Higher symmetry in Particle Physics

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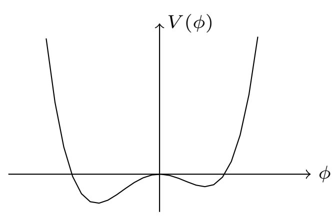
In this class we will sometimes use the word theorem, but not in a mathematically rigorous way.

1 Symmetry breaking

When discussing phenomenology and applications to particle physics, you really have to talk about symmetry breaking. Indeed, in particle physics, *symmetries are often an infrared accident*. If we take quantum gravity into account, black hole physics implies (how rigorously?) that symmetry is broken in the UV. So we will certainly have to consider patterns of symmetry breaking.

1.1 Breaking an ordinary symmetry

Breaking an ordinary symmetry is done by charged local operators. Basic example: consider a scalar field Lagrangian $\mathcal{L} = \int d^d x \big(\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\big)$ with some potential $V(\varphi) = a\varphi^4 - b\varphi^2$ that respects the $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$. Then we add a leading violating operator $\lambda\varphi^3$ with λ small, so that the symmetry is only slightly broken.[1] The gap between energies of the two local minima is proportional to λ at small λ.



1.2 Breaking one-form symmetries

Using local operators? The particularity of higher-form symmetries compared to usual symmetries is that all local operators are neutral. The symmetry cannot be violated by adding any local operator deformation to the action. There is no most-relevant charged operator, the same method will not work.

[1] Max Metlitski asks why not add $\lambda\varphi$ instead; Clay answers we can get rid of such a term by shifting φ. Debates can be had about that.

For example in Maxwell theory we have $U(1)_e^{(1)} \times U(1)_m^{(1)}$ one-form symmetries. Consider a deformed Lagrangian

$$\mathcal{L} = \frac{1}{e^2} \int d^4x \Big(f_{\mu\nu} f^{\mu\nu} + \frac{1}{\Lambda^4} (f_{\mu\nu} f^{\mu\nu})^2 \Big).$$

Since $f_{\mu\nu} = 2\partial_{[\mu} a_{\nu]}$ we still have the magnetic one-form symmetry since $\partial^\mu \varepsilon_{\mu\nu\rho\sigma} f^{\rho\sigma} = 0$. The electric one-form symmetry gets deformed but there is still a conserved current:

$$\partial^\mu \Big(f_{\mu\nu} + \frac{1}{\Lambda^4} f^2 f_{\mu\nu} \Big) = 0.$$

Using additional particles. In fact, symmetry breaking one-form symmetry requires new charged particles.

- Dynamical electric charges lead to $d \star f \neq 0$, making $U(1)_e^{(1)}$ broken.
- Dynamical magnetic charges lead to $f \neq da$ globally, making $U(1)_m^{(1)}$ broken.

What about the scale of breaking?

Claim 1. *The statement "$U(1)_e^{(1)}$ is broken by order 1 at Λ_{Pl}" is equivalent to the weak gravity conjecture.*

2 Higher-group theory and the emergence theorem

We consider the simplest scenario of higher-group. Consider a $U(1)^{(1)}$ one-form symmetry and an zero-form symmetry $G^{(0)}$ that is a connected Lie group such as $SU(N)$. These can form a higher-group (called a 2-group even though the top order of forms is just 1) with a structure constant $\kappa \in \mathbb{Z}$.

More precisely κ lies in the group cohomology $H^3(G^{(0)}, U(1))$ (same cohomology group as what characterizes Chern–Simons terms).

Many characterizations:

- **Background fields.** The one-form and zero-form symmetries have gauge fields $B^{(2)}$ and $A^{(1)}$, respectively, and the gauge-transformation with gauge parameters $\Lambda^{(1)}$ and $\lambda^{(0)}$ reads

  $$B^{(2)} \to B^{(2)} + d\Lambda^{(1)} + \frac{\kappa}{4\pi} \operatorname{Tr}(\lambda^{(0)} dA^{(1)}).$$

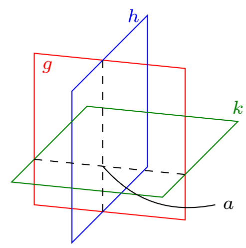
  It is possible to show that this formula is only consistent if κ is quantized.

- **Current algebras.** The symmetries have currents $J_{\mu\nu}$ and J_ν^a where a is an index for (the adjoint representation of) $G^{(0)}$. Then we have a failure of conservation at coincident points which reads

  $$\partial^\mu J_\mu^a(x) J_\nu^b(y) = \frac{\kappa \delta^{ab}}{2\pi} \partial^\lambda \delta^4(x-y) J_{\nu\lambda}(x) + \dots$$

where ... are separated-point terms. Major difference compared to an anomaly: the anomaly would just have the unit operator instead of $J_{\nu\lambda}$.

- **Symmetry defects.** We have codimension 1 symmetry operators for $G^{(0)}$, and codimension 2 symmetry operators for $U(1)^{(1)}$.



Three codimension 1 symmetry operators labeled by $g, h, k \in G^{(0)}$ generically intersect along $d-3$ dimensions (here one dimensions). The triple intersection is characterized by an element of the group cohomology $H^3(G^{(0)}, U(1)^{(1)})$, which is κ. Then such an intersection has a symmetry defect labeled by $a \in U(1)^{(1)}$. how is a built from κ, g, h, k?

Example: for $G^{(0)}$ a simple Lie group such as $SU(N)$, the parameter κ lies in $H^3(G^{(0)}, U(1)) = \mathbb{Z}$.

Importantly, this κ is not a modification of the equal-time commutations of the current algebra. In contrast, non-invertible symmetries discussed later are very different even at equal times.

Emergence theorem. One can see 2-groups $\mathbb{G}$ as higher analogues of group extensions:

$$U(1)^{(1)} \to \mathbb{G} \to G^{(0)}.$$

Just like for group extensions, $U(1)^{(1)}$ is a subgroup and $G^{(0)}$ is a quotient; it is not a subgroup (for $\kappa \neq 0$).

E.g., the OPE tells us that for $\kappa \neq 0$,

$$J_\nu J_\mu \supseteq J_{\mu\nu}.$$

In other words, as soon as you have the zero-form symmetry the one-form symmetry comes for free. (Analogy: in $SU(2)$ once you have the raising and lowering generators j_+ and j_- then $j_3 = [j_+, j_-]$ comes for free.)

Scenario. Consider $\mathbb{G}$ (with simple Lie group $G^{(0)}$) being emergent along an RG flow. In the UV effective field theory, no symmetry $\mathbb{G}$, namley some pieces of $\mathbb{G}$ have to be broken. After the RG flow, we get an IR effective field theory, and we assume that this IR theory does obey $\mathbb{G}$ with $\kappa \neq 0$.

