NLO Quarkonium Production in the FKS Subtraction Scheme

Groupement de Recherche QCD 2024

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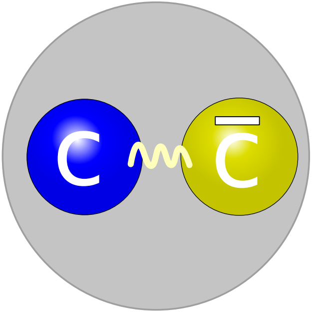
based on arXiv:2402.19221
in collaboration with Ajjath A. H. and H.-S. Shao

May 29, 2024

Quarkonium Production



Quarkonium production in NRQCD factorisation

$$\mathrm{d}\sigma(AB \to H + X) = \sum_n \Big(\sum_{a,b,X} \int \mathrm{d}x_a \mathrm{d}x_b \ f_{a/A}(x_a) f_{b/B}(x_b) \ \ \mathrm{d}\hat{\sigma}(ab \to Q\bar{Q}'[n] + X) \Big) \ \langle \mathcal{O}_n^H \rangle$$

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partonic
cross section

Partonic cross section

- short distance production of a $Q\bar{Q}'$ pair in a colour representation C, with spin S and orbital angular momentum state L
- spectroscopic notation of Fock states: $n = {}^{2S+1}L_J^{[C]}$

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PDF

partonic cross section

Parton Distribution Function (PDF)

- parton distribution functions of partons a and b in the initial hadrons A and B

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PDF — partonic cross section — LDME

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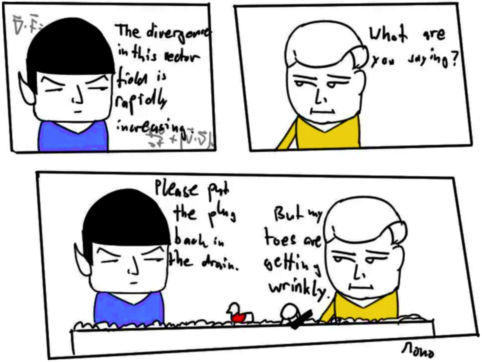
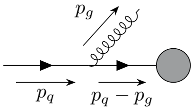
Long Distance Matrix Element (LDME)

- hadronisation of the heavy quark pair into the physical quarkonium state H

Infrared Singularities



Infrared singularities

$$|\mathcal{M}_J|^2 = \Bigg|$$


$$\Bigg|^2 \sim \frac{1}{(p_q - p_g)^2} = \frac{1}{E_q E_g (1 - \beta_q \cos\theta_{qg})}$$

with

$$\beta_q = \frac{|\vec{p}_q|}{E_q} \xrightarrow{m_q \to 0} 1$$

$$\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(4)} \sim \int_0^{E^{\max}} \mathrm{d}E_g \int_{-1}^{1} \mathrm{dcos}\,\theta_{qg} \frac{1}{E_q E_g (1 - \beta_q \cos\theta_{qg})} \to \infty \begin{cases} \text{for } E_g \to 0 \\ \text{for } \theta_{qg} \to 0 \\ \quad \wedge \beta_q \to 1 \end{cases}$$

Infrared singularities:

- soft singularity ($E_g \to 0$)
- collinear singularity ($\theta_{qg} \to 0 \wedge \beta_q \to 1$)
- soft-collinear singularity ($E_g \to 0$ and $\theta_{qg} \to 0 \wedge \beta_q \to 1$)

Subtraction schemes

$$\lim_{d\to 4}\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} \to \infty$$

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Subtraction function

find a suitable subtraction function S which

- reproduces the matrix element in the unresolved limit
- is simple to integrate over the unresolved phase space

$$\lim_{\text{IR poles}} S = \lim_{\text{IR poles}} |\mathcal{M}_J|^2 F_J$$

Subtraction schemes

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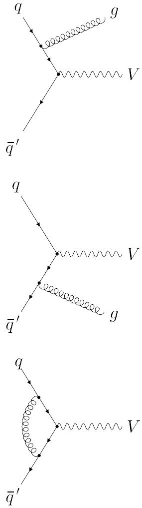
Infrared-safe cross section at NLO

