly as well and one concludes that $1+1=0$.

## 2.3 Bulk signature of Kitaev chain

### 2.3.1 Towards twist defects

If you are given a Hamiltonian, and a partition of the sites into a regions $R_1$ and $R_2$, then you can write the Hamiltonian as

$$H = H_1 + H_2 + H_{12}, \qquad H_{12} = \sum_\alpha O_1^{(\alpha)} O_2^{(\alpha)},$$

where the last term $H_{12}$ is the interaction between the two parts of the system.

If you have an on-site symmetry $U(g)$, then you can study its restriction to one part, $U_{R_2}(g) = \prod_{i \in R_2} U_i(g)$, and how it acts on the Hamiltonian:

$$U_{R_2}(g) H U_{R_2}(g)^{-1} = H_1 + H_2 + \sum_\alpha O_1^{(\alpha)} U(g) O_2^{(\alpha)} U(g)^{-1}.$$

This is of course equivalent to the Hamiltonian we started from. But we changed the interactions locally at the boundaries between $R_1$ and $R_2$.

In one space dimension, on a circle (say) the boundary is a pair of points. Now we can do something more subtle: we can perform this change in the interactions but only at one particular point, not both boundaries of $R_2$. Pick a place in your chain, split the Hamiltonian into terms that are only on one side and that concern interactions between the two sides (this only makes sense in a local Hamiltonian):

$$H = H_{\text{sides}} + H_{\text{interaction}}, \qquad H_{\text{interaction}} = \sum_\alpha O_L^{(\alpha)} O_R^{(\alpha)},$$

where $O_L$, $O_R$ are operators acting just on one side of the place you picked. Then twisting gives

$$H = H_{\text{sides}} + \sum_\alpha O_L^{(\alpha)} U O_R^{(\alpha)} U^{-1}.$$

This does not quite break translation symmetry: a combination of translation and acting with the on-site symmetry will remain.

### 2.3.2 Twist defect in the Kitaev chain

Let us insert a twist defect in the closed Kitaev chain, twisting by fermion parity. The original Hamiltonian is

$$H = \sum_{j=1}^N i\overline{\gamma}_j \gamma_{j+1}.$$

The interaction is $i\overline{\gamma}_N \gamma_1$, and the rule is that we conjugate one side by $(-1)^F$, so it becomes $i\overline{\gamma}_N (-1)^F \gamma_1 (-1)^F = -i\overline{\gamma}_N \gamma_1$.

Previously the ground state $|\text{g.s.}\rangle$ obeyed

$$i\overline{\gamma}_N \gamma_1 |\text{g.s.}\rangle = -|\text{g.s.}\rangle.$$

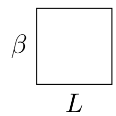
With the twist we get $i\overline{\gamma}_N \gamma_1 |\widetilde{\text{g.s.}}\rangle = |\widetilde{\text{g.s.}}\rangle$ instead.

This process is equivalent to treading a flux $\Phi = \pi$ through the circle.

### 2.3.3 Thermal trace

Consider the system on a circle and study the partition function $Z = \text{Tr}(e^{-\beta H})$, where $\beta = 1/\text{temperature}$. This is equivalent to studying the system on a torus with sides $L$ and $\beta$.

The zero-temperature limit $\beta \to +\infty$ behaves as $e^{-\beta E_{\text{g.s.}}}$, and we can normalize the ground state energy to be zero, so $Z = 1$. With a twist, we still have a single ground state, so $Z_{\Phi=\pi} = 1$. This is inserting $(-1)^F$ "vertically" in the diagram




Now we can twist in the temporal direction instead of the space direction. This amounts to inserting $(-1)^F$ in the trace, so $\widetilde{Z} = \mathrm{Tr}((-1)^F e^{-\beta H})$. Get $\pm 1$ depending on whether we also twist in the spatial direction or not. Overall

| | anti-periodic | periodic (in $x$) |
|---|---|---|
| anti-periodic | 1 | 1 |
| periodic (in $\tau$) | 1 | $-1$ |

(2)

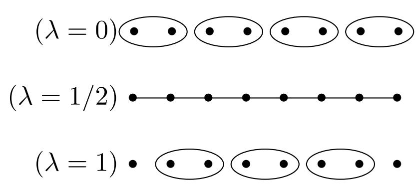
The $-1$ is a hallmark of the Kitaev chain instad of the trivial phase. There is some ambiguity in matching the periodic/anti-periodic boundary conditions with field theory; left as an exercise (in choosing conventions) to the reader.

