Exercise sheet 1: Monday June 24, 2024.
(See chapter 2 of Zohar Komargodski's notes `https://indico.ictp.it/event/7624/session/19/contribution/84/material/0/0.pdf`)

Quantum field theories here are assumed local, invariant under translations and rotations (but not necessarily reflections). They have a symmetric conserved stress-energy tensor $T_{\mu\nu}$: the operator equations $T_{\mu\nu} = T_{\nu\mu}$ and $\partial^\mu T_{\mu\nu} = 0$ are valid at separated points in correlators. We assume there is **no local gravitational anomaly:** the equations hold at coincident points too. In contrast we allow anomalies in current conservation $\langle \partial_\mu j^\mu \ldots\rangle =$ (contact terms) namely $\langle p_\mu j^\mu(p) \ldots\rangle =$ (polynomial) in momentum space.

Exercise 1. The stress-tensor two-point function is characterized by its (center of mass) momentum space expression $\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle$, which can only[1] depend on q_μ and the metric $\delta_{\mu\nu}$. (i) Using symmetry and conservation show that, in $n \geq 2$ spacetime dimensions, for a pair of scalar functions g, f,

$$\begin{aligned}\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle &= f(q^2)(q_\mu q_\nu - q^2\delta_{\mu\nu})(q_\rho q_\sigma - q^2\delta_{\rho\sigma}) \\ &\quad + g(q^2)\Big((q_\mu q_\rho - q^2\delta_{\mu\rho})(q_\nu q_\sigma - q^2\delta_{\nu\sigma})\Big)\Big|_{\text{symmetrize}(\rho,\sigma)}.\end{aligned}$$

(ii) Check that in 2d the two tensor structures coincide, so $\overset{\text{without loss of generality}}{\text{wlog}}$ $g(q^2) = 0$.

Exercise 2. Assume that the QFT is two-dimensional and scale-invariant.

(i) Show that $f(q^2) = c/q^2$ for some constant c. Check that $\langle T^\mu_\mu(q)T_{\rho\sigma}(-q)\rangle$ is polynomial in q hence T^μ_μ has a vanishing two-point function with $T_{\rho\sigma}$ at separated points.

(ii) Couple the QFT to a frozen metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ close to Euclidean. At first order this adds $\frac{1}{2}\int g_{\mu\nu}T^{\mu\nu}d^2x$ to the action. Deduce

$$\langle T^\mu_\mu(x)\rangle_{g=\delta+h} \sim c(\partial^\rho\partial^\sigma - \delta^{\rho\sigma}\Box)h_{\rho\sigma} + O(h^2).$$

This is c times the linearized Ricci scalar R of g; higher-order corrections in h come from higher-point functions of $T_{\rho\sigma}$. This is the famous 2d trace anomaly $\langle T^\mu_\mu\rangle = -\frac{c}{24\pi}R$.

(iii) In a metric $g_{\mu\nu} = e^\varphi\delta_{\mu\nu}$, check that $T'_{zz} = T_{zz} + \alpha c(-(\partial\varphi)^2 + 2\partial^2\varphi)$ is holomorphic for some value of α: use the conservation equation $\nabla^\mu T_{\mu\nu} = 0$ and $R = -4e^{-\varphi}\partial_z\partial_{\bar{z}}\varphi$.

Exercise 3. (i) Consider a chiral conserved current j_z in a 2d CFT. From $\langle j_z(z)j_z(w)\rangle = k/(z-w)^2$ (k is called the *level*) deduce $\langle j_z j_z\rangle = kq_z^2/q^2$ in momentum space.

[1] This is a slight lie: in 3d theories without reflection symmetry, one has an extra tensor structure obtained by symmetrizing $q^\lambda\varepsilon_{\lambda\mu\rho}(q_\sigma q_\nu - q^2\delta_{\sigma\nu})$ in $\mu \leftrightarrow \nu$ and also in $\rho \leftrightarrow \sigma$, where ε is the Levi–Civita tensor. It is correctly invariant under swapping $(\mu, \nu, q) \leftrightarrow (\rho, \sigma, -q)$.

(ii) Consider a $U(1)$ conserved current j_μ in a (translation & rotation invariant) 2d QFT. Show that symmetries fix (for some functions a_L, a, a_R)

$$\langle j_z j_z \rangle = q_z^2 a_L(q^2)/q^2, \qquad \langle j_z j_{\bar z} \rangle = -a(q^2), \qquad \langle j_{\bar z} j_{\bar z} \rangle = q_{\bar z}^2 a_R(q^2)/q^2.$$

Using the separated-point conservation equation $q_z j_{\bar z} + q_{\bar z} j_z =$ polynomial, and adjusting contact terms by shifting correlators by polynomial in $q_z, q_{\bar z}$, find that $a_L - a$ and $a_R - a$ are constant. If the UV and IR limits $q^2 \to +\infty, 0$ are CFTs deduce that levels of chiral currents obey $k_R^{\mathrm{UV}} - k_L^{\mathrm{UV}} = k_R^{\mathrm{IR}} - k_L^{\mathrm{IR}}$, a simple version of 't Hooft anomaly matching.

