Nonrelativistic CFTs

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References:

- Nishida, Đàm Thanh Sơn `https://arxiv.org/abs/0706.3746`
- Hammer, Đàm Thanh Sơn `https://arxiv.org/abs/2103.12610`

1 Non-relativistic QFTs

In these lectures d denotes the number of spatial dimensions. Morally, speed of light is infinite, so the light-cone is everything with $t > 0$.

1.1 A useful model

Free theory. Consider

$$S = \int dt d^d x \Big(i\psi^\dagger \partial_t \psi - \frac{|\nabla\psi|^2}{2m} \Big)$$

The equation of motion is the Schroedinger equation (not for the wavefunction, but for the field ψ).

$$i\partial_t \psi = -\frac{\nabla^2 \psi}{2m}.$$

This shows up usually when discussing second-quantization. When quantizing the theory we impose commutation (or anti-commutation if we wanted to work with fermions)

$$[\psi_x, \psi_y^\dagger] = \delta(x-y).$$

We can expand into plane waves:

$$\psi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} a_k.$$

Contrarily to relativistic theory, ψ only contains annihilation operators, no creation operators, so $\psi(x)|0\rangle = 0$. *Question:* why? *Answer:* just see that this verifies the commutation relation, and contrarily to the relativistic case we do not need to ensure causality, so there is no need to add more stuff.

This is a realtively boring theory: the Hilbert space is the Fock space $a_{k_1}^\dagger \dots a_{k_n}^\dagger|0\rangle$, with energy $E = \sum_{i=1}^n k_i^2/(2m)$. It is a superposition of plane waves.

Turning on an interaction term. We want a non-relativistic version of the ϕ^4 theory:

$$S = \int dt d^d x \Big(i\psi^\dagger \partial_t \psi - \frac{|\nabla\psi|^2}{2m} - \frac{c}{2}\psi^\dagger\psi^\dagger\psi\psi \Big).$$

We want to do some power-counting. The mass m is just there to translate from momentum to energy (just like the speed of light in relativistic theories). So we set dimensions $[m] = 0$, and $[\nabla_i] = 1$, so $[\partial_t] = 2$, so $[dt] = -2$, $[d^d x] = -d$, and overall we want $[\psi] = d/2$ to get a dimensionless action. Finally, $[c] = 2-d$. The four-point interaction behaves differently in $d < 2$, $d = 2$ and $d > 2$ dimensions.

- If $d < 2$ the interaction term is relevant.
- If $d = 2$ the interaction term is marginal.
- If $d > 2$ the interaction term is irrelevant.

We will be mostly interested in the case $d = 3$, but for the moment let us concentrate on the case $d = 2$ which looks more interesting at first.

Many questions. *Question:* why is m dimensionless? *Answer:* there is a deeper reason which is that m is a parameter in the Galilean algebra so it is not renormalized. *Question:* what if there are multiple species? *Answer:* then a combination of the masses (times the number of each particles) is non-renormalized.

Question: can such a theory be obtained as a non-relativistic limit of a relativistic one? *Answer:* yes, take ϕ^4 theory and focus on the kinematic sector of the theory where we are just above the threshold of creating n particles, so that they all have very little energy. *Question:* but the relativistic kinetic term has two time derivatives $|\partial_t\phi|^2$; where did one time derivative disappear?

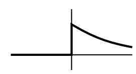
Answer: in the suitable limit we have $\phi = e^{-imt}\psi/\sqrt{2m}$ where ψ is smooth while the prefactor is oscillatory. Then inserting in the usual ϕ^4 Lagrangian, and dropping the highly oscillating terms gives the non-relativistic Lagrangian.

Question: where did the dimension of time change in this process?