## 2.4 Placing Kitaev on a manifold

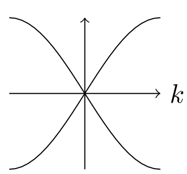
$D = 2$, Euclidean.

### 2.4.1 Interpolation from trivial to Kitaev spin chains

We will need some field theory description for Kitaev. Consider the trivial and Kitaev Hamiltonians $H_0$ and $H_1$ defined previously, and consider $H(\lambda) = (1-\lambda)H_0 + \lambda H_1$ for $\lambda \in [0,1]$. Since the two end-points are in different phases we know the gap has to close at some point. It turns out to be right in the middle, at $\lambda = 1/2$. Pictorially,




The Hamiltonian $H_{\lambda=1/2} = \frac{1}{2}\sum_{j=1}^{N}(i\gamma_j\overline{\gamma}_j + i\overline{\gamma}_j\gamma_{j+1})$ has a dispersion relation



### 2.4.2 Continuum model

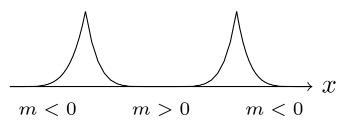
We start with

$$H = \frac{1}{2}\int \left[\chi_R(-i\partial_x)\chi_R + \chi_L(i\partial_x)\chi_L\right],$$

where $\{\chi_R(x), \chi_R(y)\} = \delta(x-y)$ and likewise for $\chi_L$. Turn on a mass term $\delta H = i\int dx \chi_R\chi_L m(x)$. Morally $m$ is $\lambda - 1/2$. This gives

$$H = \frac{1}{2}\int dx(\chi_R(-i\partial_x)\chi_R + \chi_L(i\partial_x)\chi_L) + i\int dx\chi_R\chi_L m(x).$$

For $m > 0$ we have the trivial phase. For $m < 0$ we have the Kitaev phase. So if we consider a mass $m(x)$ whose sign varies we will have transitions between the two phases, and modes localized at the points where $m = 0$. See the exercises.




### 2.4.3 Euclidean Lagrangian and curved space

$$\mathcal{L}_E = \frac{1}{2}\chi^T C^\dagger(\gamma^\mu\partial_\mu + m)\chi, \tag{3}$$

where $\mu = 0, 1$ and $\chi = (\chi_L, \chi_R)$ and (explicitly in terms of Pauli matrices)

$$C = \sigma^2, \qquad \gamma^0 = \sigma^2, \qquad \gamma^1 = \sigma^1, \tag{4}$$

which has $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and

$$C(\gamma^\mu)^* C^\dagger = -\gamma^\mu, \qquad (\gamma^\mu)^\dagger = \gamma^\mu. \tag{5}$$

Placing it on a curved space is done by changing derivatives to covariant ones and including a suitable volume form:

$$S = \frac{1}{2}\int_M d^2x\sqrt{g}\chi^T C^\dagger(\not{D} + m)\chi, \qquad \not{D} = e^\mu_a\gamma^a(\partial_\mu + i\omega_\mu) \tag{6}$$

with $M$ an oriented 2-dimensional manifold with spin structure $\omega$.

### 2.4.4 Partition function and Arf

The partition function is going to be the Pfaffian of the operator $C^\dagger(\not{D} + m)$:

$$Z_m[M, \omega] = \int \mathcal{D}\chi\ e^{-S[\chi]} = \mathrm{Pf}(C^\dagger(\not{D} + m)), \tag{7}$$

because we are simply computing a fermionic Grassmann integral, through the identity

$$\int \mathcal{D}\eta e^{-\eta^T A\eta} = \mathrm{Pf}(A). \tag{8}$$

The operator $-i\not{D}$ is Hermitian. Consider eigenfunctions $\phi_\lambda$ satisfying

$$-i\mathcal{D}\phi_\lambda = \lambda\phi_\lambda, \tag{9}$$

and normalized with respect to the Hermitian inner product:

$$\int d^2x\sqrt{g}\phi^*_\lambda\phi_{\lambda'} = \delta_{\lambda\lambda'}. \tag{10}$$