Since κ is quantized it cannot run. Thus, at least one of $G^{(0)}$ and $U(1)^{(1)}$ must emerge along the RG flow. Let us call E_0 and E_1 the energy scales of emergence of these two symmetries. Note that if $G^{(0)}$ is present at some energy scale, and $\kappa \neq 0$, then $U(1)^{(1)}$ is present at that energy scale. Thus, we have a universal inequality

$$E_0 \leq E_1.$$

This is imprecise since E_0, E_1 are only approximately defined.

Claim 2 (Emergence theorem for 2-groups)**.** *There cannot be any family of RG flows such that $E_0 \gg E_1$ parametrically.*

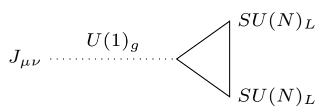
2.1 Long example

Consider $U(1)$ gauge theory with N massless fermions $\chi^{\pm,a}$, $a = 1, \ldots, N$.

	$U(1)_g$	$SU(N)_L$	$SU(N)_R$
χ^+	$+1$	$\square$	1
χ^-	-1	1	$\square$

There is also the axial symmetry to worry about, but that is a more advanced topic.

There is a non-trivial triangle diagram TODO change the dotted line to a wavy line



$$\implies \kappa_L = +1, \qquad \kappa_R = -1.$$

The electric one-form symmetry is broken by charged matter. The magnetic one-form symmetry remains. The two-group involves $U(1)^{(1)}_m$ and $SU(N)_{L,R}$. There are two two-groups, corresponding to the extensions[2]

$$U(1)^{(1)} \to \mathbb{G}_L \to SU(N)_L, \qquad U(1)^{(1)} \to \mathbb{G}_R \to SU(N)_R$$

Note that the diagonal $SU(N)_\Delta \subset SU(N)_L \times SU(N)_R$ has $\kappa = \kappa_L + \kappa_R = 0$.

Breaking the one-form symmetry. In principle we would like to add dynamical monopoles. But that is difficult. Let us embed the $U(1)$ gauge group inside a non-abelian UV gauge group.

Consider an $SU(2)$ gauge theory with an adjoint scalar φ. Then use the Higgs mechanism to give a non-trivial vev to φ, namely $\langle \mathrm{Tr}(\varphi^2) \rangle = v^2 \neq 0$. This breaks the gauge group to the $U(1)$ Cartan (diagonal) subgroup of $SU(2)$. At high-energies there will be dynamical monopoles; alternatively one can understand that the Bianchi identity $\partial_{[\mu} f_{\rho\sigma]} = 0$ of the $U(1)$ Cartan is not satisfied at high energies; instead at high energies we have an $SU(2)$ version $D_{[\mu} F_{\rho\sigma]} = 0$ including the commutator term in the covariant derivative.

The breaking of symmetry happens at the Higgsing scale $E_1 = v$.

[2]Here Clay says you don't have just one combined two-group $U(1)^{(1)} \to \mathbb{G} \to SU(N)_L \times SU(N)_R$, but I don't understand that. I guess it is less intuitive to have a non-simple group $G^{(0)}$.

General consequence. In any UV theory where the $U(1)$ gauge group is part of a non-abelian gauge group, the $U(1)_m^{(1)}$ symmetry is broken at the Higgsing scale, so both $SU(N)_L$ and $SU(N)_R$ are broken by that scale. However, the diagonal $SU(N)_\Delta$ need not be broken.

Illustration in $SU(2)$ model. We must put together the χ into some representation of $SU(2)$. The simplest option is to have a doublet of $SU(2)$. What is the flavour symmetry? We can no longer rotate just χ^+ or χ^- separately. We must rotate both simultaneously. In other words the diagonal $SU(N)_\Delta \subset SU(N)_L \times SU(N)_R$ is preserved in the UV, while $SU(N)_L$ and $SU(N)_R$ must be emergent at some scale $\leq$ the Higgsing scale.

	$SU(2)_g$	$SU(N)_\Delta$
χ	2	$\square$

2.2 Generalization to an approximate IR $G^{(0)}$

Consider an IR theory where $G^{(0)}$ is explicitly broken by some coupling $y \ll 1$. Consider a family of RG flows

$$T_{\text{UV}}(y) \xrightarrow{\text{RG}} T_{\text{IR}}(y)$$

such that $U(1)^{(1)}$ is broken in the UV, $U(1)^{(1)}$ is preserved in the IR for any y, and $G^{(0)}$ is broken by terms proportional to y. In addition, we assume that for $y = 0$ the whole 2-group $\mathbb{G}$ is restored in the IR, so the emergence theorem holds.

Note that $y \neq 0$ states come in two types:

- Massless particles in $T_{\text{IR}}(y)$, which contribute κ_0 to the parameter κ.
- Massive particles with masses that become zero at $y \to 0$, so their masses behave as $y^{\alpha_i} m_i$ for some exponents $\alpha_i > 0$ (and mass scales m_i). When $y \to 0$ these contribute $\sum_i \kappa_i$ to the parameter κ.

For $y = 0$ we have two-group symmetry $\mathbb{G}$ with $\kappa = \kappa_0 + \sum_i \kappa_i$.

Two cases:

- If $\kappa \neq 0$ then the emergence theorem forbids $G^{(0)}$ from being present in the UV.
- If $\kappa = 0$ then $G^{(0)}$ can be an exact symmetry in the UV at $y = 0$, weakly broken by y. In a sense, this is a loophole in the emergence theorem, due to the particles that become massless as $y \to 0$.

Explain more.

Example Consider our previous example of $U(1)$ gauge theory. Include in the IR Lagrangian a term $ym\chi^{+a}\chi^{-a}$, which breaks $SU(N)_{L,R}$ symmetries to the diagonal one, with the full symmetry being restored as $y \to 0$.

This admits the same UV completion as before as an $SU(2)$ gauge theory with χ and with the $SU(N)_\Delta$ symmetry in the UV. But this is boring.

Consider a more interesting completion with ρ and η a pair of doublets,

	$SU(2)$	$SU(N)_\Delta$
$\rho^a = (\chi^{+a}, \psi^{-a})$	2	$\square$
$\eta^a = (\psi^{+a}, \chi^{-a})$	2	$\overline{\square}$

In the UV, include a Yukawa term $\mathcal{L}_{\text{UV}} \supseteq y\varphi^{\mu\nu}\rho^a_\mu \eta^a_\nu$ where $\mu, \nu \in 2$ of $SU(2)_g$. For $y \to 0$ we have symmetry enhancement back to $SU(N)_L \times SU(N)_R$ since only the Yukawa breaks that symmetry to the diagonal subgroup.

