Answer sheet 1: Monday June 24, 2024.
(See chapter 2 of Zohar Komargodski's notes `https://indico.ictp.it/event/7624/session/19/contribution/84/material/0/0.pdf`)

Quantum field theories here are assumed local, invariant under translations and rotations (but not necessarily reflections). They have a symmetric conserved stress-energy tensor $T_{\mu\nu}$: the operator equations $T_{\mu\nu} = T_{\nu\mu}$ and $\partial^\mu T_{\mu\nu} = 0$ are valid at separated points in correlators. We assume there is **no local gravitational anomaly:** the equations hold at coincident points too. In contrast we allow anomalies in current conservation $\langle \partial_\mu j^\mu \dots\rangle =$ (contact terms) namely $\langle p_\mu j^\mu(p)\dots\rangle =$ (polynomial) in momentum space.

Exercise 1. The stress-tensor two-point function is characterized by its (center of mass) momentum space expression $\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle$, which can only[1] depend on q_μ and the metric $\delta_{\mu\nu}$. (i) Using symmetry and conservation show that, in $n \geq 2$ spacetime dimensions, for a pair of scalar functions g, f,

$$\begin{aligned}\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = {} & f(q^2)(q_\mu q_\nu - q^2\delta_{\mu\nu})(q_\rho q_\sigma - q^2\delta_{\rho\sigma}) \\ & + g(q^2)\Big((q_\mu q_\rho - q^2\delta_{\mu\rho})(q_\nu q_\sigma - q^2\delta_{\nu\sigma})\Big)\big|_{\text{symmetrize}(\rho,\sigma)}.\end{aligned}$$

(ii) Check that in 2d the two tensor structures coincide, so $\overset{\text{without loss of generality}}{\text{wlog}}$ $g(q^2) = 0$.

Answer Let's do the two points in the opposite order, due to how easy/hard they are.

(ii) In 2d we have $(q_\mu q_\nu - q^2\delta_{\mu\nu}) = -\widetilde{q}_\mu\widetilde{q}_\nu$ where $\widetilde{q}_\mu = \varepsilon_{\mu\rho}q^\rho$. Then the tensor structures are both $\widetilde{q}_\mu\widetilde{q}_\nu\widetilde{q}_\rho\widetilde{q}_\sigma$.

(i) The tensor structures must be constructed from the momentum q, metric δ and Levi–Civita tensor ε in a Lorentz-invariant way, meaning that indices have to be properly contracted. Since $\varepsilon_{\mu_1\dots\mu_n}\varepsilon^{\nu_1\dots\nu_n} = n!\delta^{[\nu_1}_{\mu_1}\cdots\delta^{\nu_n]}_{\mu_n}$ (where brackets denote antisymmetrization), Levi–Civita tensors can all be replaced by metrics except for zero or one.

Step 1. Let us begin with tensor structures that do not involve the Levi–Civita tensor ε. This gives the possible tensors

$$q_\mu q_\nu q_\rho q_\sigma, \qquad \delta_{\mu\nu}q_\rho q_\sigma, \qquad q_\mu q_\nu\delta_{\rho\sigma}, \qquad q_{(\mu}\delta_{\nu)(\rho}q_{\sigma)}, \qquad \delta_{\mu\nu}\delta_{\rho\sigma}, \qquad \delta_{\mu(\rho}\delta_{\sigma)\nu},$$

where we have already imposed the $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$ symmetries. We seek a linear combination of these six tensor structures that obeys conservation even at coincident point, namely such that contracting with any of q^μ, q^ν,

[1] This is a slight lie: in 3d theories without reflection symmetry, one has an extra tensor structure obtained by symmetrizing $q^\lambda\varepsilon_{\lambda\mu\rho}(q_\sigma q_\nu - q^2\delta_{\sigma\nu})$ in $\mu \leftrightarrow \nu$ and also in $\rho \leftrightarrow \sigma$, where ε is the Levi–Civita tensor. It is correctly invariant under swapping $(\mu,\nu,q) \leftrightarrow (\rho,\sigma,-q)$.

q^ρ, q^σ gives zero (rather than contact terms). Imposing conservation on the μ index leads to three tensor structures:

$$(q_\mu q_\nu - q^2\delta_{\mu\nu})q_\rho q_\sigma, \qquad (q_\mu q_\nu - q^2\delta_{\mu\nu})\delta_{\rho\sigma},$$
$$q_\mu\delta_{\nu(\rho}q_{\sigma)} - q^2\delta_{\nu(\rho}\delta_{\sigma)\mu} + q_\nu\delta_{\mu(\rho}q_{\sigma)} - \delta_{\mu\nu}q_\rho q_\sigma.$$

Then conservation on the ρ index gives two tensor structures

$$(q_\mu q_\nu - q^2\delta_{\mu\nu})(q_\rho q_\sigma - q^2\delta_{\rho\sigma}),$$
$$q_\mu\delta_{\nu(\rho}q_{\sigma)} - q^2\delta_{\nu(\rho}\delta_{\sigma)\mu} + q_\nu\delta_{\mu(\rho}q_{\sigma)} - \delta_{\mu\nu}q_\rho q_\sigma - q_\mu q_\nu\delta_{\rho\sigma} + q^2\delta_{\mu\nu}\delta_{\rho\sigma}$$

that are annihilated when contracting with q^μ or q^ρ (and by symmetry q^ν and q^σ too). A simple linear combination gives the two tensor structures in the exercise. The first matches directly. We can write the second tensor structure of the exercise as

$$\begin{aligned}
&\frac{1}{2}(q_\mu q_\rho - q^2\delta_{\mu\rho})(q_\nu q_\sigma - q^2\delta_{\nu\sigma}) + \frac{1}{2}(q_\mu q_\sigma - q^2\delta_{\mu\sigma})(q_\nu q_\rho - q^2\delta_{\nu\rho}) \\
&\quad = q_\mu q_\rho q_\nu q_\sigma - q^2\delta_{\mu(\rho}q_{\sigma)}q_\nu - q^2\delta_{\nu(\rho}q_{\sigma)}q_\mu + q^4\delta_{\mu(\rho}\delta_{\sigma)\nu} \\
&\quad = (q_\mu q_\nu - q^2\delta_{\mu\nu})(q_\rho q_\sigma - q^2\delta_{\rho\sigma}) \\
&\qquad - q^2\Big(-\delta_{\mu\nu}q_\rho q_\sigma - \delta_{\rho\sigma}q_\mu q_\nu + q^2\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu(\rho}q_{\sigma)}q_\nu + \delta_{\nu(\rho}q_{\sigma)}q_\mu - q^2\delta_{\mu(\rho}\delta_{\sigma)\nu}\Big).
\end{aligned}$$

