Generalized Symmetries and Gauging in 2d CFTs
Exercise Sheet 1

Exercise 1: Cardy formula from modular invariance

The asymptotic (high energy) density of states in a 2d CFT is controlled by the Cardy formula. We write the CFT partition on a rectangular torus $\tau = \frac{i\beta}{2\pi}$ as,

$$Z(\beta) = \int d\Delta \rho(\Delta) e^{-\beta(\Delta - c/12)}\,, \tag{1}$$

where $\rho(\Delta)$ is the density of states. In the following we derive the famous Cardy formula from modular invariance, here for imaginary τ, given by

$$Z(\beta) = Z(\frac{4\pi^2}{\beta})\,. \tag{2}$$

(a) Assuming a gap in the spectrum of Δ above the vacuum given by Δ_{gap} (i.e. $\Delta \geq \Delta_{\text{gap}}$ for any non-identity operator), the RHS of (2) is dominated by the vacuum contribution in the high temperature limit $\beta \to 0$,

$$e^{\frac{\pi^2 c}{3\beta}}(1 + \mathcal{O}(e^{-\frac{4\pi^2 \Delta_{\text{gap}}}{\beta}})) = \int d\Delta \rho(\Delta) e^{-\beta(\Delta - \frac{c}{12})} \tag{3}$$

Clearly the exponential divergence on the LHS is to be produced from the tail of the integral on the RHS at large Δ. The task here is to find the asymptotic density of states, which can be obtained from the inverse Laplace transform of the LHS in (3),

$$\lim_{\Delta \gg 1} \rho(\Delta) = \rho_0(\Delta) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\beta e^{\frac{4\pi^2 c}{12\beta}} e^{\beta(\Delta - c/12)}. \tag{4}$$

Show that

$$\rho_0(\Delta) = \frac{1}{2}\left(\frac{c}{3\Delta^3}\right)^{\frac{1}{4}} e^{2\pi\sqrt{\frac{c\Delta}{3}}}(1 + \mathcal{O}(\Delta^{-1/2}))\,, \tag{5}$$

up to exponentially suppresssed contributions.

(b) Let us define the accumulated density of states (which count the total number of states up to weight Δ)

$$F(\Delta) = \int_0^{\Delta} d\Delta' \rho(\Delta')\,. \tag{6}$$

From the previous part, we have

$$F_0(\Delta) \equiv \lim_{\Delta \gg 1} F(\Delta) = \int_0^{\Delta} d\Delta' \rho_0(\Delta') = \frac{1}{2\pi}\left(\frac{3}{c\Delta}\right)^{1/4}\left(e^{2\pi\sqrt{\frac{c}{3}\Delta}} + \mathcal{O}(\Delta^{-1/2})\right)\,. \tag{7}$$

Let us now compare this universal result with the operator spectrum in the Ising CFT. The Ising torus partition function is given by

$$Z^{\text{Ising}}_{T^2}(\tau,\bar{\tau}) = \frac{1}{2}\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \frac{1}{2}\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \frac{1}{2}\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|\,, \tag{8}$$

with $q = e^{2\pi i\tau}$. The Elliptic (Jacobi) theta functions $\theta_\alpha(\tau)$ and the Dedekind eta function $\eta(\tau)$ are given by the following explicit formulae,

$$\begin{aligned}
\theta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2/2} = 2q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+q^m)^2\,,\\
\theta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{m=1}^{\infty}(1-q^m)(1+q^{m-1/2})^2\,,\\
\theta_4(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1/2})^2\,,\\
\eta(\tau) &= q^{\frac{1}{24}}\prod_{i=1}^{\infty}(1-q^i)\,.
\end{aligned} \tag{9}$$

Use Mathematica to expand $Z^{\text{Ising}}_{T^2}$ in $q = \bar{q}$ and plot the corresponding accumulated density $F^{\text{Ising}}(\Delta)$ as a function of Δ. Compare with $F_0(\Delta)$ in (7).

Remark: For further reading including a more rigorous derivation of the Cardy formula, see hep-th:1904.06356, and generalizations hepth:1906.04184 and hepth:2212.04893.

Exercise 2: Self-duality in the Ising CFT

The Ising CFT has an interesting property that if we orbifold the CFT by its $\mathbb{Z}_2$ spin flip symmetry, we obtain the same CFT. This is commonly referred to as the self-duality of the Ising under discrete gauging.