Consider Higgsing to $\varphi \to \Lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ breaking $SU(2)_g \to U(1)_g$. Then the UV Lagrangian term becomes

$$y\Lambda(\chi^{+a}\chi^{-a}) + y\Lambda(\psi^{+a}\psi^{-a}).$$

What happens as $y \to 0$? We have enhaved flavour symmetry $SU(N)_L \times SU(N)_R$

	$U(1)_g$	$SU(N)_L$	$SU(N)_R$
χ^+	$+1$	$\square$	
χ^-	-1		$\square$
ψ^+	$+1$		$\square$
ψ^-	-1	$\square$	

So overall both UV and IR have enhanced symmetry at $y = 0$. This is achieved by having these new matter fields $\psi^\pm$ whose mass is suppressed by y. They change $\kappa_0 \neq 0$ to a total $\kappa = 0$.

3 Higher flavour symmetry in the Standard Model

Discrete quotients of gauge groups. Note that $SU(N)$ and $SU(N)/\mathbb{Z}_N$ gauge theories (without matter) have the same spectrum of gauge bosons for instance. Their one-form symmetries are:

$SU(N)$	$SU(N)/\mathbb{Z}_N$
electric $\mathbb{Z}_N^{(1)}$	magnetic $\mathbb{Z}_N^{(1)}$
charged Wilson lines	charged 't Hooft lines

Standard Model. The gauge group is $SU(3)_C \times SU(2)_L \times U(1)/\Gamma$ for Γ being trivial or $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, because the $\mathbb{Z}_6 \subset SU(3)_C \times SU(2)_L \times U(1)_Y$ acts trivially on all known particles. One way to falsify such a quotient would be to find a particle that is charged under it.

Sacrilegously (for a particle physicist), we will normalize hypercharges to have minimum charge 1 rather than 1/6. Then the particle content and representations of the different particles are as follows. Our convention is that all fermions are left-handed and Weyl.

	Q_i	$\overline{u}_i$	$\overline{d}_i$	L_i	$\overline{e}_i$	N_i	H
$SU(3)_C$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$				
$SU(2)_L$	$\mathbf{2}$			$\mathbf{2}$			$\mathbf{2}$
$U(1)_Y$	$+1$	-4	$+2$	-3	$+6$		-3
$U(1)_B$	$+1$	-1	-1				
$U(1)_L$				$+1$	-1	-1	

where the first three groups are the gauge groups and the other two are flavour symmetries. Here, Q_i is the (left-handed) up/down quark (it splits into these two only after electroweak symmetry breaking), $\overline{u}_i$ and $\overline{d}_i$ are the right-handed up and down quarks, L_i the lepton, $\overline{e}_i$ the right-handed lepton, and H the Higgs scalar. For later we also include N_i the right-handed neutrino; we don't know if they exist in our universe.

Classical 0-form symmetry. Matter comes in $N_g = 3$ generations, so without Yukawa terms we should expect[3] $U(3)_Q \times U(3)_{\overline{u}} \times U(3)_{\overline{d}} \times U(3)_L \times U(3)_{\overline{e}}$. Yukawa couplings (which become mass terms when the Higgs gets a vev)

$$\mathcal{L}_{\text{Yuk}} \supset Y^{\overline{u}}_{ij} H^* Q_i \overline{u}_j + Y^{\overline{d}}_{ij} H Q_i \overline{d}_j + Y^{\overline{e}}_{ij} H L_i \overline{e}_j$$

breaks symmetry to $U(1)_{L_1} \times U(1)_{L_2} \times U(1)_{L_3} \times U(1)_B/\mathbb{Z}_3$ where L_i is the action on each lepton separately, and B is the baryon number.

Magnetic 1-form symmetry $U(1)_y$. From the hypercharge gauge group we get a magnetic symmetry.

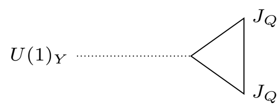
Approximate symmetries. The idea will be to keep track of

$$G^{(0)} = \prod_{o \in \{Q, \overline{u}, \overline{d}, L, \overline{e}\}} SU(3)_o$$

symmetry restoration as $y \to 0$. We don't track the $U(1)$ factors of the big symmetry group because they suffer from ABJ anomalies and that will be discussed later.

[3] We are not tracking the $U(1)$ acting on the neutrino.

Compute the 2-group structure constants. For $SU(3)_Q$, TODO change dotted to squiggly line; add middle arrows



$$\implies \kappa = q_Y \cdot 6 \cdot c_2(\square) = 6.$$

Similar work for all the fermions gives

$SU(3)^2$ flavour	Q	$\overline{u}$	$\overline{d}$	L	$\overline{e}$
κ with $U(1)^{(1)}$	$+6$	-12	$+6$	-6	$+6$

Each species flavour symmetry $SU(3)_o$ is in a non-trivial 2-group.

3.1 Emergence theorem and GUTs

The idea of GUT (grand unified theory) is to have a UV model that flows to the standard model. On top of that, an important idea was to see $U(1)_Y$ as part of a bigger non-abelian group that got Higgsed when flowing to the IR. This means that $U(1)^{(1)}$ is broken in the GUT.

The emergence theorem (without adding matter suppressed by y) tells us that *every GUT* necessarily breaks each $SU(3)_i$ independently. This is a constraint on how standard model fermions unify into multiplets.

Example. If the UV gauge group is $G_1 \times \cdots \times G_\ell$ and $U(1)_Y$ is contained in it, and if each of Q, $\overline{u}$ etc embeds in separate representations $\rho_1, \ldots, \rho_5$ etc then each ρ_i would have separate $SU(3)$ flavour symmetry. This would contradict the emergence theorem.

Temporary(?) conclusion. In a GUT, standard model fermions must combine into multiplets of the larger gauge group, in a way that makes the combination have total $\kappa = 0$. What are these combinations? Remember that we are talking about diagonal subgroups of the non-abelian groups $SU(3)$, so there is no way to make a combination with weights. For each subset of the five fermion fields, we check the corresponding total κ. We get

Number of species acted on	combination with $\kappa = 0$
1	None
2	$\{L, Q\}, \{L, \overline{d}\}, \{L, \overline{e}\}$
3	$\{\overline{u}, \overline{d}, \overline{e}\}, \{\overline{u}, \overline{e}, Q\}, \{\overline{u}, \overline{d}, Q\}$
4	None
5	$\{Q, \overline{u}, \overline{d}, L, \overline{e}\}$

Caveat: in the actual Standard Model, Yukawa couplings are not all small in this way. Instead, we should focus on those that are indeed small.