The parenthesized expression matches the second tensor structure we just found. Incidentally, conservation turns out to be enough to ensure symmetry under the isometry $x \to -x$ which maps $q \to -q$ and $(\mu,\nu) \leftrightarrow (\rho,\sigma)$.

Step 2. Regarding the footnote: what about using the Levi–Civita tensor? Antisymmetry prevents us from contracting more than one index of ε with q, and also prevents us from making the μ and ν indices of the two-point function be indices of ε, and likewise for ρ and σ. So at most we can use three of the indices. Such a term can only exist in $n \leq 3$ dimensions. Making sense of the cases $n \leq 1$ is left as an exercise to the reader.

In $n = 3$ dimensions, we are quite constrained and must complete $q^\lambda\varepsilon_{\lambda\mu\rho}$ by something with indices ν, σ. This gives tensor structures

$$q^\lambda\varepsilon_{\lambda\mu\rho}q_\nu q_\sigma, \qquad q^\lambda\varepsilon_{\lambda\mu\rho}\delta_{\nu\sigma},$$

and of course their images under $\mu \leftrightarrow \nu$ and/or $\rho \leftrightarrow \sigma$. We then impose symmetries and impose conservation, and get the expression in the footnote, namely

$$q^\lambda\varepsilon_{\lambda\mu\rho}(q_\nu q_\sigma - q^2\delta_{\nu\sigma}) + q^\lambda\varepsilon_{\lambda\mu\sigma}(q_\nu q_\rho - q^2\delta_{\nu\rho}) + q^\lambda\varepsilon_{\lambda\nu\rho}(q_\mu q_\sigma - q^2\delta_{\mu\sigma}) + q^\lambda\varepsilon_{\lambda\nu\sigma}(q_\mu q_\rho - q^2\delta_{\mu\rho}).$$

This expression is rotationally-invariant by construction, but not reflection-invariant due to ε. Under $(\mu,\nu,q)\leftrightarrow(\rho,\sigma,-q)$, the first and last terms get a sign flip from q^λ and another from ε; the second and third terms get the same sign flips and get swapped, so the expression is consistent with the symmetry exchanging the two stress-tensor operators.

In $n=2$ dimensions, it is simplest to work in (anti)holomorphic coordinates. By rotation invariance we have $\langle T_{zz}T_{zz}\rangle = b(q^2)q_z^4$. Exact conservation $q_{\overline{z}}T_{zz}+q_zT_{z\overline{z}}=0$ in correlators even at coincident point (no gravitational anomaly) implies $\langle T_{z\overline{z}}T_{zz}\rangle = -b(q^2)q_z^3q_{\overline{z}}$ (modulo a delta function at $q=0$ perhaps, but that would amount in position space to a non-decaying contribution). Exact symmetry allows us to replace $T_{z\overline{z}}=T_{\overline{z}z}$, then exact conservation $q_{\overline{z}}T_{\overline{z}z}+q_zT_{\overline{z}\overline{z}}=0$ fixes the correlator $\langle T_{\overline{z}\overline{z}}T_{zz}\rangle$. Likewise we can use conservation on the second factor. Overall we find

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = b(q^2)\widetilde{q}_\mu\widetilde{q}_\nu\widetilde{q}_\rho\widetilde{q}_\sigma,$$

where $\widetilde{q}_z=\varepsilon_{z\overline{z}}\delta^{\overline{z}z}q_z=q_z$ and $\widetilde{q}_{\overline{z}}=\varepsilon_{\overline{z}z}\delta^{z\overline{z}}q_{\overline{z}}=-q_{\overline{z}}$. This is the unique tensor structure that is listed in the exercise. No need to fully track the epsilon tensors etc.

The approach with abstract indices like in higher dimensions is more daunting. We either have $\varepsilon_{\mu\rho}$ times a two-index tensor, so $\varepsilon_{\mu\rho}q_\nu q_\sigma$ or $\varepsilon_{\mu\rho}\delta_{\nu\sigma}$, or we have the contraction $\varepsilon_{\lambda\upsilon}q^\upsilon =: \widetilde{q}_\lambda$ times a suitable tensor. This gives

$$\begin{array}{lllll} \varepsilon_{\mu\rho}q_\nu q_\sigma, & \varepsilon_{\mu\rho}\delta_{\nu\sigma}, & \widetilde{q}_\mu q_\nu q_\rho q_\sigma, & \widetilde{q}_\mu q_\nu\delta_{\rho\sigma}, & \widetilde{q}_\mu q_\rho\delta_{\nu\sigma}, \\ & & q_\mu q_\nu q_\rho\widetilde{q}_\sigma, & \delta_{\mu\nu}q_\rho\widetilde{q}_\sigma, & q_\mu\delta_{\nu\rho}\widetilde{q}_\sigma, \end{array}$$

or rather, their symmetrization under $\mu\leftrightarrow\nu$ and $\rho\leftrightarrow\sigma$.

Actually, this is a redundant set of tensor structures: one can check $q_\mu\widetilde{q}_\sigma=\widetilde{q}_\mu q_\sigma - q^2\varepsilon_{\mu\sigma}$ component by component, which allows to recast the sixth and last tensor structures in terms of the others (symmetrized in $\mu\leftrightarrow\nu$ and $\rho\leftrightarrow\sigma$).