Below we will show this explicitly at the level of the torus partition function. We start with the vanilla Ising torus partition function,

$$Z^{\text{Ising}}_{T^2}(\tau,\bar\tau) = |\chi_0(\tau)|^2 + |\chi_{\frac{1}{2}}(\tau)|^2 + |\chi_{\frac{1}{16}}(\tau)|^2\,. \tag{10}$$

where

$$\chi_0(\tau) = \frac{\sqrt{\theta_3(\tau)}+\sqrt{\theta_4(\tau)}}{2\sqrt{\eta(\tau)}}\,,\ \chi_{\frac{1}{2}}(\tau) = \frac{\sqrt{\theta_3(\tau)}+\sqrt{\theta_4(\tau)}}{2\sqrt{\eta(\tau)}}\,,\ \chi_{\frac{1}{16}}(\tau) = \frac{\sqrt{\theta_2(\tau)}}{\sqrt{2\eta(\tau)}}\,. \tag{11}$$

We would like to compute the torus partition function for the $\mathbb{Z}_2$ orbifold

$$Z^{\text{Ising}/\mathbb{Z}_2}(\tau,\bar\tau) = \frac{1}{2}\sum_{g_1,g_2\in\mathbb{Z}_2} Z^{\text{Ising}}[g_1,g_2](\tau,\bar\tau)\,, \tag{12}$$

where $Z^{\text{Ising}}[g_1,g_2](\tau,\bar\tau)$ is the torus partition function for the Ising CFT with twist g_1 along the space direction x^1 and twist g_2 along the time direction x^2,

$$Z^{\text{Ising}}[g_1,g_2](\tau,\bar\tau) = \operatorname{tr}_{\mathcal{H}^{g_1}_{S^1}}\left(g_2 q^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}\right). \tag{13}$$

Note that a temporal twist introduces a symmetry operator acting on the Hilbert space, whereas a spatial twist modifies the Hilbert space (introducing the twisted sector).

(a) Perform q expansions of the Ising characters $\chi_0, \chi_{\frac{1}{2}}, \chi_{\frac{1}{16}}$ in (11) and convince yourself that the degeneracies match your expectations for unitary irreps of the $c = \frac{1}{2}$ Virasoro algebra.

(b) We start with the partition function with only a twist in the time direction,

$$Z^{\text{Ising}}[0,1](\tau,\bar\tau) = |\chi_0(\tau)|^2 + |\chi_{\frac{1}{2}}(\tau)|^2 - |\chi_{\frac{1}{16}}(\tau)|^2\,. \tag{14}$$

which follows from the fact that both the identity $\mathbb{1}$ and the energy operator ϵ are even under the $\mathbb{Z}_2$ spin flip symmetry and only the spin operator σ is $\mathbb{Z}_2$ odd.

Using modular transformation of the Ising characters, show that

$$Z^{\text{Ising}/\mathbb{Z}_2}(\tau,\bar{\tau}) = Z^{\text{Ising}}(\tau,\bar{\tau})\,. \tag{15}$$

(c) The modern interpretation of the Ising self-duality is that it originates from a generalized symmetry in the Ising CFT beyond the familiar $\mathbb{Z}_2$ spin flip symmetry. Conversely, any CFT with such a generalized symmetry is guaranteed to exhibit self-duality. Here we will uncover this generalized symmetry by simple considerations at the level of the torus partition function.

As is the case for usual symmetries, a generalized symmetry is defined by how the topological defect acts on local operators in the CFT. Let us call this topological defect D. Including a temporal twist by D on the torus as in (14), we have

$$Z^{\text{Ising}}_{T^2}[0,D](\tau,\bar{\tau}) \equiv \operatorname{tr}_{\mathcal{H}_{S^1}}(D q^{L_0-\frac{1}{48}} \bar{q}^{\bar{L}_0-\frac{1}{48}}) = d_{\mathbb{1}}|\chi_0(\tau)|^2 + d_\epsilon|\chi_{\frac{1}{2}}(\tau)|^2 + d_\sigma|\chi_{\frac{1}{16}}(\tau)|^2\,. \tag{16}$$

where $d_{\mathbb{1},\epsilon,\sigma}$ are numbers which capture the "charges" of the local operators under this putative symmetry. In the special case where D is the generator for the spin flip $\mathbb{Z}_2$, we have $d_{\mathbb{1}} = d_\epsilon = -d_\sigma = 1$.

