IHES Problems
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For details and solutions to the following problems see 2011.09600.

1) Consider an axion a, a periodic scalar field with identification $a \sim a + 2\pi f$ where f is a constant. We work in four spacetime dimensions. The action is assumed to be free:

$$S = \frac{1}{2} \int da \wedge *da \tag{1}$$

- Identify the charged defects under the $U(1)^{(0)}$ and $U(1)^{(2)}$ symmetries.
- Write the coupling to background fields $A^{(1)}$ and $A^{(3)}$.
- Show that these symmetries have a mixed anomaly.

2) Now couple the axion to a dynamical $U(1)$ gauge field:

$$S = \frac{1}{2} \int da \wedge *da + \frac{1}{2g^2} \int F \wedge *F - \frac{iK}{8\pi^2 f} \int aF \wedge F \ , \tag{2}$$

where g is the gauge coupling and K is a coupling constant.

- Show that K is quantized to be an integer.
- Construct the symmetry defect for the unbroken $\mathbb{Z}_K^{(0)}$.
- Construct the symmetry defect for the unbroken electric symmetry $\mathbb{Z}_K^{(1)}$.

3) Consider the Chern-Simons theory $U(1)_k$ with action:

$$S = \frac{ik}{4\pi} \int b \wedge db \ . \tag{3}$$

For simplicity take k to be even below.

- Show that the Wilson lines define a discrete one-form symmetry $\mathbb{Z}_k^{(1)}$
- Write the coupling of this one-form symmetry to an appropriate background gauge field.
- Deduce the anomaly of this one-form symmetry and write the inflow action.

4) Exercises on gauging and duality:

- Consider a d-dimensional QFT T which is invariant under gauging a $\mathbb{Z}_k^{(p)}$ symmetry. (Here d denotes the spacetime dimension). For which d and p is this possible?

- Working in d=2, with a $\mathbb{Z}_k^{(0)}$ symmetry. Write the action for T with $\mathbb{Z}_k^{(0)}$ gauged in a half-space $x \geq 0$. Assuming Dirichlet boundary conditions this defines a topological duality defect $\mathcal{D}$.

- Consider a situation where the theory is on an infinite cylinder $S^1 \times \mathbb{R}$ with defects $\mathcal{D}$ and $\mathcal{D}^\dagger$ wrapped around the circle. Following the logic of the lecture the fusion $\mathcal{D} \times \mathcal{D}^\dagger$ can be viewed geometrically as the partition function of $\mathbb{Z}_k^{(0)}$ gauge theory in a cylinder $S^1 \times I$ with Dirichlet boundary conditions at the ends of the interval. Evaluate the partition function to show that:

$$\mathcal{D} \times \mathcal{D}^\dagger = \sum_{i=0}^{k-1} L^i \ , \tag{4}$$

  where L is the invertible symmetry line generating the $\mathbb{Z}_k^{(0)}$ symmetry. (Wrapped around S^1.)