The NLO cross section can be split in three parts

$$\mathrm{d}\sigma^{\mathrm{NLO}} = \underbrace{\mathrm{d}\sigma^{\mathrm{R}}}_{\text{contains IR poles from phase-space integration}} + \overbrace{\mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{PDF}}}^{\text{contain explicit poles}}$$



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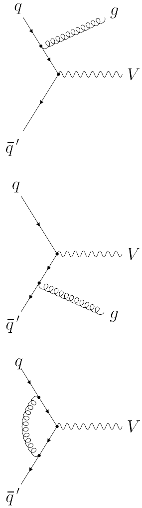
Subtraction method makes **IR poles explicit**:

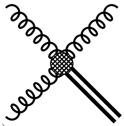
- poles cancel among the four parts
- analytic formula for finite remainder
- phase-space integration can be performed numerically



FKS subtraction scheme

Frixione-Kunszt-Signer

$$\mathrm{d}\sigma(r) = \mathrm{d}\sigma(r)$$

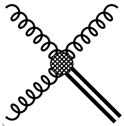


FKS subtraction scheme

Frixione-Kunszt-Signer

$$\mathrm{d}\sigma(r) = (1 - \mathcal{S}_4)\,\mathrm{d}\sigma(r)$$

$$+\,\mathcal{S}_4\,\mathrm{d}\sigma(r)$$

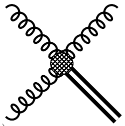


1. regulate soft singularities: $\mathcal{S}_4 = \lim_{E_4 \to 0}$
   - soft gluon can be emitted from both initial state gluons **and** from the $Q\bar{Q}'$ pair

FKS subtraction scheme

Frixione-Kunszt-Signer

$$
\begin{aligned}
\mathrm{d}\sigma(r) &= (1-\mathcal{S}_4)\,(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))\;\mathrm{d}\sigma(r)\\
&+(1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})\,\mathrm{d}\sigma(r)\\
&+\mathcal{S}_4\,\mathrm{d}\sigma(r)
\end{aligned}
$$



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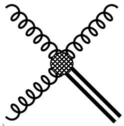
2. regulate collinear singularities: $\mathcal{C}_{4i} = \lim_{\theta_{i4}\to 0}$
   - collinear gluon can be emitted from both initial state gluons, but **not** from the $Q\bar{Q}'$ pair

FKS subtraction scheme

Frixione-Kunszt-Signer

$$\mathrm{d}\sigma(r) = (1-\mathcal{S}_4)\,(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))\;\mathrm{d}\sigma(r) + (1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})\,\mathrm{d}\sigma(r) + \mathcal{S}_4\,\mathrm{d}\sigma(r)$$



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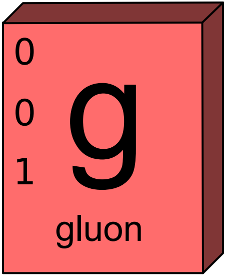
FKS for quarkonium production

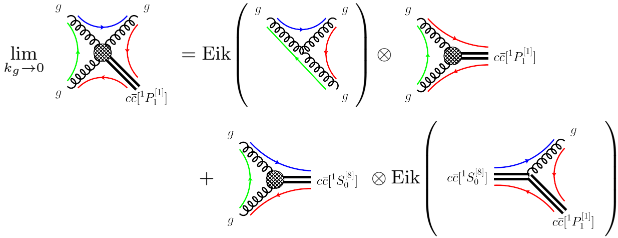
- **new soft subtraction terms** w.r.t. production of elementary particles
- **collinear subtraction terms** are **unchanged**