(iii) In a background gauge field A, show $\langle \partial_\mu j^\mu \rangle = (1/2)(a_L - a_R)\epsilon^{\mu\nu} F_{\mu\nu} + O(A^2)$ for a suitable choice of contact terms.

Exercise 4. Consider a Poincaré-invariant 2d QFT with a $U(1)$ conserved current j. Understand how charge conjugation and time-reversal acts on j and T. Show that $\langle j_\mu T_{\nu\rho} \rangle = 0$. (More generally, no mixed anomaly between these symmetries: gauging either one does not spoil the other.)

Exercise 5. (i) In 2d, take currents j_μ and j'_μ with $\langle j_\mu j'_\nu \rangle = q_\mu \varepsilon_{\nu\rho} q^\rho / q^2$. Show j'_μ is exactly conserved while conservation of j_ν has contact terms. By adding contact terms to $\langle j_\mu j'_\nu \rangle$ make $\partial^\mu j_\mu = 0$ exact and see that $\partial^\mu j'_\mu$ gets contact terms.

(ii) In $n = 2k$ dimensions, same questions with $k+1$ currents and $\left\langle j^{(0)}_{\mu_0}(q^{(0)}) j^{(1)}_{\mu_1}(q^{(1)}) \dots j^{(k)}_{\mu_k}(q^{(k)}) \right\rangle = \varepsilon_{\mu_1 \dots \mu_k \nu_1 \dots \nu_k} q^{(1)\nu_1} \dots q^{(k)\nu_k} q^{(0)}_{\mu_0} / q^{(0)2}$.

(iii) Turn on backgrounds $A^{(i)}$ for $j^{(i)}$, $i = 1, \dots, k$, and compute the effect of the previous line on $\langle \partial^\mu j^{(0)}_\mu \rangle$ in terms of field strengths of $A^{(i)}$.

Exercise 6. Switch to 4d. Left-handed fermions of the Standard Model transform in (three generations of) $(\mathbf{1},\mathbf{2})_{c_1} + (\mathbf{1},\mathbf{1})_{c_2} + (\mathbf{3},\mathbf{2})_{c_3} + (\overline{\mathbf{3}},\mathbf{1})_{c_4} + (\overline{\mathbf{3}},\mathbf{1})_{c_5}$ under the gauge symmetry $SU(3) \times SU(2) \times U(1)$, where the notation $(\mathbf{a},\mathbf{b})_c$ denotes the tensor product of a representation of $SU(3)$ of dimension a, of $SU(2)$ of dimension b, and of a charge c representation of $U(1)$. Denoting generators of the gauge group by t_α, the gauge anomaly for any triplet of generators $t_\alpha, t_\beta, t_\gamma$ can be calculated by a triangle Feynman diagram, and is proportional to

$$\sum_{\text{fermion representation } \mathcal{R}} \mathrm{Tr}_{\mathcal{R}}(t_\alpha t_\beta t_\gamma + t_\alpha t_\gamma t_\beta).$$

Check that the anomalies involving $SU(3)$ and $SU(2)$ generators vanish. Check that the remaining gauge-anomaly cancellations (together with the gauge-gravitational anomaly $2c_1 + c_2 + 6c_3 + 3c_4 + 3c_5 = 0$) only allow for two possible hypercharge assignments up to scaling. One of them is the Standard Model answer $c_1 = 1/2$, $c_2 = -1$, $c_3 = -1/6$, $c_4 = 2/3$, $c_5 = -1/3$.

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Exercise 7 (Abelian Duality in Diverse Dimensions). Much intuition about phases and transitions can be gleaned from mean-field theory. Let us consider the mean-field (i.e. semiclassical) dynamics of a real scalar field ϕ with a $\mathbb{Z}_2$ Ising symmetry. Since we are working at leading order in the semiclassical expansion, we just minimize the potential and ignore loop corrections. Thus the discussion applies in any spacetime dimension D. (Whether or not this is a good description depends on D.)