It seems that imposing q^μ conservation gives only two tensor structures,

$$\begin{aligned} &q_{(\nu}\varepsilon_{\mu)(\rho}q_{\sigma)}+\delta_{\mu\nu}q_{(\rho}\widetilde{q}_{\sigma)}-\widetilde{q}_{(\mu}\delta_{\nu)(\rho}q_{\sigma)}, \\ &\qquad \widetilde{q}_{(\mu}q_{\nu)}q_\rho q_\sigma - q^2\widetilde{q}_{(\mu}\delta_{\nu)(\rho}q_{\sigma)}. \end{aligned}$$

Then it seems that there is no linear combination that is killed by contraction with q^ρ. Clearly this is more difficult than the (anti)holomorphic approach.

Exercise 2. Assume that the QFT is two-dimensional and scale-invariant.

(i) Show that $f(q^2)=c/q^2$ for some constant c. Check that $\langle T^\mu_\mu(q)T_{\rho\sigma}(-q)\rangle$ is polynomial in q hence T^μ_μ has a vanishing two-point function with $T_{\rho\sigma}$ at separated points.

(ii) Couple the QFT to a frozen metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ close to Euclidean. At first order this adds $\frac{1}{2}\int g_{\mu\nu}T^{\mu\nu}d^2x$ to the action. Deduce

$$\langle T^\mu_\mu(x)\rangle_{g=\delta+h} \sim c(\partial^\rho\partial^\sigma - \delta^{\rho\sigma}\Box)h_{\rho\sigma} + O(h^2).$$

This is c times the linearized Ricci scalar R of g; higher-order corrections in h come from higher-point functions of $T_{\rho\sigma}$. This is the famous 2d trace anomaly $\langle T^\mu_\mu\rangle = -\frac{c}{24\pi}R$.

(iii) In a metric $g_{\mu\nu} = e^\varphi\delta_{\mu\nu}$, check that $T'_{zz} = T_{zz} + \alpha c(-(\partial\varphi)^2 + 2\partial^2\varphi)$ is holomorphic for some value of α: use the conservation equation $\nabla^\mu T_{\mu\nu} = 0$ and $R = -4e^{-\varphi}\partial_z\partial_{\bar{z}}\varphi$.

Answer **(i)** The stress-tensor has dimension $n = 2$ so the real-space two-point function $\langle T_{\mu\nu}T_{\rho\sigma}\rangle$ scales as length^{-4}. The Fourier transform is an integral d^2x so the momentum-space two-point function scales as length^0. But there is a momentum-conservation delta function δ(total momentum), which has dimension momentum^{-2}, so what remains has dimension momentum^2. Since the tensor structure is quartic in q it has to be cancelled by a $1/q^2$.

Then we just need to contract indices μ, ν in the tensor structure, and remember $\delta^\mu_\mu = 2$, to get

$$\langle T^\mu_\mu(q)T_{\rho\sigma}(-q)\rangle = -c(q_\rho q_\sigma - q^2\delta_{\rho\sigma}),$$

which is indeed polynomial in q. Fourier-transforming gives derivatives of delta functions, namely contact terms.

(ii) In flat space $\langle T^\mu_\mu\rangle = 0$ because it has non-trivial scaling dimension but there is no length scale available. Now turn on a non-trivial metric:

$$\begin{aligned}\langle T^\mu_\mu(x)\rangle_{g=\delta+h} &= \langle T^\mu_\mu(x)e^{-\frac{1}{2}\int h_{\rho\sigma}T^{\rho\sigma}d^2x}\rangle_{g=\delta} + O(h^2)\\ &= -\frac{1}{2}\int h_{\rho\sigma}(y)\langle T^\mu_\mu(x)T^{\rho\sigma}(y)\rangle_{g=\delta}dy + O(h^2)\\ &= -\frac{c}{2}\int h_{\rho\sigma}(y)(\partial_{y_\rho}\partial_{y_\sigma} - \delta_{\rho\sigma}\Delta_y)\delta(x-y)dy + O(h^2)\end{aligned}$$

where $\Delta_y = \delta^{\nu\lambda}\partial_{y_\nu}\partial_{y_\lambda}$. Note that $\partial_{y_\mu}\delta(x-y) = -\partial_{x_\mu}\delta(x-y)$ so we can trade y derivatives for x derivatives, pull them out of the integral, then perform the y integral of $h_{\rho\sigma}(y)\delta(x-y)$ to get the desired answer.

(iii) to be completed, time-permitting

Exercise 3. (i) Consider a chiral conserved current j_z in a 2d CFT. From $\langle j_z(z)j_z(w)\rangle = k/(z-w)^2$ (k is called the *level*) deduce $\langle j_zj_z\rangle = kq_z^2/q^2$ in momentum space.

(ii) Consider a $U(1)$ conserved current j_μ in a (translation & rotation invariant) 2d QFT. Show that symmetries fix (for some functions a_L, a, a_R)

$$\langle j_z j_z\rangle = q_z^2 a_L(q^2)/q^2, \qquad \langle j_z j_{\overline{z}}\rangle = -a(q^2), \qquad \langle j_{\overline{z}} j_{\overline{z}}\rangle = q_{\overline{z}}^2 a_R(q^2)/q^2.$$

Using the separated-point conservation equation $q_z j_{\overline{z}} + q_{\overline{z}} j_z =$ polynomial, and adjusting contact terms by shifting correlators by polynomial in $q_z, q_{\overline{z}}$, find that $a_L - a$ and $a_R - a$ are constant. If the UV and IR limits $q^2 \to +\infty, 0$ are CFTs deduce that levels of chiral currents obey $k_R^{\mathrm{UV}} - k_L^{\mathrm{UV}} = k_R^{\mathrm{IR}} - k_L^{\mathrm{IR}}$, a simple version of 't Hooft anomaly matching.