Familiar GUT: Georgi–Glashow. This model has $SU(5)$ with $\overline{5}$+10 fermion. Higgsing reduces $SU(5)$ to $SU(3) \times SU(2)$. The representations split as $5 \to (3,1) + (1,2)$ and $\overline{5} \to (\overline{3},1) + (1,2) = \{L, \overline{d}\}$, so the antisymmetric is

$$10 = \Lambda^2 5 \to \Lambda^2\big((3,1) + (1,2)\big) = (\overline{3},1) + (3,2) + (1,1) = \{\overline{u}, \overline{e}, Q\}.$$

As expected, these two sets $\{L, \overline{d}\}$ and $\{\overline{u}, \overline{e}, Q\}$ appear in our table.

Familiar GUT: $SO(10)$**.** This model has $SO(10)$ gauge group. The easiest way to think about the Higgsing is to go from $SO(10)$ to $SU(5)$.

In general, when Higgsing $SO(2N)$ to $SU(N)$, the vector representation splits as $2N \to N + \overline{N}$. The Dirac spinor representation of $SO(2N)$ is 2^N dimensional, it is a sum of two chiral spinor representations of $SO(2N)$, which are 2^{N-1} dimensional. They split as

$$\begin{aligned} 2^N &= 1 + N + \Lambda^2 N + \cdots + N + 1 \\ 2^{N-1} &= 1 + \Lambda^2 N + \Lambda^4 N + \ldots \text{(every other term)} \end{aligned}$$

For $N = 5$ we get

$$16 \to 1 + 10 + \overline{5},$$

which gives all species at once.

Pati–Salaam model. Gauge group $G_{\text{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R$ where the first is Higgsed to $SU(3)$, the second is not, and the third factor is Higgsed to the Cartan; the $U(1)_Y$ is a combination of Cartans. Then under this Higgsing,

$$\begin{aligned} &(4,2,1) \to (3,2) + (1,2) = \{Q, L\}, \\ &(\overline{4},2,1) \to (\overline{3},1) + (\overline{3},1) + (1,1) + (1,1) = \{\overline{u}, \overline{d}, \overline{e}, N\}, \end{aligned}$$

which was also in our table (except for the N which is uncharged under $U(1)_Y$ anyway).

What about $\{L, \overline{e}\}$ **and** $\{\overline{u}, \overline{d}, Q\}$**?** There seems to be no known GUT that has this pattern of unification. It would be interesting because it would lead to no proton decay at leading order because all the quarks lie in the same multiplet of the GUT.

3.2 Exploiting the $y \neq 0$ loophole

Trinification. The gauge group is $G_{\text{trin}} = SU(3)_C \times SU(3)_L \times SU(3)_R$. The Higgsing to the Standard Model is subtle and responsible for the funny effects below. Matter

$$\begin{aligned} \Psi_Q &= (3,\overline{3},1) \to \{Q, d'\} \\ \Psi_{\overline{Q}} &= (\overline{3},1,3) \to \{\overline{u}, \overline{d}, \overline{d}'\} \\ \Psi_L &= (1,3,\overline{3}) \to \{L, \overline{e}, N\} \end{aligned}$$

where $d' \in (3,1)$ and $\overline{d}' = (\overline{3},1)$ are new fields. Why are people happy with this unification even though there are extra vector-like (together they form a Dirac spinor, so they don't introduce additional anomalies) down quarks $d', \overline{d}'$ on top of the Standard Model matter content.

In the UV Lagrangian we have a term $y\Phi\Psi_Q\Psi_{\overline{Q}}$ where Φ is the Higgs field. This gives

$$y\Lambda d'\overline{d}' + yH^*Q\overline{u} + yHQ\overline{d}$$

Crucially the same y that breaks flavour symmetries in the Standard Model must appear in the mass, so the scale of new particles $d', \overline{d}'$ is suppressed by y compared to the Higgsing scale.

3.3 Models of axions

We have a periodic scalar a with $a \sim a + 2\pi f$ where f has mass dimension 1. The action is $S = \frac{1}{2}\int da \wedge \star da$. Currents:

- $J_\mu = \partial_\mu a$ for a $U(1)^{(0)}$ shift symmetry,
- $J_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\lambda}\partial^\lambda a$ for a $U(1)^{(2)}$ symmetry.

Let us couple the axion to a $U(1)_g$ gauge field: the new action is

$$S = \frac{1}{2}\int da \wedge \star da + \frac{1}{2e^2}\int F \wedge \star F + \frac{ik}{8\pi^2 f}\int aF \wedge F,$$

where e is the gauge coupling, f is the scale of periodicity of the axion, k is a quantized coupling. Here, the last term is an irrelevant (dimension 5) operator, suppressed by the mass scale f. Lots of things are known about possible UV completions, for instance a could be the phase of some scalar field that acquires a vev through Higgsing. See the exercise.

Equations of motion are easily expressed in terms of the currents $J^{(3)}$ (for the two-form symmetry of the axion), $J_e^{(2)}$, $J_m^{(2)}$ (for the electric and magnetic one-form symmetries of the gauge field we just added), and $J^{(1)}$ (shift symmetry of the axion):

$$d{\star}J^{(3)} = 0, \qquad d{\star}J_m^{(2)} = 0, \qquad d{\star}J^{(1)} = \frac{k}{8\pi^2}F \wedge F, \qquad d{\star}J_e^{(2)} = \frac{-k}{4\pi^2 f}da \wedge F.$$

Crucially, the $da \wedge F$ source term cannot be absorbed by shifting $J_e^{(2)}$ by aF because a is not globally well-defined.

This implies various symmetry breaking.

- The $U(1)^{(0)}$ shift symmetry is broken to $\mathbb{Z}_k^{(0)}$ (shift the axion by $2\pi f/k$)
- The $U(1)_e^{(1)}$ electric one-form symmetry is broken to $\mathbb{Z}_k^{(1)}$,

Higher-group. There is a higher-group here (for $k > 1$). It is a 3-group, namely a combination of 0-form, 1-form, 2-form symmetry. Couple to backgrounds: $A^{(3)}$ for $U(1)^{(2)}$, and (discrete) $B^{(2)}$ for $\mathbb{Z}_k^{(1)}$. The first is easy: contract $A^{(3)}$ with $\star da$, which gives a term

$$S \supset \frac{i}{2\pi f} \int da \wedge A^{(3)}.$$

How to couple the discrete background gauge field $B^{(2)}$? We saw that with Thomas Dumitrescu: replace F by $F - B$.