Then $\phi_\lambda$ and $C\phi^*_\lambda$ have the same eigenvalue so each eigenvalue appears twice. We split as follows, where the prime denotes the fact that we sum over pairs of repeated eigenvalues,

$$\chi = {\sum_\lambda}' (\eta^{(1)}_\lambda \phi_\lambda(x) + \eta^{(2)}_\lambda C\phi^*_\lambda(x)). \tag{11}$$

Then the action can be written quite simply as

$$S = \sum_\lambda \begin{pmatrix} \eta^{(1)}_\lambda \\ \eta^{(2)}_\lambda \end{pmatrix}^T \begin{pmatrix} 0 & i\lambda + m \\ -(i\lambda + m) & 0 \end{pmatrix} \begin{pmatrix} \eta^{(1)}_\lambda \\ \eta^{(2)}_\lambda \end{pmatrix} \tag{12}$$

Therefore:

$$Z_m(M) = \pm {\prod_\lambda}' (i\lambda + m) \tag{13}$$

(again, prime indicates that eigenvalues come in pairs, so we count them once in the above expressions).

How to determine the sign? To cancel it, we consider for some positive mass $M > 0$, the ratio

$$\frac{Z_m}{Z_{|M|}} = {\prod_\lambda}' \frac{i\lambda + m}{i\lambda + |M|}. \tag{14}$$

Why should the unknown sign be the same for $Z_m$ as for $Z_{|M|}$? Because on a compact spacetime manifold (finite system) we do not expect to have a singularity at $m = 0$.

In addition, if we consider that $M > 0$ is the trivial phase, we should expect that we can take $Z_{|M|} = 1$ by tuning local counterterms, namely terms such as

$$S = \int d^2x\sqrt{g} R \log\frac{\Lambda}{|M|} + \dots \tag{15}$$

Varying the mass $M$ itself (while keeping it positive) should also amount to such local counterterms.

So effectively we can think of the ratio $Z_m/Z_M$ as the partition function we care about, and we can even tune to $Z_m/Z_{-m}$. We define the partition function of the non-trivial Kitaev phase to be

$$Z_{\text{Kitaev}}(M,\omega) = \frac{Z_{-|m|}}{Z_{|m|}} = \prod_\lambda \frac{i\lambda - |m|}{i\lambda + |m|} \tag{16}$$

Using the commutation with the chirality matrix $\gamma^3$, one can show that the eigenvalues actually comes in pairs of positive and negative pairs of values (e.g., $1, 1, -1, -1$). Therefore we can write:

$$Z_{\text{Kitaev}}(M,\omega) = \prod_{\lambda>0}{}' \frac{(i\lambda - |m|)(-i\lambda - |m|)}{(i\lambda + |m|)(-i\lambda + |m|)} \prod_{\lambda=0}{}'(-1) = (-1)^{N_0(\not{D})/2} \tag{17}$$

where $N_0(\not{D}) = \dim\ker\not{D}$. Thus the partition function is a $\pm1$ topological invariant. It is exactly the Arf invariant of $M$ with the spin structure $\omega$, i.e., $\text{Arf}(M,\omega)$. This is the Arf CFT.

**The torus case** On the torus, eigenvalues are simply

$$\lambda = \pm\sqrt{k_x^2 + k_y^2}. \tag{18}$$

the only zero modes are $k_x = k_y = 0$ which can only appear by having periodic boundary conditions. We find again the values given in (2) for the four spin structures on the torus, namely $Z = +1$ except for both-periodic boundary conditions, for which $Z = -1$.

### 2.4.5 Continuum theory with boundary

Consider the theory on $\mathcal{M}$ with boundary $\partial\mathcal{M}$. We expect a single Majorana mode on the boundary, with boundary action

$$S = \int_{\partial M} d\tau \psi \partial_\tau \psi.$$

Naively

$$Z_{\text{boundary}} = \text{Pf}(d/d\tau),$$

which again has a sign ambiguity. What we can write instead is

$$Z_{\text{bulk+boundary}} = |\operatorname{Pf}(d/d\tau)|(-1)^{N_0(\not{D},\mathcal{M})/2}$$

where $N_0(\not{D},\mathcal{M})$ is calculated with non-local APS boundary condition [Witten, Yonekura, 1909.08775].