(iii) In a background gauge field A, show $\langle \partial_\mu j^\mu\rangle = (1/2)(a_L - a_R)\epsilon^{\mu\nu}F_{\mu\nu} + O(A^2)$ for a suitable choice of contact terms.

Answer (i) Compute (where $p_z, p_{\overline{z}}$ are simply the (anti)holomorphic components of p, nothing to do with z, sorry for the bad notation[2])

$$\begin{aligned}\langle j_z(q) j_z(p)\rangle &= \iint e^{iq_z z + iq_{\overline{z}}\overline{z} + ip_z w + ip_{\overline{z}}\overline{w}} \langle j_z(z) j_z(w)\rangle d^2z\, d^2w \\ &= k\delta(q+p)\int \frac{e^{iq_z z + iq_{\overline{z}}\overline{z}}}{z^2} dz\, d\overline{z}.\end{aligned}$$

Intuitively, rescaling the z and $\overline{z}$ integration variables to absorb q_z and $q_{\overline{z}}$ shows that it must scale as $q_z/q_{\overline{z}} = q_z^2/q^2$. A more proper approach is to split $z = re^{i\theta}$ and $q_z = |q|e^{i\psi}$ and $q_{\overline{z}} = |q|e^{-i\psi}$, and evaluate

$$\begin{aligned}\langle j_z(q) j_z(-q)\rangle &= k\int_0^{+\infty}\int_0^{2\pi} e^{2ir|q|\cos(\theta+\psi)} e^{-2i\theta} d\theta\, \frac{dr}{r} \\ &= -2\pi e^{2i\psi} k \int_0^{+\infty} J_2(2r|q|)\frac{dr}{r} = -\pi e^{2i\psi} k = -\pi \frac{q_z^2}{q^2} k.\end{aligned}$$

Up to unimportant numerical factors that are not properly taken into account this gives the desired answer.

(ii) $\langle j_z j_z\rangle$ has charge 2 under rotations so we need something with charge 2, built from the momentum components q_z and $q_{\overline{z}}$ which have charges ± 1. This means q_z^2 times some function of $q_z q_{\overline{z}}$. We write the function as $a_L(q^2)/q^2$ for convenience. Same story for the other two-point functions: just track the $U(1) = SO(2)$ charge. Then conservation says that

$$\langle j_z(q_{\overline{z}} j_z + q_z j_{\overline{z}})\rangle = q_{\overline{z}}\langle j_z j_z\rangle + q_z\langle j_z j_{\overline{z}}\rangle = q_z\big(a_L(q^2) - a(q^2)\big)$$

[2] In this notation, components of x^μ are $x_{\overline{z}} = x^z = z$ and $x_z = x^{\overline{z}} = \overline{z}$, and the scalar product is $p\cdot q = p_z q_{\overline{z}} + p_{\overline{z}} q_z$. Factors of 2 are probably wrong.

must be a polynomial in $q_z, q_{\bar{z}}$. This means $a_L - a$ is polynomial in q^2. Likewise $a - a_R$ is a polynomial due to the polynomiality of the two-point function of $j_{\bar{z}}$ and $(q_{\bar{z}} j_z + q_z j_{\bar{z}})$. Any term of order q^{2l}, $l \geq 1$, in these polynomials can be eliminated by shifting a_L and a_R by them: this shifts two-point functions by $q_z^2 q^{2l-2}$ and $q_{\bar{z}}^2 q^{2l-2}$, respectively. There remains the constants. Only the constant $a_L - a_R$ matters since a can be freely shifted by a constant.

If the UV limit $q^2 \to +\infty$ is a CFT then $a_L \to k_L^{\mathrm{UV}}$ and $a_R \to k_R^{\mathrm{UV}}$ in that limit. Same in the IR. Thus $a_L - a_R$, which is a constant, takes the value $k_L^{\mathrm{UV}} - k_R^{\mathrm{UV}}$ in the UV and $k_L^{\mathrm{IR}} - k_R^{\mathrm{IR}}$ in the IR.

(iii) Without background A, $\langle j^\mu \rangle$ vanishes by rotation invariance (there is no momentum available), hence $\langle \partial_\mu j^\mu \rangle = 0$. In a background gauge field A,

$$\langle \partial_\mu j^\mu(x) \rangle_A = \int A_\nu(y) \langle \partial_\mu j^\mu(x) j^\nu(y) \rangle \, dy + O(A^2).$$

We need to look at contact terms $\langle j_z (q^\nu j_\nu) \rangle = q_z(a_L - a)$ and $\langle j_{\bar{z}} (q^\nu j_\nu) \rangle = q_{\bar{z}}(a - a_R)$. Then

$$\begin{aligned}
\langle \partial_\mu j^\mu(x) \rangle_A &= \int A_\nu(y) \langle \partial_\mu j^\mu(x) j^\nu(y) \rangle \, dy \\
&= (a_L - a) i \partial_z \int A_{\bar{z}}(y) \delta(x-y) dy + (a - a_R) i \partial_{\bar{z}} \int A_z(y) \delta(x-y) dy \\
&= (a_L - a) i \partial_z A_{\bar{z}}(x) + (a - a_R) i \partial_{\bar{z}} A_z(x).
\end{aligned}$$

In general it is not very nice, but if we adjust the constant part of a (which is a contact term) such that $a_L - a = a - a_R = (a_L - a_R)/2$, we get the field strength of A (up to some sign mistake somewhere in my calculations).

Exercise 4. Consider a Poincaré-invariant 2d QFT with a $U(1)$ conserved current j. Understand how charge conjugation and time-reversal acts on j and T. Show that $\langle j_\mu T_{\nu\rho} \rangle = 0$. (More generally, no mixed anomaly between these symmetries: gauging either one does not spoil the other.)

Answer Missing; ask Zohar when he comes in the second week.

