Diophantine Approximation, Fractal Geometry and Related topics / Approximation diophantienne, géométrie fractale et sujets connexes

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Parametric geometry of numbers and simultaneous approximation to geometric progressions

An important problem in Diophantine approximation is to determine, for a given positive integer n, the supremum $\hat{\lambda}_n$ of the exponents $\hat{\lambda}_n(\xi)$ of uniform simultaneous rational approximation to geometric progressions $(1, \xi, \xi^2, \ldots, \xi^n)$ whose ratio ξ is either a transcendental real number or an algebraic real number of degree $> n$. In 1969, Davenport and Schmidt provided an upper bound on $\hat{\lambda}_n$ and, via geometry of numbers, they deduced a corresponding lower bound on the exponent of best approximation to such ξ by algebraic integers of degree at most $n + 1$. The same general transference principle applies to other classes of numbers, like approximation to ξ by algebraic units of degree at most $n + 2$, as Teulié showed in 2001. Recall that Dirichlet's theorem on simultaneous rational approximation yields $\hat{\lambda}_n \geq 1/n$. However, we still don't know, for any $n \geq 3$, if $\hat{\lambda}_n$ is equal to $1/n$ or strictly greater.
Inthistalk,weconcentrateonthecases$n=2$and$n=3$. For$n=2$,Ishowedin 2003 that the upper bound of Davenport and Schmidt for $\hat{\lambda}_2$ is best possible, namely that $\hat{\lambda}_2 = 1/\gamma \cong 0.618$, where γ stands for the golden ratio. Then, for many years, I thought that $\hat{\lambda}_3$ could be equal to the positive root $\lambda_3 \cong 0.4245$ of the polynomial $T^2 - \gamma^3 T + \gamma$, until I realized that it is strictly smaller. As the argument lead only to a very small improvement on the upper bound, I simply published, in 2008, the proof that $\hat{\lambda}_3 \leq \lambda_3$.
In the presentation, we take the point of view of parametric geometry of numbers. We first recall the basic facts that we need about n-systems and dual n-systems. For $n = 2$, we explain why a point $(1,\xi,\xi^2)$ with optimal exponent $\hat{\lambda}_2(\xi) = 1/\gamma$ admits a very simple self-similar dual 3-system, we give generic algebraic relations between the points of Z^3 that realize this map up to a bounded difference, and we show how these in turn determine the point $(1, \xi, \xi^2)$. One can hope that a similar phenomenon holds for each $n \geq 2$. For $n = 3$, assuming that $\hat{\lambda}_3(\xi) = \lambda_3$, we find an interesting self-similar dual 4-system attached to the point $(1,\xi,\xi^2,\xi^3)$ and algebraic relations with similar properties between the points that realize it up to bounded difference. However, they eventually lead to a contradiction. . .
In general, the theory attaches a dual n-system $P = (P_1,\ldots,P_n)\colon [0,\infty) \longrightarrow R^n$ to any non-zero point u of R^n, and P is unique up to bounded difference. This encodes most of the Diophantine approximation properties of u. For a geometric progression $u = (1,\xi,\xi^2,\xi^3)$ in R^4 with $\hat{\lambda}_3(\xi) > \sqrt{2} - 1 \cong 0.4142$, we can show that the behavior of P is qualitatively much simpler than that of a general dual 4-system. Moreover, the differences $P_3(q) - P_1(q)$ and $P_4(q) - P_2(q)$ both tend to infinity with q. Based on this, we deduce the existence of a sequence of integral bases of R^4 which, in a simple way, realize P up to a bounded difference. We propose this as a tool to improve the present upper bound λ_3 on $\hat{\lambda}_3(\xi)$. By contrast, the current way of studying $\hat{\lambda}_n(\xi)$ for a general n is to form a sequence of so-called minimal points for $u = (1,\xi,\ldots,\xi^n)$, which can be loosely described as a sequence of points of Z^{n+1} that realize the first component P_1 of P up to bounded difference.