**Question by Slava Rychkov:** why do you say different words about this anomaly compared to Thomas Dumitrescu? **Answer.** In 4d free Maxwell is well-defined, and the problem only arrives when trying to couple the theory to backgrounds for the symmetry. Here the Majorana on the boundary is not really well-defined; it is a sort of global gravitational anomaly, an anomaly in coupling with the spin connection.

# 3 Lecture 3

## 3.1 Bordisms and invertible TQFTs

$$Z[M,\omega] = (-1)^{N_0(\not{D})/2} \qquad \text{bordism invariant} \tag{19}$$

(Atiya-P-Singer; Witten 1508.04715; Nakahara, Geometry Topology and Physics) Closed $d$-manifolds, $X_d$ and $Y_d$ are bordant if exists $(d+1)$-manifold $M_{d+1}$ such that $\partial M_{d+1} = X_d \cup \overline{Y}_d$. We say that $Z[X]$ is a bordism invariant if $Z[X_d] = Z[Y_d]$ for X and Y bordant.

On $d$-manifolds/(bordism equivalence) form an abelian group denoted by $\Omega_d$:

- Addition is disjoint union: $X \cup Y$.
- The zero is the empty set, $0 = \emptyset$.
- The inverse is the orientation reversal: $-X = \overline{X}$.

For invertible TQFT Z is a bordism invariant:

$$Z : \Omega_d \to \mathbb{C}. \tag{20}$$

As a consequence:

$$Z(X) = Z^*(X), \qquad Z[X \cup Y] = Z[X]Z[Y], \tag{21}$$

$$1 = Z(\mathrm{pt}) = Z(X \cup \overline{X}) = Z(X)Z(\overline{X}) = |Z(x)|^2 \tag{22}$$

and therefore:

$$Z : \Omega_d \to U(1). \tag{23}$$

Bosons, no-symmetry: $\Omega_d^{SO}$.
Fermions, no-symmetry: $\Omega_d^{\text{spin}}$.
Unitary symmetry $G$, $\Omega_d^{SO}(BG)$.
Time reversal, bosons $\mathbb{Z}_2^T$.
Fermions with time reversal: $T^2 = (-1)^F$, $T^2 = 1$, $\Omega_d^{\text{Pin}+}$ and $\Omega_d^{\text{Pin}-}$.

### 3.1.1 Fermions in 1+1d with no symmetry

$$\Omega_{d=2}^{\text{spin}} = \mathbb{Z}_2 \tag{24}$$

It is generated by $T(P,P)$ (torus with periodic boundary conditions). Now, we can think about the cobordism classification. That is, maps from $\Omega_2^{\text{spin}} \to \mathbb{C}$.

$$Z_{\text{Kitaev}}(T(P,P)) = -1. \tag{25}$$

### 3.1.2 Bosons in 1+1d no symmetry

$$\Omega^{SO}_{d=2} = \mathbb{Z}_1 \tag{26}$$

### 3.1.3 Bosons in 1+1d with $\mathbb{Z}_2^T$

$$\Omega^{O}_{d=2} = \mathbb{Z}_2 \tag{27}$$

It is generated by $\mathbb{R}P^2$:

$$Z[\mathbb{R}P^2] = -1 \tag{28}$$

Haldane chain.

### 3.1.4 Fermions in 1+1d with $T^2 = +1$

$$\Omega^{\mathrm{Pin}_-}_{d=2} = \mathbb{Z}_8 \tag{29}$$

generated by $\mathbb{R}P^2$ (has 2 spin structures):

$$Z_k(\mathbb{R}P^2) = e^{\pm 2\pi i k/8} \tag{30}$$

(plus or minus depending on the spin structure).

### 3.1.5 Fermions in 3+1d with $T^2 = (-1)^F$

Non-interacting classification $\mathbb{Z} \to \mathbb{Z}_{16}$:

$$\Omega^{\mathrm{Pin}_+}_{d=4} = \mathbb{Z}_{16} \tag{31}$$

generated by $\mathbb{R}P^4$

$$Z_{k=1}[\mathbb{R}P^4] = e^{\pm 2\pi i/16}. \tag{32}$$

## 3.2 More generally

$$\Omega_d = \mathbb{Z}_{n_1} \otimes \mathbb{Z}_{n_2} \cdots \otimes \mathbb{Z}_{n_k} \otimes \mathbb{Z}^n \tag{33}$$

Then:

- $\mathbb{Z}_n \to U(1)$, $1 \mapsto e^{\frac{2\pi i}{n} k}$, $k = 0, 1, \ldots, n-1$
- $\mathbb{Z} \to U(1)$, $1 \to e^{i\theta}$, $\theta \in [0, 2\pi)$.