Exercise 5. (i) In 2d, take currents j_μ and j'_μ with $\langle j_\mu j'_\nu \rangle = q_\mu \varepsilon_{\nu\rho} q^\rho / q^2$. Show j'_μ is exactly conserved while conservation of j_ν has contact terms. By adding contact terms to $\langle j_\mu j'_\nu \rangle$ make $\partial^\mu j_\mu = 0$ exact and see that $\partial^\mu j'_\mu$ gets contact terms.

(ii) In $n = 2k$ dimensions, same questions with $k+1$ currents and $\left\langle j^{(0)}_{\mu_0}(q^{(0)}) j^{(1)}_{\mu_1}(q^{(1)}) \ldots j^{(k)}_{\mu_k}(q^{(k)}) \right\rangle = \varepsilon_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k} q^{(1)\nu_1} \ldots q^{(k)\nu_k} q^{(0)}_{\mu_0} / q^{(0)2}$.

(iii) Turn on backgrounds $A^{(i)}$ for $j^{(i)}$, $i = 1, \ldots, k$, and compute the effect of the previous line on $\langle \partial^\mu j^{(0)}_\mu \rangle$ in terms of field strengths of $A^{(i)}$.

Answer **(i)** Contracting $\langle j_\mu j'_\nu \rangle = q_\mu \varepsilon_{\nu\rho} q^\rho / q^2$ with q^ν gives zero since $q^\nu \varepsilon_{\nu\rho} q^\rho = 0$. Contracting with q^μ gives $\varepsilon_{\nu\rho} q^\rho$, which is polynomial in q hence is a contact term.

The identity

$$q_\mu \varepsilon_{\nu\rho} q^\rho / q^2 - q_\nu \varepsilon_{\mu\rho} q^\rho / q^2 = -\varepsilon_{\mu\nu}$$

can be checked by noting that both sides are antisymmetric in $\mu \leftrightarrow \nu$ and the $(\mu,\nu) = (1,2)$ component is $-q_1 q_1/q^2 - q_2 q_2/q^2 = -1$. We can subtract it from the current two-point function (it is a contact term). Then $\langle j_\mu j'_\nu \rangle = q_\nu \varepsilon_{\mu\rho} q^\rho / q^2$. The roles of j and j' were just swapped, so that now j is exactly conserved.

(ii) In $n = 2k$ dimensions, we have $k+1$ momenta summing to $q^{(0)} + \cdots + q^{(k)} = 0$ by momentum conservation. Contracting the $(k+1)$-point function given in the exercise by $q^{(i)\mu_i}$ to check conservation of the (i)-th current, for $i = 1, \ldots, k$, gives us an expression with two $q^{(i)}$ contracted with the Levi–Civita tensor. By antisymmetry this vanishes. On the other hand, contracting with $q^{(0)\mu_0}$ gives the following contact term. This contact term can be usefully written as (using momentum conservation)

$$\begin{aligned}
&\varepsilon_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k} q^{(1)\nu_1} \ldots q^{(k)\nu_k} \\
&= -\varepsilon_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k} \left(q^{(0)} + q^{(2)} + \cdots + q^{(k)} \right)^{\nu_1} q^{(2)\nu_2} \ldots q^{(k)\nu_k} \\
&= -\varepsilon_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k} q^{(0)\nu_1} q^{(2)\nu_2} \ldots q^{(k)\nu_k}
\end{aligned}$$

where the last step uses that each term $q^{(i)}$ with $i = 2, \ldots, k$ in the sum vanishes by antisymmetry of ε. We can eliminate this term in the conservation of $j^{(0)}$ by adding the contact term $\varepsilon_{\mu_1 \ldots \mu_k \mu_0 \nu_2 \ldots \nu_k} q^{(2)\nu_2} \ldots q^{(k)\nu_k}$ to the correlator. This changes the correlator to

$$\begin{aligned}
&\left\langle j^{(0)}_{\mu_0}(q^{(0)}) j^{(1)}_{\mu_1}(q^{(1)}) \ldots j^{(k)}_{\mu_k}(q^{(k)}) \right\rangle \\
&= \varepsilon_{\mu_1 \ldots \mu_k \rho \nu_2 \ldots \nu_k} q^{(2)\nu_2} \ldots q^{(k)\nu_k} \left(q^{(1)\rho} q^{(0)}_{\mu_0} / q^{(0)2} + \delta^\rho_{\mu_0} \right).
\end{aligned}$$

This choice obeys exact conservation of $j^{(i)}$ for $i \neq 1$, but $j^{(1)}$ is not conserved: contracting with $q^{(1)\mu_1}$ eliminates the first term by antisymmetry of ε and yields the contact term $\varepsilon_{\mu_1 \ldots \mu_k \mu_0 \nu_2 \ldots \nu_k} q^{(1)\mu_1} q^{(2)\nu_2} \ldots q^{(k)\nu_k}$. This is the same type of failure as before, with $j^{(0)}$ and $j^{(1)}$ interchanged. It would be good to see if the new correlator $\langle j^{(0)} j^{(1)} \ldots \rangle$ that we wrote matches on the nose the image of the original correlator under $j^{(0)} \leftrightarrow j^{(1)}$. It is a lot of bookkeeping.

(iii) When expanding $\langle \partial^\mu j^{(0)}_\mu \rangle$ in powers of the backgrounds $A^{(i)}$, we have to worry about all the low-point correlators involving $\partial^\mu j^{(0)}_\mu$ together with any number of currents. It's not obvious why the first term that matters is the $(k+1)$-point function in the exercise, but the reader can try thinking about it. At the end of the day, we get

$$\langle \partial^\mu j_\mu^{(0)}(x)\rangle_A = \int d^{2k}y A^{\mu_1}(y^{(1)}) \dots A^{\mu_k}(y^{(k)}) \langle \partial^\mu j_\mu^{(0)}(x) j_{\mu_1}^{(1)}(y^{(1)}) \dots j_{\mu_k}^{(k)}(y^{(k)})\rangle_A + \text{lower}.$$

The correlator is purely contact terms, as we just saw, involving antisymmetrized $q^{(i)}$, namely position derivatives, and delta functions at $y^{(i)} = x$. We get

$$\langle \partial^\mu j_\mu^{(0)}(x)\rangle_A = \sqrt{-1}^{\,?} \varepsilon_{\mu_1\dots\mu_k\nu_1\dots\nu_k} \partial^{\nu_1} A^{\mu_1}(y^{(1)}) \dots \partial^{\nu_k} A^{\mu_k}(y^{(k)}) + \text{lower}.$$

Up to a phase and some factors of 2 this is $F^{(1)} \wedge \dots \wedge F^{(k)}$.