- Analyze the vacuum structure as a function of the mass $m^2 \in \mathbb{R}$ given a quartic potential of the form

$$V(\phi) = m^2\phi^2 + \lambda_4\phi^4, \qquad \lambda_4 > 0.$$

  In particular discuss the order of the transition at $m^2 = 0$. (In applications to the classical, finite-temperature Ising model $m^2 \sim T - T_c$, but the discussion also applies to quantum phase transitions at zero temperature in the Ising universality class, in which case m^2 is some coupling in the Hamiltonian.)

- Show that the transition can be made 1st order by breaking the $\mathbb{Z}_2$ symmetry via a linear perturbation $\Delta V = h\phi$. In the Ising model $h \in \mathbb{R}$ is an external magnetic field. Sketch the phase diagram as a function of m^2, h. Argue that generically the only way for a line of 1st order phase transition to genuinely end (rather than turn into some other lines(s) of transitions) is in a 2nd order point.

- Consider the Ising model with $\mathbb{Z}_2$ symmetry and a sextic potential, $V(\phi) = m^2\phi^2 + \lambda_4\phi^4 + \lambda_6\phi^6$. Imagine that $\lambda_6 > 0$, so that the potential is stable, but that $m^2, \lambda_4 \in \mathbb{R}$ can have either sign. Analyze the phase diagram and show that the sign of λ_4 controls the order of the phase transition as we dial m^2. The point $m^2 = \lambda_4 = 0$ at which the order of the phase transition changes is called a multi-critical point. Here it is also called a tri-critical point since we are dialing two parameters (rather than the single parameter to reach a generic critical point). Is the tri-critical point described by the same physics as the line of second-order Ising transitions at $\lambda_4 > 0$?

- The previous point shows that a first order line can change into a second order line at a multi-critical point. Are there other possible behaviors for a 1st order line other than this and ending in a second order point? Hint: think of the phase diagram of water.

Exercise 8 (Abelian Duality in Diverse Dimensions).

Generalize the derivation of electric-magnetic duality in $D = 4$ reviewed in lecture to the following settings. For low D these occur in many QFT applications. The case $D > 4$ is interesting in the context of string theory, supergravity, holography etc.

- Start with ordinary $U(1)$ Maxwell theory with field strength $f^{(2)} = da^{(1)}$ as above, but now work in D spacetime dimensions (with $D \geq 3$). Carry out the duality explicitly and show that the dual gauge field $\widetilde{a}^{(D-3)}$ that must be introduced as a Lagrange multiplier is a $(D-3)$-form gauge field with $(D-2)$-form field strength $\widetilde{f}^{(D-2)} = d\widetilde{a}^{(D-3)}$. Spell out explicitly the gauge transformations and flux quantization rule for $\widetilde{a}^{(D-3)}$. Hint: if the general case is confusing, first do $D = 3$.

- Given a compact boson $\chi \sim \chi + 2\pi$ in any dimension D, show how to dualize it into a $D-1$-form gauge field. Hint: this case overlaps with the $D = 3$ limit of the pervious point. In $D \geq 3$ such a a compact boson is necessarily a Nambu-Goldstone boson for its broken shift symmetry, while in $D = 2$ the compact boson is not a Goldstone boson (in agreement with the Coleman-Mermin-Wagner theorem on the absence of spontaneously continuous symmetry breaking in 2d).

- Both Maxwell theory and a compact boson are examples of a p-form gauge field. In general, a p-form gauge field $a^{(p)}$ has field strength $f^{(p+1)} = da^{(p)}$ and gauge transformations $a^{(p)} \to a^{(p)} + d\lambda^{(p-1)}$. Here $\lambda^{(p-1)}$ is itself a $(p-1)$-form gauge field (defined recursively in p). Thus $\lambda^{(p-1)}$ has integer fluxes on $(p-1)$-cycles,
$$\frac{1}{2\pi}\int_{\Sigma_{p-1}} \lambda^{(p-1)} \in \mathbb{Z} \ ,$$
and applying the Dirac argument in this case we learn that $f^{(p+1)}$ has integer fluxes on $(p+1)$-cycles,
$$\frac{1}{2\pi}\int_{\Sigma_{p+1}} f^{(p+1)} \in \mathbb{Z} \ .$$
We take the action to be of generalized Maxwell type:
$$S = \frac{1}{2e^2}\int_{\mathcal{M}_D} f^{(p+1)} \wedge * f^{(p+1)} \ .$$
What is the mass dimension of the coupling e^2? Show that $a^{(p)}$ can be dualized into a $(D-p-2)$-form gauge field $\widetilde{a}^{(D-p-2)}$ with dual field strength $\widetilde{f}^{(D-p-1)} = d\widetilde{a}^{(D-p-2)}$.