Invertible TQFTs in d-dimensions, $\mathrm{Tor}(\Omega_d)$.

Fermions in 2+1d, invertible phases $\mathbb{Z}$: $p+ip$ sc (2+1d).

$$\mathcal{L}_E = \frac{1}{2}\chi^T C^\dagger(\gamma^\mu\partial_\mu + m)\chi \tag{34}$$

boundary

$$\mathcal{L} = \frac{1}{2}\chi_R(\partial_\tau - i\partial_x)\chi_R, \qquad c = \frac{1}{2}. \tag{35}$$

We have

$$\frac{Z_{-|m|}}{Z_{|m|}} = \frac{\mathrm{Pf}(C^\dagger \not{D} - |m|)}{\mathrm{Pf}(C^\dagger \not{D} + |m|)} = \prod_\lambda \frac{i\lambda - |m|}{i\lambda + |m|} \tag{36}$$

The computation is easier when $m \to \infty$:

$$\lim_{m\to\infty} \frac{Z_{-|m|}}{Z_{|m|}} = e^{\frac{i\pi}{2}\eta(-i\not{D})} = Z_{p+ip} \tag{37}$$

It turns out that this is not strictly a topological invariant.

$\eta$ depends on the metric $g_{\mu\nu}$. It says that

$$\eta(-i\not{D}) = \frac{1}{4}\frac{-1}{24\pi^2}\int_{M_4}\mathrm{tr}(R\wedge R), \qquad \mathrm{mod}\ 4 \tag{38}$$

if one is not careful, one would write:

$$\int_{X_3}(\omega\wedge d\omega + \frac{2}{3}\omega^3) \tag{39}$$

which is not correct because $\omega$ is not globally defined:

$$Z_{p+ip} = e^{CS_g[X_3,\sigma]}, \qquad CS_g = \frac{2\pi}{16}\frac{-1}{24\pi^2}\int_{M_4}\mathrm{tr}(R\wedge R) \tag{40}$$

This theory, $Z_{p+ip}$ gives an invertible phase of fermions in 2+1d but with phase depending on the metric. The signature $\sigma$ of $Y_4$ is an invariant of 4-dimensional manifolds, and it is also a bordsim invariant.

$$\Omega^{\mathrm{Spin}}_{d=4} = \mathbb{Z} \tag{41}$$

The general belief is that all invertible phases in $d$-dimensions:

$$\mathrm{Tor}(\Omega_d) \oplus \mathrm{Free}(\Omega_{d+1}) \tag{42}$$

the second part is Chern-Simons terms in $d$.

### 3.3 Why bordsms should classify invertible phases?

Invertible unitary $\mathrm{TQFT}_d$ with $Z(s^d) = 1$, $Z$ is a bordism invariant. Proved by Moore-Freed, hep-th/0403135, Yonekura, 1803.10796.

The prove uses surgery. Take a manifold $X_d$ and remove from it $S^d \times D^{d-p}$ and then glue back $D^{p+1} \times S^{d-p-1}$. Crucially, $\partial(S^p \times D^{d-p})\partial(D^{p+1} \times S^{d-p-1}) = S^p \times S^{d-p-1}$.

Any bordism can be decomposed into a sequence of surgery operations. By proving that it does not change by jumping, where one uses invertibility and that $Z(S^d) = 1$.

# 4 Lecture 4

## 4.1 Invertible TQFT is a bordism invariant

Here we will prove that for an invertible TQFT with $Z(S^d) = 1$ Z is a bordism invariant.

#### 4.1.1 Bordism can be decomposed by a sequence of surgery

Lets prove that any bodism $Y_d$ can be decomposed by a sequence of surgery operations.

We cut $Y_d = (X_d \mathrm{cut}(S^p \times D^{p-p}) \cup D^{p+1} \times S^{d-p-1}$ (they have the same boundary). Suppose $d = 2$, and $p = 0$,

(two manifolds are bordant if and only if they can be connected by surgery.)