Exercise 6. Switch to 4d. Left-handed fermions of the Standard Model transform in (three generations of) $(\mathbf{1},\mathbf{2})_{c_1} + (\mathbf{1},\mathbf{1})_{c_2} + (\mathbf{3},\mathbf{2})_{c_3} + (\overline{\mathbf{3}},\mathbf{1})_{c_4} + (\overline{\mathbf{3}},\mathbf{1})_{c_5}$ under the gauge symmetry $SU(3) \times SU(2) \times U(1)$, where the notation $(\mathbf{a},\mathbf{b})_c$ denotes the tensor product of a representation of $SU(3)$ of dimension a, of $SU(2)$ of dimension b, and of a charge c representation of $U(1)$. Denoting generators of the gauge group by t_α, the gauge anomaly for any triplet of generators $t_\alpha, t_\beta, t_\gamma$ can be calculated by a triangle Feynman diagram, and is proportional to

$$\sum_{\text{fermion representation } \mathcal{R}} \mathrm{Tr}_{\mathcal{R}}(t_\alpha t_\beta t_\gamma + t_\alpha t_\gamma t_\beta).$$

Check that the anomalies involving $SU(3)$ and $SU(2)$ generators vanish. Check that the remaining gauge-anomaly cancellations (together with the gauge-gravitational anomaly $2c_1 + c_2 + 6c_3 + 3c_4 + 3c_5 = 0$) only allow for two possible hypercharge assignments up to scaling. One of them is the Standard Model answer $c_1 = 1/2$, $c_2 = -1$, $c_3 = -1/6$, $c_4 = 2/3$, $c_5 = -1/3$.

Answer With only $SU(3)$ and $SU(2)$ generators, we have either two generators t_α, t_β from one group and one generator t_γ from the other, in which case the trace vanishes because $\mathrm{Tr}(t_\alpha t_\beta t_\gamma) = \mathrm{Tr}(t_\alpha t_\beta)\,\mathrm{Tr}(t_\gamma) = 0$ since $\mathrm{Tr}(t_\gamma) = 0$ as the matrices are traceless. So we only need to worry about $SU(3)^3$ and $SU(2)^3$ anomaly cancellation. Fermions that do not transform under the given group cannot contribute to the anomaly since t_α simply vanishes in that representation. So for $SU(3)^3$ we only get contributions from $2 \times \mathbf{3} + 2 \times \overline{\mathbf{3}}$, which vanishes because t_α in a representation and its dual are conjugate transpose of each other. Same story for $SU(2)^3$, together with $\overline{\mathbf{2}} \simeq \mathbf{2}$.

For the mixed anomalies involving $U(1)$, the same argument as above shows that we cannot have a single $SU(3)$ or $SU(2)$. So the anomalies to consider are $U(1)\times SU(3)^2$ and $U(1)\times SU(2)^2$ and $U(1)^3$. In the first two cases the traces factorize as $\mathrm{Tr}_c(t_\alpha)\,\mathrm{Tr}_{\mathbf{a}}(t_\beta t_\gamma)$ or similar. Now $\mathbf{3}$ and $\overline{\mathbf{3}}$ contribute the same since the trace is quadratic in $SU(3)$ generators, and similarly $\mathbf{2}$ contributes non-trivially for the $U(1)\times SU(2)^2$ anomaly. We get three equations

$$2c_3+c_4+c_5=0,\qquad c_1+3c_3=0,\qquad 2c_1^3+c_2^3+6c_3^3+3c_4^3+3c_5^3=0,$$

in addition to the $U(1)\times(\text{gravity})^2$ anomaly. For that anomaly all matter contributes the same to the gravity part so we just the total $U(1)$ charge of all fermions:

$$2c_1+c_2+6c_3+3c_4+3c_5=0.$$

We solve the three linear equations and get $c_2=-2c_1=6c_3$ and $c_5=-c_4-2c_3$. Plugging into the cubic equation gives

$$\begin{aligned}0&=2(-3c_3)^3+(6c_3)^3+6c_3^3+3c_4^3+3(-c_4-2c_3)^3=3\Big(56c_3^3+c_4^3+(-c_4-2c_3)^3\Big)\\&=18c_3(2c_3-c_4)(4c_3+c_4),\end{aligned}$$

thus three solutions. The $c_4=2c_3$ and $c_4=-4c_3$ solutions are the same up to swapping $c_4\leftrightarrow c_5$. They give the Standard Model solution up to a suitable normalization of the hypercharge. The $c_3=0$ solution gives $c_1=c_2=c_3=0$ and $c_5=-c_4$. (This amounts to the Cartan subgroup of an isospin symmetry acting on right-handed quarks I think.)

Answer sheet Dumitrescu Lecture 1.

Exercise 7 (Abelian Duality in Diverse Dimensions). Much intuition about phases and transitions can be gleaned from mean-field theory. Let us consider the mean-field (i.e. semiclassical) dynamics of a real scalar field ϕ with a $\mathbb{Z}_2$ Ising symmetry. Since we are working at leading order in the semiclassical expansion, we just minimize the potential and ignore loop corrections. Thus the discussion applies in any spacetime dimension D. (Whether or not this is a good description depends on D.)

- Analyze the vacuum structure as a function of the mass $m^2 \in \mathbb{R}$ given a quartic potential of the form

$$V(\phi) = m^2\phi^2 + \lambda_4\phi^4, \qquad \lambda_4 > 0.$$

  In particular discuss the order of the transition at $m^2 = 0$. (In applications to the classical, finite-temperature Ising model $m^2 \sim T - T_c$, but the discussion also applies to quantum phase transitions at zero temperature in the Ising universality class, in which case m^2 is some coupling in the Hamiltonian.)