The most interesting term $aF \wedge F$ changes to $a(F-B)\wedge(F-B)$. This implies that the gauge-invariant field strength of $A^{(3)}$ is no longer $dA^{(3)}$ (variation with respect to a), but actually

$$G^{(4)} = dA^{(3)} + \frac{k}{4\pi} B^{(2)} \wedge B^{(2)}.$$

The quadratic term signals the existence of a higher group:

$$U(1)^{(2)} \to \mathbb{G} \to \mathbb{Z}_k^{(1)}.$$

Emergence theorem. Consider an RG flow from UV without $\mathbb{G}$ to an IR theory that has $\mathbb{G}$ symmetry. Denote by E_2 and E_1 the scales of emergence of the 2-form and 1-form symmetries. We get $E_1 \leq E_2$, where

- E_1 is the scale of dynamical charged particles;
- E_2 is the scale of dynamical axion strings.

4 Non-invertible chiral symmetry

4.1 ABJ anomaly

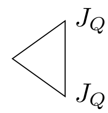
Our setting: $3+1$ dimensional $U(1)$ gauge theory with field strength $f = dc$ where c is the dynamical gauge field. We also have some matter (not specified here) such that the theory classically has a $U(1)_{\text{chiral}}$ global 0-form symmetry with current J. We also assume that the current is subject to an anomaly: J will not be conserved at a quantum level, due to the ABJ (Adler–Bell–Jackiw) anomaly

$$d \star J = \frac{N}{8\pi^2} f \wedge f \tag{1}$$

Many examples:

- **Massless QED** with chiral fermions $\chi^{\pm,a}$ with gauge charge ± 1 and $a = 1, \ldots, N$. The chiral symmetry with current J is the one that maps

$\chi^{+,a} \to e^{i\theta}\chi^{+,a}$ and leaves $\chi^{-,a}$ invariant[4] The anomaly comes from the flavour–gauge–gauge triangle[5]



- **Axion electrodynamics** $a \sim a + 2\pi\mu$ with action including $S \supset \frac{iN}{8\pi^2\mu}\int af \wedge f$, with shift symmetry current $J = da$. This current has the same type of anomaly.
- There are also non-abelian gauge theory examples, with discrete versions of this story.

From now on, take $N = 1$ (smallest possible value for the anomaly); otherwise various discrete group factors creep in.

4.2 Puzzle

Puzzle: does J obeying (1) a generate conserved charge?

- No? The current J is not conserved!
- Yes? See below.

This "yes" camp is led by 't Hooft, who should be credited with this observation. Despite the non-zero divergence let us consider nevertheless the integral $Q(t) = \int_{\mathbb{R}^3} d^3x J_0(x,t)$. Then

$$Q(t=+\infty) - Q(t=-\infty) = \int dt \partial_t Q = \int_{\text{spacetime}} d \star J = \frac{1}{8\pi^2}\int_{\text{spacetime}} f \wedge f,$$

where f is a differential form, and the cohomology class of $f/2\pi$ lies in $H^2(\text{spacetime}, \mathbb{Z})$, meaning that the integral over any two-cycle is an integer.[6] So on a topologically trivial spacetime, e.g., by regulating $\mathbb{R}^4$ as S^4, the right-hand side vanishes. Thus, chiral symmetry selection rules hold. Correlators of local operators on $\mathbb{R}^4$, and standard S-matrix elements obey selection rules.

Somewhat surprisingly: selection rules can be violated

- on a rich enough spacetime like $S^2 \times S^2$;
- in the presence of a 't Hooft line, characterized by $\int_{S^2} \frac{f}{2\pi} = m \in \mathbb{Z}$ magnetic charge.

[4]Often one writes the chiral symmetry with an opposite action on $\chi^{-,a}$, but this might lead perhaps (I think) to issues of factors of 2 in the quantization of some charges.

[5]Yesterday we had a flavour–flavour–gauge triangle instead, which has a very different consequence.

[6]Remember there are no instantons for $U(1)$, just fluxes on two-manifolds.

In the second setting, line operator insertions are creating non-trivial topology for us.

We need to develop a language to talk about symmetries that have such different consequences in these different contexts. Key clue: $f \wedge f$ is the square of something. We have $df = 0$, so that f generates a $U(1)^{(1)}_m$ magnetic one-form symmetry. *So the anomaly equation* (1) *is trying hard to be a current algebra.*

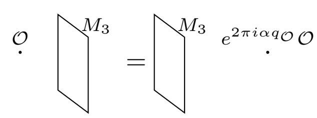
4.3 Construction of non-invertible symmetry

Modern definition: **a symmetry is a topological operator**.

Goal: define a topological operator that implements chiral symmetry. Without anomaly it is easy:

$$U_\alpha(M_3) \coloneqq \exp(2\pi i\alpha \int_{M_3} \star J).$$

This generates a finite chiral transformation with angle $2\pi\alpha$. It is defined on a closed codimension 1 manifold M_3. It is topological: if M_3 is changed continuously without crossing other operators, then the correlator is unmodified. On the other hand, if M_3 is moved and crosses a local operator, it acts by symmetry transformations:



where $q_{\mathcal{O}}$ is the charge of $\mathcal{O}$ under chiral symmetry. Most aspects of symmetry can be studied in terms of such $U_\alpha(M_3)$ without ever going back to an explicit expression in terms of an integrated current. But today we have an anomaly: the operator $U_\alpha(M_3)$ on its own is not topological and we have to modify it somehow, which requires looking at its expression more closely than usual.

With the anomaly. For J obeying (1), consider the naive "conserved current"

$$\star J - \frac{1}{8\pi^2} c \wedge dc.$$

Can we make sense of

$$\exp\Big(2\pi i\alpha \int_{M_3} \Big(\star J - \frac{1}{8\pi^2} c \wedge dc\Big)\Big)$$

It is formally topological, but not gauge-invariant! The problem is the Chern–Simons term $c \wedge dc$ with continuous level α.

First option: take $\alpha \in \mathbb{Z}$ to make the Chern–Simons term be well-defined. But integer α is the trivial operation, so nothing can be said about it.

Novelty: $\alpha \in \mathbb{Q}$ also makes sense! (It is not so novel for condensed matter physicists.)

Rational level does not make sense as an invertible/classical term in the action, but it makes sense as a *response*.

Discussion about Chern–Simons theory. Consider e.g. $U(1)$ Chern–Simons theory on M_3 at level k, with dynamical gauge field b, and couple it to a source c (which will later be identified to our aforementioned gauge field). The partition function is

$$Z^k[dc] = \int [Db] \exp\biggl(\frac{ik}{4\pi}\int_{M_3} b\wedge db + \frac{i}{2\pi}\int_{M_3} b\wedge dc\biggr)$$

where c is background and b dynamical. This is a non-invertible TQFT (it has many operators, many ground states on general manifolds).