1.2 Beta function

Feynman rules. The Green function in non-relativistic theories is retarded:

$$G(t,x)\begin{cases} = 0 & t<0, \\ \neq 0, & t>0. \end{cases}$$

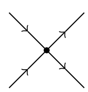


Indeed

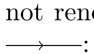
$$\langle 0|T\psi(t,x)\psi^\dagger(0,0)|0\rangle = \langle 0|\psi^\dagger(0,0)\psi(t,x)|0\rangle = 0 \qquad t<0$$

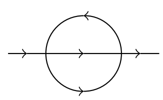
since $\psi|0\rangle = 0$. After Fourier transform one finds a propagator with the following $i\epsilon$ prescription, and a four-point vertex:

$$G(\omega,p) = \frac{i}{\omega - \frac{p^2}{2m} + i\epsilon},$$



$$\text{vertex} = -2ic.$$

Non-renormalization of the mass. Then we can check that the mass is not renormalized (at least at first order) by drawing the leading correction to :

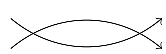


$$= (\text{coef})\big(G(t_x - t_y, x-y)\big)^2 G(t_y - t_x, y-x)$$

there is always one of the propagators going in the wrong time direction.

Renormalization of the four-point vertex. Corrections to the four-point vertex:

$$= 0,$$



$$\neq 0.$$

In $d=2$ we eventually get the beta function

$$\beta(c) = \frac{c^2}{2\pi} \qquad \text{for } d=2.$$

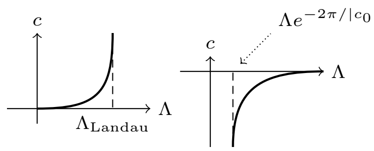
Then the flow can be integrated explicitly by solving

$$\frac{\partial c(\Lambda)}{\partial \log\Lambda} = \beta(c), \qquad c(\Lambda_0) = c_0.$$

This gives

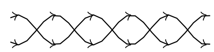
$$c(\Lambda) = \frac{c_0}{1 + \frac{c_0}{2\pi}\log\frac{\Lambda_0}{\Lambda}}$$

If the coupling constant starts positive, $c_0 > 0$, then in the IR, $c \to 0$, but at finite $\Lambda_{\text{Landau}} = \Lambda_0 e^{2\pi/c_0}$ we get a Landau pole. Starting instead from $c_0 < 0$ we get an IR divergence



Question: isn't $c < 0$ sick because the potential is unbounded? *Answer:* no vacuum instability because in this theory the number of particles is fixed.

Two-particles potential. The two-particles potential can be computed by Feynman diagrams of the form



because the particle number $N = \int dx \psi^\dagger \psi$ is conserved by the evolution. The Hilbert space splits into a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ where $\mathcal{H}_n$ has n particles.

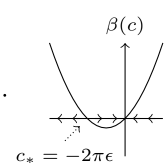
For some critical value of c_0 one finds a confining potential, leading to a bound state.

1.3 Non-relativistic conformal theories

1.3.1 Epsilon expansion

We consider $d = 2 + \epsilon$ dimensions. The beta function is

$$\beta(c) = \epsilon c + \frac{c^2}{2\pi}.$$



For positive c, or small enough negative c the RG flow make $c \to 0$. For negative enough c there is a particular value of c_0 that gives a fixed point.

1.3.2 Schrödinger (non-relativistic conformal) symmetry

What are the invariances of the Schrödinger equation $i\partial\psi/\partial_t = -\nabla^2\psi/(2m)$?

- Spatial translations $\psi(t, x) \to \psi'(t, x) = \psi(t, x + a)$, also spatial rotation, reflection.

- Phase rotation $\psi = e^{i\alpha}\psi$.
- Galilean boost[1] K_i defined by $v_i K_i \colon \psi \to \psi' = e^{im\,v\cdot x - imv^2 t/2}\psi(t, x - vt)$ for a vector v. The phase factor cancels the effect of how time derivative now acts on the (boosted) spatial coordinate.
- Dilatation $\psi \to \psi' = \frac{1}{\lambda^{d/2}}\psi(\lambda^2 t, \lambda x)$.
- Proper conformal transformation[2]

$$C \colon \psi \to \psi'(t, x) = \frac{1}{(1+\alpha t)^{d/2}} e^{\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}} \psi\Big(\frac{t}{1+\alpha t}, \frac{x}{1+\alpha t}\Big).$$

1.3.3 Non-relativistic theory from light-cone restriction of a relativistic CFT

In principle, it would be good to directly take a non-relativistic limit of a relativistic theory. But this is tricky to do generally because the relativistic theory wants to generate particles whereas we would want a limit with constant particle number.