- Show that the transition can be made 1st order by breaking the $\mathbb{Z}_2$ symmetry via a linear perturbation $\Delta V = h\phi$. In the Ising model $h \in \mathbb{R}$ is an external magnetic field. Sketch the phase diagram as a function of m^2, h. Argue that generically the only way for a line of 1st order phase transition to genuinely end (rather than turn into some other lines(s) of transitions) is in a 2nd order point.

- Consider the Ising model with $\mathbb{Z}_2$ symmetry and a sextic potential, $V(\phi) = m^2\phi^2 + \lambda_4\phi^4 + \lambda_6\phi^6$. Imagine that $\lambda_6 > 0$, so that the potential is stable, but that $m^2, \lambda_4 \in \mathbb{R}$ can have either sign. Analyze the phase diagram and show that the sign of λ_4 controls the order of the phase transition as we dial m^2. The point $m^2 = \lambda_4 = 0$ at which the order of the phase transition changes is called a multi-critical point. Here it is also called a tri-critical point since we are dialing two parameters (rather than the single parameter to reach a generic critical point). Is the tri-critical point described by the same physics as the line of second-order Ising transitions at $\lambda_4 > 0$?

- The previous point shows that a first order line can change into a second order line at a multi-critical point. Are there other possible behaviors for a 1st order line other than this and ending in a second order point? Hint: think of the phase diagram of water.

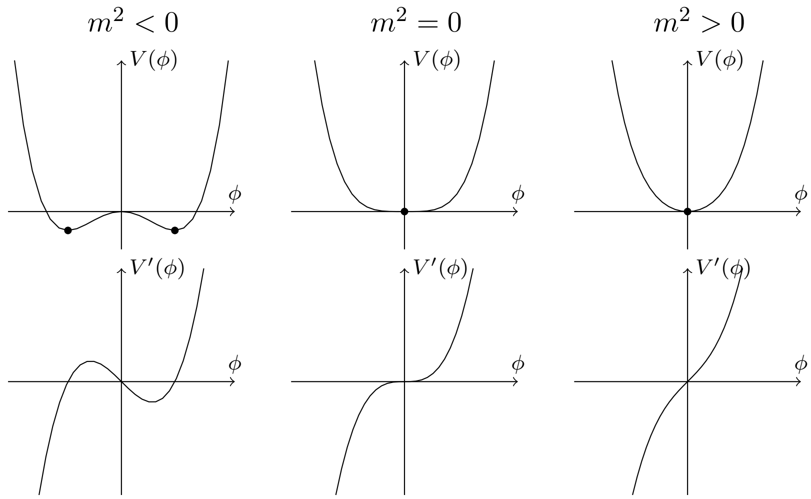

Figure 1: Depiction of $V(\phi) = m^2\phi^2 + \lambda_4\phi^4$ and $V'(\phi)$ for various signs of m^2. Minima are indicated by black dots on the plot of V.

Answer **(i)** For $\lambda_4 > 0$ the potential $V(\phi) = m^2\phi^2 + \lambda_4\phi^4$ is positive at infinity. It has extrema at $V'(\phi) = (2m^2 + 4\lambda_4\phi^2)\phi = 0$. For $m^2 \geq 0$ the only real solution is $\phi = 0$; it is a minimum since $V(0) = 0$ and $V(\phi) > 0$ for $\phi \neq 0$. For $m^2 < 0$ there are three solutions $\phi = 0$ and $\phi = \pm\sqrt{-m^2/(2\lambda_4)}$. One easily checks that $\phi = 0$ is a local maximum and the other two are local minima. These local minima are degenerate (have the same value of V) by $\phi \to -\phi$ symmetry. See Figure 1 ....second-order phase transition....

(ii) For $V = h\phi + m^2\phi^2 + \lambda_4\phi^4$ the condition for an extremum is for $V'(\phi) = h + 2m^2\phi + 4\lambda_4\phi^3$ to vanish. For $m^2 \geq 0$ this goes monotonically from $-\infty$ to $+\infty$ so V has a unique minimum; there is a unique ground state. For $m^2 < 0$ the cubic $V'(\phi)$ can have one or three roots, corresponding to a minimum, or two local minima and a maximum, as depicted in Figure 2. To understand the phase diagram we must focus on the global minimum. For $h = 0$ and $m^2 < 0$ we had two degenerate minima. For $0 < h < h_1$ the local minima have different values of $V(\phi)$. For $h = h_1$ the local maximum merges with one of the local minima, destabilizing it. Inevitably, the other local minimum is lower than that local maximum, so this transition at h_1 does not concern the global minimum. The resulting phase diagram is depicted in Figure 3.

For a general potential $V(\phi)$, consider a first-order phase transition, namely a pair of local minima whose energy difference goes from negative to positive across the transition. The minima are degenerate at the phase