If $M_3 = S^3$ without any operator insertion, then b, c are no longer connections, they are simply globally-defined one-form. We can simply evaluate the path integral since it is a Gaussian integral: just solve the equation of motion, $b = -c/k$. Plugging it gives[7]

$$Z^k_{S^3}[dc] = \exp\biggl(\frac{-i}{4\pi k}\int_{M_3} c\wedge dc\biggr),$$

which looks very much like an effective $1/k$ Chern–Simons term.

On a general manifold M_3, we **define "level $1/k$"** by this path integral $Z^k_{M_3}[dc]$.

How unique is this definition? Could I take some other TQFT that coincides with that TQFT on S^3, and use that as my definition? Remarkably, any such TQFT has the property that it decomposes into this one, tensored wiht a part that does not couple to c at all.

Definition of composite topological operator. For any $k \geq 1$ we define

$$D_{1/k}(M_3) \coloneqq U_{1/k}(M_3) Z^k_{M_3}[dc] = \int [Db] \exp\biggl(\frac{2\pi i}{k}\int_{M_3} \star J + \frac{ik}{4\pi}\int_{M_3} b\wedge db + \frac{i}{2\pi}\int_{M_3} b\wedge dc\biggr).$$

This is easily generalized to rational α, but let us not discuss it explicitly.

Note that this technology was not necessary when $M_3 = S^3$. This harks back to the fact that we had good selection rules for correlators of local operators on $\mathbb{R}^4$ or S^4: indeed, for these cases it is enough to use symmetry operators along various S^3.

Comment: if the anomaly coefficient is $N > 1$ instead, there is a non-anomalous $\mathbb{Z}_N$ subgroup, but for other rationals in $U(1)$ we need the same construction of composite topological operators.

There are generalizations to gravity.

Three different possibilities for the divergence of a current (to be checked)

- $d\star J = F' \wedge F''$ where F' and F'' are background fields; then there is just a 't Hooft anomaly, with an obstruction to gauging.

[7] There is actually a prefactor (one-loop determinant), which can be partially eliminated by local counterterms: there remains a meaningful overall constant (which is c-independent) that we will come back to.

- $d\star J = F\wedge F'$ where F is an operator and F' a background: then there is a symmetry as long as F' is turned off, but it is a weird kind of symmetry, leading to two-groups.
- $d\star J = F\wedge F$ where F is dynamical: this is today's discussion, you need to dress the symmetry operators and get non-invertible symmetries.

Alternative viewpoint on coupling with a TQFT. What is wrong with $U_{1/k}$? Place it along $x=0$.

$$\begin{aligned} U_{1/k} &= \exp\left(\frac{2\pi i}{k}\int_{x=0}\star J\right) \\ &= \exp\left(\frac{2\pi i}{k}\int_{x\geq 0} d\star J\right) \\ &= \exp\left(\frac{2\pi i}{k}\int_{x\geq 0}\frac{f\wedge f}{8\pi^2}\right). \end{aligned}$$

This is an effective θ term on one side of the defect. The $x<0$ and $x>0$ theories are not the QFTs: $\theta_+ = \theta_- + 2\pi/k$. We compensate this by adding here a TQFT on the interface, with two key properties:

- it must have $\mathbb{Z}_k$ one-form symmetry (abelian anyons)
- it must have an anomaly of $\mathbb{Z}_k^{(1)}$: denoting by $B^{(2)}$ the two-form background field for this 1-form symmetry, the anomaly inflow must be (spin of generator=$1/(2k)$)

$$\exp\left(\frac{2\pi i}{k}\int_{x\geq 0}\frac{B^{(2)}\wedge B^{(2)}}{2}\right)$$

Then we couple bulk and TQFT by $B^{(2)} = f/2\pi$.

Student question: why do we need a TQFT rather than some other QFT? Because otherwise we would not get a topological symmetry operator, we would end up with a much more complicated theory on one side.

Alternative construction via half-gauging. Consider gauging $Z_k^{(1)}$ subgroup of the magnetic $U(1)^{(1)}$. To do that, we need two new $U(1)$ fields because discrete gauging is not so easy. The fields are $\beta^{(2)}$ and $\gamma^{(1)}$ and we have a term

$$S \supset \frac{ik}{2\pi}\int d\beta^{(2)}\wedge\gamma^{(1)},$$

Here $\gamma^{(1)}$ is a Lagrange multiplier. The equations of motion of $\gamma^{(1)}$ give $d\beta^{(2)} = 0$, and restricts holonomy of $\beta^{(2)}$ to $\mathbb{Z}_k^{(1)}$ instead of general $U(1)^{(1)}$.

We modify the bulk action by adding the two new dynamical fields $\beta^{(2)}, \gamma^{(1)}$

$$S_{\text{bulk}} + \frac{i}{2\pi}\int\beta^{(2)}\wedge f + \frac{ik}{2\pi}\int\beta^{(2)}\wedge d\gamma^{(1)} + \frac{ik}{4\pi}\int\beta^{(2)}\wedge\beta^{(2)}$$

where the first term says $\beta^{(2)}$ is sourcing the magnetic one-form symmetry, the second is there to make $\beta^{(2)}$ discrete, and the last is a discrete torsion term (SPT phase). Integrate out γ, β, namely solve the equations of motion. This gives $\beta^{(2)} = -f/k$. We get

$$S_{\text{bulk}} - \frac{2\pi i}{k} \int \frac{f \wedge f}{8\pi^2},$$

so this $\mathbb{Z}_k^{(1)}$ gauging sent $\theta \to \theta - 2\pi/k$. Then use an anomalous chiral rotation to undo the shift in θ angle.

Surprising conclusion: the theory T with ABJ anomaly is invariant under gauging $\mathbb{Z}_k^{(1)}$ with appropriate SPT. It is a self-duality of the theory under this finite gauging.

In general, if a theory T is invariant under discrete gauging, then it admits a duality defect (or maybe it should be called Dirichlet defect). The idea is that you gauge the symmetry on one side of a wall. In general this defines an interface between theory T and the gauged theory $T/\mathbb{Z}_k^{(1)}$. But if that latter theory is the same as T then we have a topological defect in theory T. Explicit action in our case (with defect at $x = 0$),

$$S = S_{\text{bulk}} + \frac{i}{2\pi} \int_{x\geq 0} \Big(\beta \wedge f + k\beta \wedge d\gamma + \frac{\kappa}{2}\beta \wedge \beta\Big) + \left(\frac{2\pi i}{k} \int_{x=0} \star J + \frac{2\pi i}{k} \int_{x\geq 0} \frac{f \wedge f}{8\pi^2}\right).$$

This is almost the same action as what we did before: instead of gauging everywhere (which we realized did nothing), we gauge only on one half-space, and on top of that we add

$$\frac{2\pi i}{k} \int_{x=0} \star J + \frac{2\pi i}{k} \int_{x\geq 0} \frac{f \wedge f}{8\pi^2},$$

which is trivial by (1).