Consider $f : M_{d+1} \to [0, 1]$. We have $\partial_\mu f(p_i) = 0$ with $f(p_i) \neq f(p_j)$ for $i \neq j$. The inverse $f^{-1}(s)$ is smooth except inear the critical points.

Focus on $\partial_\mu f(p) = 0$ and $f(p) = v$. We compare $f^{-1}(v + \epsilon)$ and $f^{-1}(v - \epsilon)$:

$$f(x) = v - \sum_{i=1}^{n} x_i^2 + \sum_{i=n+1}^{d+1} x_i^2 \tag{43}$$

$f^{(-1)}(v + \epsilon)$ and $f^{(-1)}(v - \epsilon)$ agrees except near $\epsilon = 0$.

$$N = f^{-1}\mathrm{cut}\{\sum_{i=1}^{d+1} x_i^2 \leq \delta\} \tag{44}$$

the boundary is $S^d = \{x : \sum_{i=1}^{d+1} x_i^2 = \delta\}$. $S^d = R_+ \cup R_-$ and $f(R_+) \geq v$ and $f(R_-) \leq v$.

$$\sum_{i=n+1}^{d+1} x_i^2 - \sum_{i=1}^{n} x_i^2 \geq 0, \qquad \sum_{i=n+1}^{d+1} x_i^2 + \sum_{i=1}^{n} x_i^2 = \delta \tag{45}$$

therefore:

$$\sum_{i=1}^{n} x_i^2 = \frac{\delta}{2}, \qquad D^n \tag{46}$$

We conclude that $R_+ \cong D^n \times S^{d-n}$ and $R_- \cong S^{n-1} \times D^{d-n+1}$.

Let $X_+ = N \cup R_+$ vs $X_- = N \cup R_-$

$$X_+ \cong f^{-1}(v+\epsilon), \qquad X_- \cong f^{-1}(v-\epsilon) \tag{47}$$

But $X_+ \cong (X_-\text{cut } R_-) \cup R_+$. So we proved that a bordism can be constructed by surgery.

### 4.1.2 Proof

Now lets prove the main result. Suppose $Y_d = (X_d\text{cut } S^p \times D^{d-p}) \cup D^{p+1} \times S^{d-p-1}$. Claim $Z(Y_d) = Z(X_d)$. Indeed:

$$Z(X_d) = \langle S^p \times D^{d-p} | X_d \text{ cut } S^p \times D^{d-p} \rangle \tag{48}$$

$$Z(Y_d) = \langle D^{d+1} \times S^{d-p-1} | X_d \text{ cut } S^d \times D^{d-p} \rangle \tag{49}$$

Now we use invertibility:

$$\begin{aligned} Z(X_d) &= \frac{\langle S^p \times D^{d-p} | D^{p+1} \times S^{d-p-1} \rangle \langle D^{p+1} \times S^{d-p-1} | X_d \text{ cut } S^p \times D^{d-p} \rangle}{\langle D^{p+1} \times S^{d-p-1} | D^{p+1} \times S^{d-p-1} \rangle} \\ &= \frac{Z(S^d) Z(Y_d)}{Z(S^{p+1} \times S^{d-p-1})} \\ &= Z(Y_d) \end{aligned} \tag{50}$$

Where in the last line we used the lemma: if $Z(S^d) = 1$ then $Z(S^p \times S^{d-p}) = 1$. pf:

$$\begin{aligned} Z(S^d) &= \langle D^p \times S^{d-p} | S^{p-1} \times D^{d-p} \rangle \\ &= \sqrt{Z(S^p \times S^{d-p}) Z(S^{p-1} \times S^{d-p+1}} \end{aligned} \tag{51}$$

Then $Z(S^p \times S^{d-p}) = Z(S^{p-1} \times S^{d-p+1})^{-1}$.

Claim: for odd $d$, $Z(S^d) = 1$.

$$\begin{aligned} Z(S^3) &= \langle D^2 \times S^1 | S^1 \times D^2 \rangle = \sqrt{\langle D^2 \times S^1 | D^2 \times S^1 \rangle \langle S^1 \times D^2 | S^1 \times D^2 \rangle} \\ &= Z(S^1 \times S^2) \\ &= 1 \end{aligned} \tag{52}$$

$Z(S^1 \times M_{d-1}) = 0$ or $M_{d-1} = 1$. For d even we can modify the theory:

$$Z(X_d) \to \lambda^{\chi(X_d)} Z(X_d) \tag{53}$$

$\lambda > 0$ and $\chi(X_d)$ is the Euler character:

$$\chi(X_d) = \int_{X_d} e(\Omega) \tag{54}$$

e.g.

