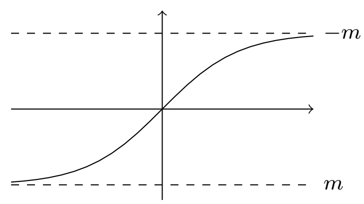
Exercise sheet Metlitski Tuesday June 25, 2024.

Exercise 1 (Edge Majorana zero-modes from continuum). Consider $H = \int dx \big(\chi_R(-i\partial_x)\chi_R + \chi_L(i\partial_x)\chi_L + im\chi_R\chi_L\big)$ on the real line, with $m(x)$ having the asymptotic behaviour $m(x) \to \pm m$ for $x \to \pm\infty$:



Show that there is a Majorana zero mode localized near $x = 0$.

Exercise 2 (Kitaev + Kitaev = 0). Fill in the steps for the proof that two copies of the Kitaev chain are continuously connected to a trivial phase.

Exercise 3 (Kitaev enriched by time-reversal). Consider a Kitaev chain enriched by an anti-unitary (so $i \to -i$) time-reversal symmetry $T\colon c_j \to c_j^\dagger$, $i \to -i$, $t \to -t$. Here we work in real time. If $c_j = \gamma_j + i\overline{\gamma}_j$ then

$$T\colon \gamma_j \to \gamma_j, \qquad \overline{\gamma}_j \to -\overline{\gamma}_j, \qquad i \to -i.$$

The Kitaev Hamiltonian $H = i\sum_j \overline{\gamma}_j\gamma_{j+1}$ is invariant under T. Suppose you have k copies of the Kitaev chain with T symmetry acting in the same way on all copies. On an open chain you will have k Majorana modes at each edge. Say, at the left edge, we have γ_L^α, $\alpha = 1, \dots, k$ and $T\colon \gamma_L^\alpha \to \gamma_L^\alpha$, $i \to -i$. Below we drop the subscript L.

(a) Observe that any Hermitian quadratic edge perturbation $\Delta H = i\sum_{\alpha,\beta} t^{\alpha\beta}\gamma^\alpha\gamma^\beta$ for $t^{\alpha\beta} \in \mathbb{R}$ and $t^{\alpha\beta} = -t^{\beta\alpha}$ is prohibited by T. So at free fermion theory with T symmetry the Kitaev chain generates a $\mathbb{Z}$ classification.

(b) For $k = 2$, the only perturbation on the edge we can write down is quadratic $\Delta H = i\gamma^1\gamma^2$, which is prohibited by T. Further, show that T switches the fermion parity of the left edge here, i.e., effectively

$$\{T_L, (-1)^{F_L}\} = 0. \tag{1}$$

where T_L and $(-1)^{F_L}$ are effective actions of the symmetry on the left edge. The algebra (1) means that the symmetry is realized projectively on the edge, and there must be two ground states related by T with opposite fermion parity.

(c) Consider $k = 4$. Show that effectively time-reversed acts on the left edge via T_L with $(T_L)^2 = -1$. Show that this implies that the edge is doubly-degenerate.

(d) Show that for $k = 8$ the edge can be gapped out with interactions. This implies that interactions reduce the $\mathbb{Z}$ classification to $\mathbb{Z}_8$.