Note that β, γ are dynamical fields defined only for $x \geq 0$. At $x = 0$ we choose Dirichlet boundary conditions $\beta|_{x=0} = 0$. This defect is topological (check it thanks to $d\beta = 0$).

Claim 3. *The edge modes of β, γ system realize $U(1)_k$.*

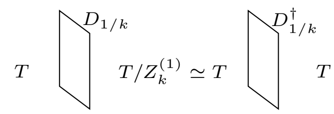
To see this, integrate out β. (We will not keep track of the coupling to f but it is easy to do.) The equation of motion is $d\gamma + \beta = 0$. This ends up giving us

$$-\frac{ik}{4\pi} \int_{x\geq 0} d\gamma \wedge d\gamma = \frac{ik}{4\pi} \int \gamma \wedge d\gamma$$

as announced in the claim. The coupling to f makes it so that we get a $\mathbb{Z}_k^{(1)}$ duality defect $U_{1/k}(M_3) Z^k_{M_3}[f] = D_{1/k}(M_3)$.

4.4 Defect fusion algebra

The price of the ABJ anomaly is that $D_{1/k}$ is non-invertible. Let us study the fusion of $D_{1/k}$ and $D^\dagger_{1/k}$ (obtained by reversing the orientation). The set-up is



This amounts to gauging $Z_k^{(1)}$ on $M_3 \times I$ with Dirichlet boundary conditions at the ends of the interval.

Now, $Z_k^{(!)}$ gauge equivalence classes are given by relative cohomology, relative to the boundary because we are working with Dirichlet boundary conditions. Thus,

$$\beta \in H^2(M_3 \times I, \partial(M_3 \times I), \mathbb{Z}_k).$$

We apply Lefshetz duality (Poincaré duality for relative cohomology):

$$H^2(M_3 \times I, \partial(M_3 \times I), \mathbb{Z}_k) \simeq H_2(M_3 \times I, \mathbb{Z}_k) \overset{\text{homotopy}}{\simeq} H_2(M_3, \mathbb{Z}_k)$$

Thus, non-trivial β are characterized by a surface $S \in H_2(M_3, \mathbb{Z}_k)$ where it has holonomy.

If we use the equation of motion which relates β with f, we can look at a Wilson surface of β, namely $\exp(-i \oint_S \beta) = \exp(\frac{2\pi i}{k} \oint_S \frac{f}{2\pi})$. These are nothing but the $\mathbb{Z}_k^{(1)}$ symmetry defects. The gauging amounts to a sum over Wilson surfaces, namely a sum over symmetry defects.

We want to compute the partition function of the $\mathbb{Z}_k^{(1)}$ gauge theory. It is simply a sum over the possible β:

$$D_{1/k} \times D^\dagger_{1/k} = \# \sum_{S \in H_2(M_3, \mathbb{Z}_k)} \exp\Big(\frac{i}{k} \oint_S f\Big)$$

where # is a numerical coefficient that is related to a standard normalization for $\mathbb{Z}_k^{(1)}$ gauge theory, coming from quotienting by gauge transformations (counted by H^1 below) but in a non-redundant way (counted by H^0):

$$\# = \frac{|H^0(M \times I, \partial(M \times I), \mathbb{Z}_k)|}{|H^1(M \times I, \partial(M \times I), \mathbb{Z}_k)|} = \frac{1}{k}.$$

Interpretation of fusion. The right-hand side here can be understood as condensation of $\mathbb{Z}_k^{(1)}$ symmetry defects; we've only applied the gauging on the given hypersurface, not the full theory. Local operators cannot be charged under the Wilson surfaces, so only $S = 0$ is relevant and we get

$$D_{1/k} \times D^\dagger_{1/k} = 1/k \text{ on local operators.}$$

In some sense, this is a codimension 2 defect that tries very hard to be codimension 1. It is a sort of mesh of the codimension 2 operators; it is "porous".

Note that if $D_{1/k}$ had an inverse A then $AD_{1/k}D^\dagger_{1/k}A^\dagger$ would be the identity defect. We will now prove that $D_{1/k}D^\dagger_{1/k}$ has a bunch of zero eigenvalues, which establishes that such an A does not exist.

Consider a state with a magnetic charge, e.g., on a spatial slice $S^2 \times S^1$. The Hilbert spaces decomposes as a direct sum of spaces $\mathcal{H}_m$ where flux is m,

$$\mathcal{H}_{S^2\times S_1} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m, \qquad \int_{S^2} \frac{f}{2\pm} = m.$$

On $\mathcal{H}_m$:

$$D_{1/k} \times D^\dagger_{1/k} \overset{\text{on } \mathcal{H}_m}{=} \frac{1}{k}\sum_{l=0}^{k-1} \exp\Big(\frac{2\pi i \ell m}{k}\Big) = \begin{cases} 1 & \text{if } k|m, \\ 0 & \text{else.} \end{cases}$$

Thus, we see that $D_{1/k}$ includes a sort of projection onto Hilbert space sectors where the flux (on all surfaces, actually) is a multiple of k.

Question: what is a **simple defect**? When we think of operators acting on a Hilbert space, we can always do arbitrary linear combinations. When we think of them as defects, which we can insert them at a place in space and spread out at all times, then it makes sense to talk about the Hilbert space with that defect insertion. But then in that context we can only do sums of defects (with non-negative integer coefficients), not arbitrary linear combinations. So there is a notion of simple defect, which are those that cannot be written as sums of other defects.

Action on operators and selection rules Local operators are blind to the TQFT $U(1)_k$, and are only sensitive to $U_{1/k}$, so

$$D_{1/k}(\mathcal{O}) = \exp\Big(\frac{2\pi i q_{\mathcal{O}}}{k}\Big)\mathcal{O}.$$

This works for any k, leading to the full chiral symmetry selection rules as long as topology is trivial. This is consistent with what we said at the beginning.