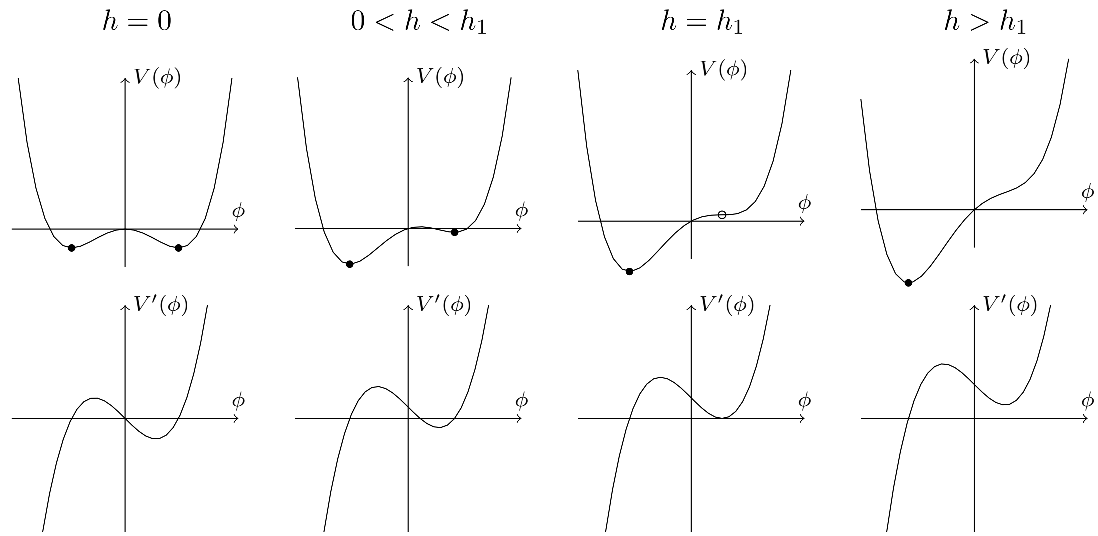

Figure 2: Depiction of $V(\phi) = h\phi + m^2\phi^2 + \lambda_4\phi^4$ for $m^2 < 0$ and varying values of h. We only plot for $h \geq 0$ due to the $h \to -h$ and $\phi \to -\phi$ invariance. The cross-over value h_1 beyond which one of the local minima destabilizes is $h_1 = \sqrt{-8m^6/(27\lambda_4)}$ (value where the discriminant of V' vanishes).

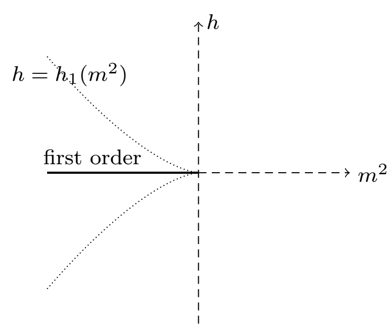

Figure 3: Phase diagram for $V(\phi) = h\phi + m^2\phi^2 + \lambda_4\phi^4$. The only phase transition is along $h = 0$ and $m^2 < 0$. The dotted lines (where a sub-leading local minimum disappears) are just drawn for reference; they do not affect the low-energy physics. Dashed lines are just the coordinate axes.

transition itself. As we move along (not across!) the phase transition (remaining at the phase transition itself), the minima remain degenerate. The situation can only simplify if somehow the minima merge. But this means that the potential (minus the energy of these minima) has a pair of double zeros that merge, hence a quadruple zero. Such a transition is second-order. Alternatively the local minima can stop being global minima: this happens if some other minimum goes from larger energy to the same energy as them, which would then be some other first-order phase transition line. In other words, a first-order phase transition line can either end at a second-order phase transition or when it intersects with some other first-order phase transition line, where the situation is more complicated.

(iii) Now $V(\phi) = Q(\phi^2)$ with $Q(X) = m^2 X + \lambda_4 X^2 + \lambda_6 X^3$. What matters is the global minimum (and degeneracy thereof) of $Q(X)$ for $X \in [0, +\infty)$. Such a minimum lies either at the end-point $X = 0$ of the interval (not at $X \to +\infty$ since $\lambda_6 > 0$), or at a point with vanishing $Q'(X) = m^2 + 2\lambda_4 X + 3\lambda_6 X^2$, namely $X = X_\pm$ with

$$X_\pm \coloneqq \frac{1}{3\lambda_6}\Big(-\lambda_4 \pm \sqrt{\lambda_4^2 - 3\lambda_6 m^2}\Big),$$

provided these are real and positive. This leads to a case distinction. Let me be sloppy about which of the following inequalities should be strict.

- If $m^2 < 0$ then $X_- < 0 < X_+$ so that on the interval $[0, +\infty)$ the polynomial Q is decreasing then increasing. Thus, the potential V has a local maximum at $\phi = 0$ and a $\mathbb{Z}_2$ pair of global minima at $\pm\sqrt{X_+}$.
- If $m^2 > \min(0, \lambda_4)^2/(3\lambda_6)$ then $Q'(X) > 0$ for $X > 0$, hence V has a unique minimum at $\phi = 0$.
- If $\lambda_4 < 0$ and $0 < m^2 < \lambda_4^2/(3\lambda_6)$ then the two roots $X_\pm$ of Q' are positive, so that on the interval $[0, +\infty)$ the polynomial Q is increasing, decreasing, and increasing. The potential V has three local minima at $\phi = 0$ and $\phi = \pm\sqrt{X_+}$, and the key question is which of $V(0) = 0$ and $V(\pm\sqrt{X_+})$ is the smallest. We evaluate

$$\begin{aligned} V(\pm\sqrt{X_+}) &= Q(X_+) = Q(X_+) - X_+ Q'(X_+) \\ &= (1/3) X_+^2 \Big(-\lambda_4 - 2\sqrt{\lambda_4^2 - 3\lambda_6 m^2}\Big). \end{aligned}$$

The parenthesized term has the following sign (in the first step we multiply by something positive since $\lambda_4 < 0$)

$$\begin{aligned}\operatorname{sgn}\Big(-\lambda_4 - 2\sqrt{\lambda_4^2 - 3\lambda_6 m^2}\Big) &= \operatorname{sgn}\prod_{\pm}\Big(-\lambda_4 \pm 2\sqrt{\lambda_4^2 - 3\lambda_6 m^2}\Big) \\ &= \operatorname{sgn} 3\Big(4\lambda_6 m^2 - \lambda_4^2\Big)\end{aligned}$$

so the transition is actually at $m^2 = \lambda_4^2/(4\lambda_6^2)$.

In summary, if $m^2 > \min(0,\lambda_4)^2/(4\lambda_6)$ then V has a global minimum at $\phi = 0$, and if $m^2 < \min(0,\lambda_4)^2/(4\lambda_6)$ then V has a pair of global minima at $\phi = \pm\sqrt{X_+}$.