References:

Wang, Wen, Witten, 1705.06728, section 4.1–4.3

Levin, Gu, 1202.3120, section VI B, C

1. i) Consider the Dijkgraaf-Witten model on an open manifold X_d

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g\}} \underbrace{\prod_{\Delta_d} \nu(g_{v_0}, g_{v_1} \ldots g_{v_d})^{s(\Delta_d)}}_{e^{-S}}$$

Show that e^{-S} does not depend on g_v when v is in the interior of X_d. In fact, show that we may replace all interior vertices by a single vertex and place some reference group element g_* on it.

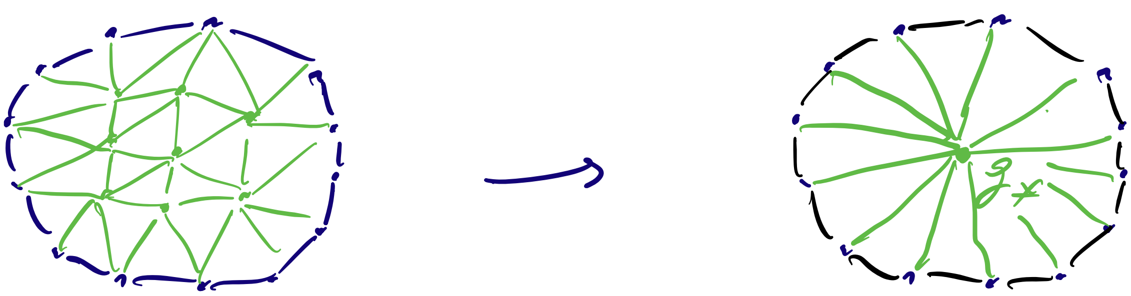


Fig 1.

Thus,

$$Z \overset{(1)}{=} \frac{1}{|G|^{N_b}} \sum_{\{g_{bound}\}} \prod_{\Delta_{d-1} \in \partial X_d} \nu(g_{v_0}, g_{v_1}, \ldots g_{v_{d-1}}, g_x)^{s(\Delta_{d-1})}$$

where N_b is the number of boundary vertices and $\{g_{bound}\}$ are values of g on ∂X_d.

Note: that Z in (1) is a local action on the boundary manifold ∂X_d. However, because of the choice of g_x it is not manifestly symmetric under

$g_i \to h g_i\,, \quad h \in G.$

(Choosing to instead sum over g_* in (1) would make the boundary action non-local).

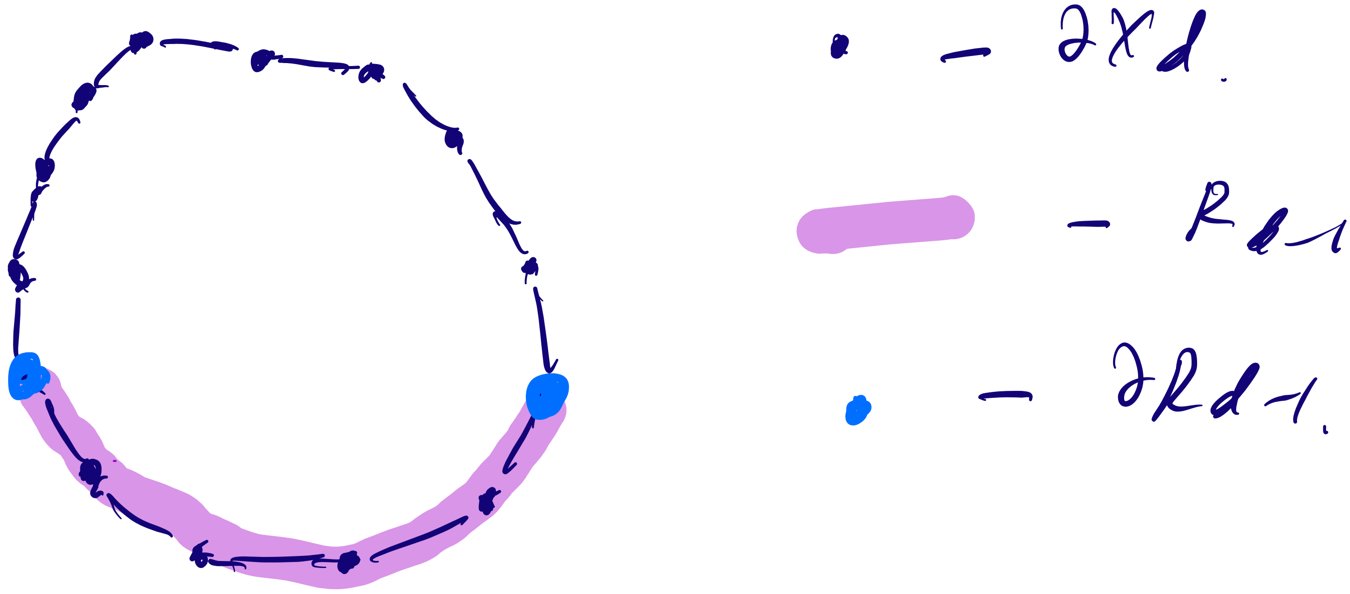
The lack of manifest symmetry in (1) leads to an unusual symmetry action on the Hilbert space, when (1) is converted to a Hamiltonian form.

Let's explore this.

ii) Consider $R_{d-1} \subset \partial X_d$.