Next consider the 't Hooft line $M(\ell)$ on the line ℓ, which imposes a flux $m = \int_{S^2} f/(2\pi)$ around it. Recall the **Witten effect**, namely the effect of the θ angle on electric/magnetic charge. Near a 't Hooft line, with f_{mon} denoting the standard monopole field strength, we split $f = f_{\text{mon}} + dc_{\text{flu}}$ with dc_{flu} being a dynamical part. Then inside the action we have

$$\begin{aligned} S &\supset \frac{i\theta}{8\pi^2}\int f\wedge f = \frac{i\theta}{8\pi^2}\int (f_{\text{mon}} + dc_{\text{flu}}) \wedge (f_{\text{mon}} + dc_{\text{flu}}) \\ &\supset \frac{i\theta}{2\pi}\Big(\int_{S^2}\frac{f_{\text{mon}}}{2\pi}\Big)\Big(\int_{\ell, r\geq 0} dc_f\Big) = \frac{i\theta m}{2\pi}\int_\ell c_f. \end{aligned}$$

The θ term induces an electric charge $m\theta/(2\pi)$.

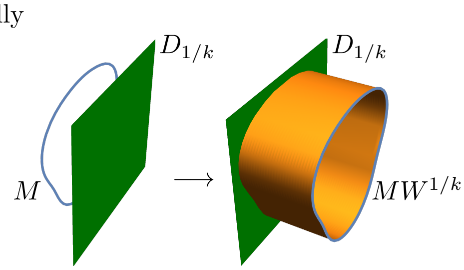
Now, across $D_{1/k}$ the θ term jumps as $\theta \to \theta + 2\pi/k$, so the 't Hooft loop M gets an electric charge $1/k$: schematically,

$$D_{1/k}(M(\ell)) = M(\ell)W(\ell)^{1/k},$$

where M denotes the minimal 't Hooft line and the right-hand side is an *improperly quantized* dyon. What it means is that it is not a strict line:

$$\text{heuristically } W^{1/k} = \exp\left(\frac{i}{k}\oint_\ell c\right),$$
$$\text{rigorously } W^{1/k} = \exp\left(\frac{i}{k}\int_\Sigma f\right),$$

where $\partial\Sigma = \ell$. It is an open surface operator, with a very mild dependence on the surface: only a topological dependence on the surface (whereas the boundary is not topological). Pictorially



This may seem strange at first, but it is in fact a general feature of non-invertible symmetry: it can change the type of operator. Here it maps lines to open topological surfaces.

Analogous to the order to disorder map of Kramers–Wannier duality.

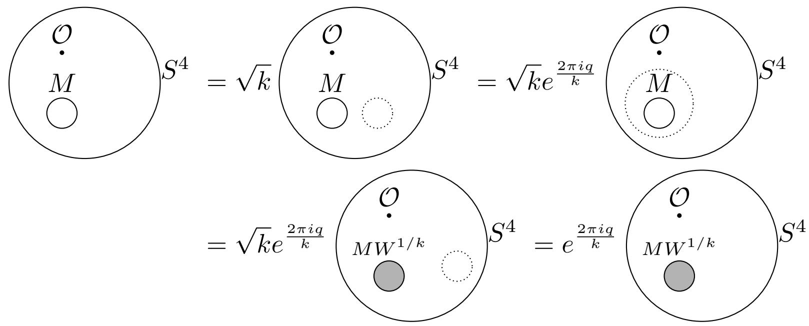
Analogous to an improperly-quantized Diract string.

Analogous to a branch cut in a (locally) holomorphic function: the position of the branch cut does not matter, but tracking where you are with respect to these branch cuts matters, and how they connect, etc.

Note: you can combine the first day of these lectures to the second day to get very intricate structures, with higher-groups and non-invertible symmetries at the same time.

4.5 Selection rules

Consider a correlator on S^4 of some operator $\mathcal{O}$ and of the M 't Hooft line.



This selection rule is a new result.

4.6 Non-invertible symmetry breaking

Let us think about symmetry breaking. We will use the fusion rule

$$D_{1/k} \times D^{\dagger}_{1/k} = \frac{1}{k} \sum_{S} \exp\Big(\frac{i}{k} \oint_S f \Big).$$

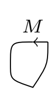
Somehow we have a non-trivial algebra between the chiral symmetry and the one-form symmetry.

As we did in an earlier lecture, we think of an RG flow from UV where $D_{1/k}$ is broken to IR where $D_{1/k}$ emerges.

Novel breaking possibility: dynamical monopoles break $U(1)^{(1)}_m$. This implies that $D_{1/k}$ has to break.

Let us estimate energy scales by coupling to monopole worldline. The monopole mass goes roughly as Λ/g where Λ is the Higgsing scale (if we embed the abelian theory inside a non-abelian one) and g the gauge coupling. The other thing to know is the IR cutoff.

The infrared effective action is the $U(1)$ gauge theory. What is the typical size of corrections to this effective action. The first correction is due to a loop of monopoles:[8]



$$\sim \exp(-S_{\text{worldline}}) \sim \exp(-m\delta t) \sim \exp(-\#/g^2)$$

which means we should expect the loop of monopole gives a non-perturbative violation of $D_{1/k}$.

[8]Subtlety: for this loop to be an operator rather than a number you need some more work, taking into account fermion zero modes etc.

Pedestrian viewpoint The monopole can arise in a non-abelian gauge theory. In the IR we have $d\star J = \frac{1}{8\pi^2} f \wedge f$. In the UV it can be $d\star J = \frac{1}{8\pi^2}\,\mathrm{Tr}(f \wedge f)$. In that non-abelian theory the chiral symmetry is violated directly by instantons on S^4. The monopole effect mentioned above is the right size to be an instanton effect.

Example with exponentially small mass. In the UV $SU(2)$ plus two doublets χ^i, while φ is in the adjoint. There is no one-form symmetry. Then φ gets a vev to Higgs $SU(2)$ to $U(1)$. In the IR we have $\chi^{\pm i}$ with non-invertible chiral symmetry.

The single-instanton background $\frac{1}{8\pi^2}\int \mathrm{Tr}(f \wedge f) = 1$ supports one fermion zero mode for each doublt. Thus, the instanton generates a multi-fermion operator that saturates zero modes. The UV Lagrangian includes

$$\mathcal{L}_{\mathrm{UV}} \supset \int d^4x \exp\Big(-\frac{8\pi^2}{g^2}\Big)\chi^1\chi^2$$
$$\mathcal{L}_{\mathrm{IR}} \supset \int d^4x \exp\Big(-\frac{8\pi^2}{g^2}\Big)\Big(\chi^{+,1}\chi^{-,2} - \chi^{-,1}\chi^{+,2}\Big).$$

This is a technically natural, exponential small, chiral symmetry violation.

This is a good starting point for model-building and for phenomenology. The exponential smallness of the symmetry violation is useful especially when you want to explain some small numbers in nature.