Now in general the numbers $d_{\mathbb{1},\epsilon,\sigma}$ are constrained by the modular covariance of the torus partition function,

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau}) = Z^{\text{Ising}}_{T^2}[0,D](-1/\tau,-1/\bar{\tau})\,. \tag{17}$$

and the LHS now involves a spatial twist by D. Consequently, $Z^{\text{Ising}}_{T^2}[D,0]$ counts operators in the twisted Hilbert space $\mathcal{H}^D_{S^1}$ and must have the following decomposition,

$$Z^{\text{Ising}}_{T^2}[D,0](\tau,\bar{\tau}) \equiv \operatorname{tr}_{\mathcal{H}^D_{S^1}}(q^{L_0-\frac{1}{48}} \bar{q}^{\bar{L}_0-\frac{1}{48}}) = \sum_{i,j\in\{0,\frac{1}{2},\frac{1}{16}\}} n_{ij}\chi_i(\tau)\chi_j(\bar{\tau})\,. \tag{18}$$

with $n_{ij} \in \mathbb{Z}_{\geq 0}$.

Find the most general solution to $d_{\mathbb{1},\epsilon,\sigma}$ subject to the above constraints.

Exercise 3: Orbifolds and anomalies

The compact boson at a generic radius R has the following global symmetry

$$G = (U(1)_m \times U(1)_w) \rtimes \mathbb{Z}_2^C\,, \tag{19}$$

which consists of the momentum and winding $U(1)$ symmetries together with a $\mathbb{Z}_2^C$ charge conjugation symmetry. The CFT torus partition function is given by

$$Z_{T^2}(\tau,\bar{\tau}) = \frac{1}{\eta(\tau)\eta(\bar{\tau})} \sum_{p,w\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{p}{R}+\frac{wR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{p}{R}-\frac{wR}{2}\right)^2}\,, \tag{20}$$

which receives contributions from Virasoro primaries including the momentum-winding operators

$$V_{p,w} =: e^{i\left(\frac{p}{R}+\frac{wR}{2}\right)X_L} e^{i\left(\frac{p}{R}-\frac{wR}{2}\right)X_R}:\,, \quad h = \frac{1}{2}\left(\frac{p}{R}+\frac{wR}{2}\right)^2,\ \bar{h} = \frac{1}{2}\left(\frac{p}{R}-\frac{wR}{2}\right)^2 \tag{21}$$

where their conformal weights $(h, \bar{h})$ are given, and a tower of operators built from $\partial X, \bar{\partial} X$ and their derivatives,

$$j_{n^2} \bar{j}_{m^2}\,, \quad n, m \in \mathbb{Z}_+\,, h = n^2\,, \bar{h} = m^2\,. \tag{22}$$

For example,

$$j_1 = \partial X\,,\ j_4 = j_1^4 - 2j_1 \partial^2 j_1 + \frac{3}{2}(\partial j_1)^2\,. \tag{23}$$

In this exercise, we study discrete gauging (orbifold) of the compact boson with respect to certain discrete subgroups of G.

(a) Prove that (20) is invariant under modular S-transformation,

$$Z_{T^2}(-1/\tau, -1/\bar{\tau}) = Z_{T^2}(\tau, \bar{\tau})\,. \tag{24}$$

Hint: Use the Poisson resummation formulae for lattice sums.

(b) Proceed as in the Exercise 2 to compute the orbifold partition function of the compact boson at radius R with respect to the $\mathbb{Z}_2$ subgroup of $U(1)_m$ generated by translation $X \to X + \pi R$. Under this $\mathbb{Z}_2$, the operators in (22) are invariant while the momentum-winding operators transform as

$$V_{p,w} \to (-1)^p V_{p,w}\,. \tag{25}$$

Show that the answer agrees with the partition function of the compact boson at radius $R/2$.

(c) Consider a different $\mathbb{Z}_2$ subgroup of G coming from the diagonal subgroup of $U(1)_m \times U(1)_w$. Under this $\mathbb{Z}_2$, the operators in (22) are invariant while the momentum-winding operators transform as

$$V_{p,w} \to (-1)^{p+w} V_{p,w}\,. \tag{26}$$

Compute the partition function with a $\mathbb{Z}_2$ twist in the temporal direction $Z_{T^2}[0,1](\tau, \bar{\tau})$, and from modular S and T transformations of $Z_{T^2}[0,1](\tau, \bar{\tau})$, derive the twisted partition functions with spatial twist $Z_{T^2}[1,0](\tau, \bar{\tau})$, and with simultaneous twists in the temporal and spatial directions $Z_{T^2}[1,1](\tau, \bar{\tau})$. Is the answer you find for $Z_{T^2}[1,1](\tau, \bar{\tau})$ invariant under a modular S-transformation? What does this impliy about this $\mathbb{Z}_2$ symmetry?