Imagine fixing the values of g on ∂R_{d-1} and performing the sum in (1) only over g_v with v in the interior $\mathring{R}_{d-1}$ of R_{d-1}, i.e. consider

$$\Psi(g|_{\partial R_{d-1}}) \overset{(2)}{=} \frac{1}{|G|^{N_{\mathring{R}}}} \sum_{\{g_{\mathring{R}_{d-1}}\}} \prod_{\Delta_{d-1} \in R_{d-1}} \nu(g_{v_0}, g_{v_1}, \ldots g_{v_{d-1}}, g_*)^{s(\Delta_{d-1})}$$



We can think of (2) as a kind of wave-function on ∂R_{d-1}.

For instance, if $R_{d-1} = M_{d-2} \times I$ where $I = \{ s, \ s \in [0,T] \}$.

We can think of

$$\psi = \langle g'' | e^{-HT} | g' \rangle$$

g'' and g' are values of

g at $s=T$ and $s=0$.

Show that

$$\psi(h \cdot g|_{\partial R_{d-1}}) = \prod_{\Delta_{d-2} \in \partial R_{d-1}} \nu(g_0, g_1, \ldots, g_{d-2}, g_*, h^{-1}g_*)^{s(\Delta_{d-2})} \cdot \psi(g|_{\partial R_{d-1}}). \quad (3)$$

(Hint: adjoin vertex with $h^{-1}g_*$ on it to

Fig 1.

We now pass to Hamiltonian formalism for boundary dynamics.

We start with state sum (1) for $\partial X_d = M_{d-2} \times \mathbb{R}$.

The boundary Hilbert space is spanned by kets $|\{g\}\rangle$, where $\{g\}$ labels group elements on vertices of M_{d-2}.

(3) implies that the symmetry G acts on this Hilbert space via

$$U(h) = U_1(h) \cdot U_0(h) \qquad (4)$$

where

$$U_0(h)\,|\{g\}\rangle = |\{h \cdot g\}\rangle$$

and $U_1(h)\,|\{g\}\rangle =$

$$= \prod \nu(g_0, g_1, \dots g_{d-2}, h g_*, g_*)^{-s(\Delta_{d-2})}$$

$$\Delta_{d-2} \in M_{d-2}$$

iii) Consider the case $d=2$.

If $X_d = I \times \mathbb{R}$ the spatial boundary M_{d-2} is just two points

Show that the symmetry action on each of these points (4) V is just a projective rep. of G with cocycle $\omega \in H^2(G, U(1))$, with ω determined by ν,

i.e. at each endpoint

$$U(g)\,U(h) = \omega(g,h)\,U(g\cdot h).$$

iv) Consider $d=3$ and $M_{d-2}=S^1$

Then $U(h) = U_1(h)\,U_0(h)$

with $U_1(h) = \prod_{j=1}^{N} \nu(g_j, g_{j+1}, h g_*, g_*)$.

When ν is a non-trivial element of $H^3(G, U(1))$, $U(h)$ is a non-onsite symmetry, i.e. it <u>cannot</u> be factored as

$$U(h) = \prod_b U_b(h)$$

where b labels disjoint blocks of sites and U_b satisfies the G-group law.

As an example, consider $G = \mathbb{Z}_2$.

and $\omega(g_1, g_2, g_3) \overset{(5)}{=} (-1)^{g_1 g_2 g_3} \in H^3(\mathbb{Z}_2, U(1))$
(we discussed this example in lecture 9)
Recall $\nu(g_1, g_2, g_3, g_4) = \omega(g_1^{-1} g_2, g_2^{-1} g_3, g_3^{-1} g_4)$

Our Hilbert space is a chain with

elements $\{0, 1\}$ on it.

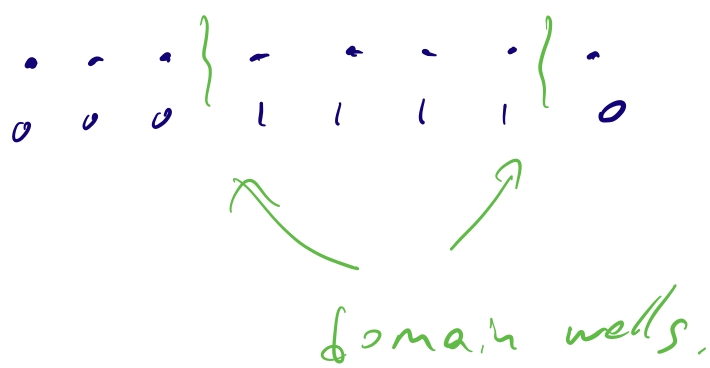
Show that $U_1 = (-1)^{N_{dw}/2}$ where

N_{dw} is the number of "domain walls"

on the chain (a domain wall is a

boundary between a 0 "domain" and a "1

domain"



Show that the following boundary Hamiltonian commutes with the $\mathbb{Z}_2$ symmetry

$$H = -\sum_j \left(\sigma^x_j - \sigma^z_{j-1} \sigma^x_j \sigma^z_{j+1} \right)$$

Here σ's are Pauli's and we think of $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$.
It can be shown that H realizes a $c=1$ boson CFT where both the $e^{i\varphi}$ and $e^{i\theta}$ operators are charged under $\mathbb{Z}_2$.
(See Levin-Gu reference.)
This means that this symmetry of $c=1$ theory is anomalous and the anomaly is characterized by the bulk DW theory corresponding to our $\omega \in H^3(\mathbb{Z}_2, U(1))$ in (5).