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R \tag{55}$$

Now we can choose $\lambda$ so that $Z(S^d) = 1$ (why is $Z(S^p) > 0$? Can be obtained by gluing together two hemispheres, so it is equal to the inner product $\langle D^p | D^p \rangle$. Could $|D^p\rangle = 0$? If so, everything would be zero, because $Z(X_d) = \langle D^p | X_d \text{ cut } D^p \rangle$. Violates invertibility, which implies that there is a single ground state in every manifold).

## 4.2 Dijkgraaf-Witten theories

Start with invertible phase without symmetry, add a symmetry $G$. In particular, starting with trivial invertible phase (no symmetry). One gets to symmetry protected topological phase (SPT). There are protected edge states.

For dimension $d \leq 4$ it is believed that it constructs all SPTs for bosons with $G$-unitary. (reference: Dijkgraaf and Witten, Comm. Math. Phys. 129, 393 (1990); Chan, Ga, Lu, Wen, 1106.4772 (relation to lattice phases)).

Construct $Z$ for $M_d$: $s[v_0, \ldots, v_d] = \pm 1$. In each edge one places $g \in G$:

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} e^{-S[\{g_v\}]}, \quad e^{-S} = \prod_{\Delta_d} \nu(g_{v_0}, g_{v_2}, \ldots, g_{v_d})^{s(\Delta)} \tag{56}$$

with $\nu \in H^d(G, U(1))$.

Group cohomology: A homogeneous co-chain $f \in \tilde{C}^n(G, A)$:

$$f : C^{n+1} \to A \tag{57}$$

satisfying $f(gg_0, gg_1, \ldots, gg_n) = f(g_0, g_1, \ldots, g_n)$ for any $g \in G$. Now we define $d : \tilde{C}^n \to \tilde{C}^{n+1}$:

$$(df)(g_0, g_1, \ldots, g_{n+1}) = \prod_{i=0}^{n+1} f(g_0, g_1, \ldots \hat{g}_i, \ldots, g_{n+1})^{(-1)^i} \tag{58}$$

with $\hat{g}_i$ excluded. Then $\tilde{H}^n(G, A) = \ker(d_n)/\mathrm{im}(d_{n-1})$.

Inhomogeneous cochains: $w \in C^n(G, A)$.

$$w(g_1, g_2, \ldots, g_n) = f(1, g_1, g_1 g_2, \ldots, g_1 g_2, \ldots g_n). \tag{59}$$

If $X_d$ is closed, $Z(X_d) = 1$. Proof: $\nu(v_0, v_1, \ldots, v_d) = \nu(g_{v_0}, g_{v_1}, \ldots, g_{v_d})$. $d\nu = 0$ (simplicial cohomology sense).

$$1 = \int_{X_d \cup \nu_\infty} d\nu = \int_{X_d} \nu. \tag{60}$$

To diagnose the non-triviality we need to couple to background $G$-gauge field. They live in the edges: with $u_{ij} \in G$ obeying a flatness condition and local gauge transformations. To couple it we replace $\nu(g_0, g_1, \ldots, g_d) \to \omega(g_0^{-1} u_{01} g_1, g_1^{-1} u_{12} g_2, \ldots, g_{d-1}^{-1} u_{d-1,d} g_d)$. IF you just consider $\omega(u_{01}, u_{12}, \ldots, u_{d-1,d})$. Can show that

$$\int_{X_d} \omega = \int_{X_d} \omega'. \tag{61}$$

Then

$$Z(X_d, u) = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \int_{X_d} \omega = \prod_{\Lambda \in X_d} \omega(u_{01}, u_{12}, \ldots, u_{d-1,d})^{S(\Delta)} \tag{62}$$

For example: $G = \mathbb{Z}_2$:

$$\omega(g_1, g_2, g_3) = (-1)^{g_1 g_2 g_3}, \qquad g_i \in \{0, 1\}. \tag{63}$$

$$Z[X_3, a] = (-1)^{\int_{X_3} a \cup a \cup a} \tag{64}$$

for $X_3 = \mathbb{R}P^3$ we get $Z(\mathbb{R}P^3) = -1$.