1 Kramers-Wannier symmetry

1.1 Sequential quantum circuit

The non-invertible Kramers-Wannier operator takes the form

$$\mathsf{D} = \sqrt{2} e^{-\frac{2\pi i L}{8}} U_{\text{KW}} \frac{1 + \prod_{j=1}^{L} X_j}{2} \,. \tag{1.1}$$

Here

$$U_{\text{KW}} = \left(\prod_{j=1}^{L-1} \frac{1 + iX_j}{\sqrt{2}} \frac{1 + iZ_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 + iX_L}{\sqrt{2}} \tag{1.2}$$

is a unitary, sequential linear circuit. Prove that

$$U_{\text{KW}} X_j U_{\text{KW}}^{-1} = \begin{cases} Z_j Z_{j+1} \,, & j \neq L \,, \\ (\prod_{j=1}^{L} X_j) Z_L Z_1 \,, & j = L \,. \end{cases} \tag{1.3}$$

Therefore, U_{KW} does not act locally on the $\mathbb{Z}_2$-even local operators. Furthermore, U_{KW} does not commute with the lattice translation. From this it is clear that the multiplication by the projection factor $\frac{1+\prod_{j=1}^{L} X_j}{2}$ removes this issue.

1.2 Self-dual deformation

Prove that the deformed Hamiltonian

$$H = -\sum_{j=1}^{L} X_j - \sum_{j=1}^{L} Z_j Z_{j+1} + \frac{\lambda}{2} \sum_{j=1}^{L} (X_{j-1} Z_j Z_{j+1} + Z_{j-1} Z_j X_{j+1}) \tag{1.4}$$

at $\lambda = 1$ has the following three product states

$$|++...+\rangle \,, |00...0\rangle \,, |11...1\rangle \tag{1.5}$$

as exactly degenerate ground states. (It is more challenging to prove that this Hamiltonian at $\lambda = 1$ is gapped and has no other ground states.)

Next, in this three-dimensional ground space, find the simultaneous eigenbasis for η and D.

1.3 Matrix product operator

The Kramers-Wannier operator admits the following MPO presentation with bond dimension 2:

$$\begin{aligned} \mathsf{D} &= \text{Tr} \left(\mathbb{U}^1 \mathbb{U}^2 \cdots \mathbb{U}^L \right) \,, \\ \mathbb{U}^j &= \begin{pmatrix} |0\rangle\langle +|_j & |0\rangle\langle -|_j \\ |1\rangle\langle -|_j & |1\rangle\langle +|_j \end{pmatrix} \end{aligned} \tag{1.6}$$

Prove that

$$\mathbb{U}^j X_j = \mathbb{Z} \mathbb{U}^j \mathbb{Z} \,, \quad Z_j \mathbb{U}^j = \mathbb{Z} \mathbb{U}^j \,, \tag{1.7}$$

where $\mathbb{Z} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Using the above, show that $\mathsf{D} X_j = Z_j Z_{j+1} \mathsf{D}$. Similarly, compute $X_j \mathbb{U}^j$ and $\mathbb{U}^j Z_j$, and show that $\mathsf{D} Z_j Z_{j+1} = X_{j+1} \mathsf{D}$.

1.4 Non-invertible reflection

Define

$$\mathbb{U}^{j\prime} = \mathbb{H}\mathbb{U}^j\mathbb{H} = \begin{pmatrix} |+\rangle\langle 0|_j & |-\rangle\langle 1|_j \\ |-\rangle\langle 0|_j & |+\rangle\langle 1|_j \end{pmatrix} \,, \qquad \mathbb{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \,. \tag{1.8}$$

We see that $\mathbb{U}^{j\prime} = (\mathbb{U}^j)^{\mathrm{T}\dagger}$, where the transpose is taken in the virtual/bond space and the dagger is taken in the physical space.

Define the reflection operator P so that it acts as $\mathsf{P}O_j\mathsf{P}^{-1} = O_{-j}$. Using the MPO presentation of D, show that

$$\mathsf{P}\mathsf{D}\mathsf{P}^{-1} = \mathsf{D}^\dagger \,. \tag{1.9}$$

Next, define a non-invertible reflection operator

$$\mathsf{D}' = \mathsf{P}\mathsf{D} \,. \tag{1.10}$$

Show that

$$(\mathsf{D}')^2 = 1 + \eta \,. \tag{1.11}$$

Unlike D, the non-invertible reflection operator does not mix with lattice translations.

2 Rep(D_8)

In this section we assume L to be even, and consider the non-invertible operator for the Rep(D_8) fusion category:

$$\mathsf{D} = T^{-1}\mathsf{D}^{\mathrm{e}}\mathsf{D}^{\mathrm{o}} \tag{2.1}$$

where

$$\mathsf{D}^{\mathrm{e}} = \sqrt{2}e^{-\frac{2\pi i L}{16}}\frac{1+iX_2}{\sqrt{2}}\frac{1+iZ_2Z_4}{\sqrt{2}}\cdots\frac{1+iX_L}{\sqrt{2}}\frac{1+\eta^{\mathrm{e}}}{2} \,, \qquad \mathsf{D}^{\mathrm{o}} = \sqrt{2}e^{-\frac{2\pi i L}{16}}\frac{1+iX_1}{\sqrt{2}}\frac{1+iZ_1Z_3}{\sqrt{2}}\cdots\frac{1+iX_{L-1}}{\sqrt{2}}\frac{1+\eta^{\mathrm{o}}}{2} \,, \tag{2.2}$$

are the KW operators on the even and the odd sites, respectively. (In this section D stands for the Rep(D_8) operator rather than the KW operator.) Here

$$\eta^{\mathrm{e}} = \prod_{j:\text{even}} X_j \,, \quad \eta^{\mathrm{o}} = \prod_{j:\text{odd}} X_j \,, \tag{2.3}$$

generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and $T : j \to j+1$ is the lattice translation.

2.1 Fusion rule

Using the fusion rule for the KW operators, prove that

$$\mathsf{D}\eta^{\mathrm{e}} = \eta^{\mathrm{e}}\mathsf{D} = \mathsf{D}\eta^{\mathrm{o}} = \eta^{\mathrm{o}}\mathsf{D} = \mathsf{D} \,, \quad \mathsf{D}^2 = (1+\eta^{\mathrm{e}})(1+\eta^{\mathrm{o}}) \,. \tag{2.4}$$

Also show that $\mathsf{D} = \mathsf{D}^\dagger$.

2.2 Cluster state as a non-invertible SPT state

Using $Z_{j-1}X_jZ_{j+1}\left|\text{cluster}\right\rangle = \left|\text{cluster}\right\rangle$, prove that

$$\mathsf{D}\left|\text{cluster}\right\rangle = 2\left|\text{cluster}\right\rangle \ . \tag{2.5}$$