Consider a $(d+1)+1$ dimensional Minkowski space. The Klein–Gordon equation reads (with $i = 1, \ldots, d$)

$$\Big(-\partial_t^2 + \partial_i\partial_i + \frac{\partial^2}{\partial y^2}\Big)\phi = 0.$$

Switch to light-cone coordinates $x^\pm = (t \pm y)/\sqrt{2}$. Then the equation becomes

$$(-2\partial_+\partial_- + \partial_i\partial_i)\phi = 0.$$

Then require the field to take the form $\phi = e^{imx^-}\phi(x^+, x^i)$ then the equation becomes

$$\Big(-2im\frac{\partial}{\partial x^+} + \partial_i\partial_i\Big)\phi(x^+, x^i) = 0,$$

which is the Schrödinger equation. The Schrödinger algebra should arise by taking a similar operation on the $(d+1)+1$ dimensional conformal algebra $\mathfrak{so}(d+2, 2)$: select generators that commute with one light-cone momentum P^+ (this forms a Lie algebra, which incidentally includes P_+ itself). Conversion from the relativistic conformal symmetry to the non-relativistic (here M is the total mass, N is the particle number)

relativistic	Schrödinger
P^+	$M = mN$
P^-	H
$D + M^{+-}$	D
M^{i+}	K^i
$K^+/2$	C

[1] The precise expression needs to be checked.

[2] It looks like a special conformal transformation, a combination of inversion, translation, inversion.

1.3.4 A few facts

In the Schrödinger algebra,

- N is central, namely $[N, \text{anything}] = 0$;
- $[K_i, P_j] = i\delta_{ij} M$ where M is the total mass $M = mN$;
- $[D, P_i] = iP_i,\ [D, K_i] = -iK_i,\ [D, H] = 2iH,\ [D, C] = -2iC$ scaling dimensions, consistent with the earlier naive dimension assignment;
- $[C, H] = iD$ so C, D, H form a $SO(2,1)$ algebra.

Question: can a theory be invariant under P_i, K_i, H, D (and M), but not C? *Answer:* ?

Introduce some operators. From $\psi(x)$ and $\psi^\dagger(x)$ we build

$$n(x) = \psi_x^\dagger \psi_x$$
$$j(x) = \frac{-i}{2} \psi^\dagger \overleftrightarrow{\nabla} \psi.$$

Then

$$[n(x), n(y)] = 0, \qquad [n(x), j^i(y)] = -in(y)\nabla^i \delta(x-y),$$
$$[j^i(x), j^j(y)] = -i(j^j(x)\partial_i + j^i(y)\partial_j)\delta(x-y).$$

These are related to diffeomorphism invariance. (Bruno is lost.) But these are just operators, they typically don't commute with the Hamiltonian since we are doing quantum mechanics.

Then we can express many symmetry generators (but not the Hamiltonian for instance) in terms of these currents as

$$N = \int dx\, n(x), \qquad K_i = \int dx\, x_i n(x), \qquad C = \int dx\, x^2 n(x),$$
$$P = \int dx\, j(x), \qquad D = \int dx\, x \cdot j.$$

We have $\partial_t n + \nabla \cdot j = 0$, which lets us compute the time derivative of the moments N, K_i and C. We have $[H, K_i] \sim \int dx x \nabla j \sim P_i$. We can compute all the commutators between these operators. The most non-trivial aspect is how H commutes with D. This is what distinguishes theories that are scale-invariant from those who are not.

Claim: at the critical point (at the fixed point) the commutator of the Hamiltonian and dilation is the one we expect (the scale-invariant one).

Claim: if the theory is constructed from ψ and $\psi^\dagger$, then scale-invariance implies Schrödinger symmetry.