Soft Gluon Emission



Color-singlet spin-singlet P-wave

Soft limit of the amplitude



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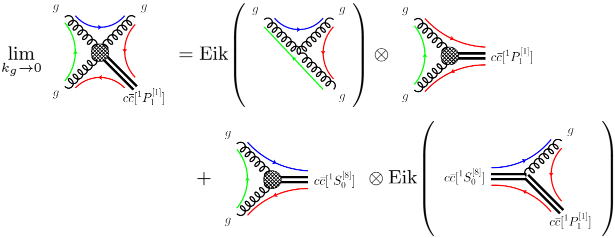

$$\lim_{k_i \to 0} \mathcal{A}^{(n+1)}_{\{[1],0,1,1\}}(r) = \sum_{j=n_I}^{n_L^{(R)}+n_H} g_s \frac{k_j \cdot \varepsilon^*_{\lambda_i}(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}_j) \mathcal{A}^{(n)}_{\{[1],0,1,1\}}(r^{\not{i}})$$

$$+ g_s \left[\frac{\varepsilon^*_{\lambda_l}(K) \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i) k_i \cdot \varepsilon^*_{\lambda_l}(K)}{(K \cdot k_i)^2} \right] \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{A}^{(n)}_{\{[8],0,0,0\}}(r^{\not{i}})$$

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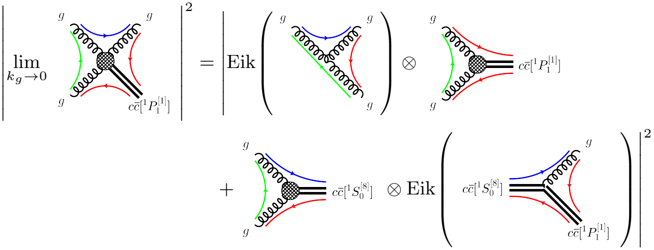
Soft limit of the squared amplitude



$$\lim_{k_i \to 0} \mathcal{M}_{\{[1],0,1,1\}}(r) = g_s^2 \left\{ \sum_{\substack{k,l=n_I \\ k,l\neq i, k\leq l}}^{n_L^{(R)}+n_H} \frac{k_k \cdot k_l}{k_k \cdot k_i k_l \cdot k_i} \mathcal{M}_{kl}(r^{\not{i}}) \right.$$

$$+ \sum_{\substack{k=n_I \\ k\neq i}}^{n_L^{(R)}+n_H} \left[\frac{k_{k,\mu}}{k_k \cdot k_i K \cdot k_i} - \frac{K \cdot k_k k_{i,\mu}}{k_k \cdot k_i \, (K \cdot k_i)^2} \right] \mathcal{M}^{\mu}_{k[18]}(r^{\not{i}}, r_1^{\not{i}})$$

$$\left. - \frac{2\epsilon - 2}{(K \cdot k_i)^2} C_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{M}(r_1^{\not{i}}) \right\}$$

Color-singlet spin-singlet P-wave

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$$- \frac{2\epsilon - 2}{\left(K \cdot k_i\right)^2} C_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{M}(r_1^{\not{i}}) \Bigg\}$$

Subtracted squared matrix element

$$\mathcal{M}_{\{[1],0,1,1\}}(r) - \sum_{i\in g} \lim_{k_i \to 0} \mathcal{M}_{\{[1],0,1,1\}}(r)$$

- free of soft singularities
- (numerical) integration in $d = 4$ dimensions possible

Color-singlet spin-singlet P-wave

Integrated soft counterterms

Reminder: subtraction schemes

$$\lim_{d\to 4} \int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} = \underbrace{\int \left(|\mathcal{M}_J|^2 F_J - S\right) \mathrm{d}\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d\to 4} \int S \, \mathrm{d}\Pi^{(d)}}_{\text{analytic solution}}$$

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Analytic solution:

$$d\hat{\sigma}^{(S)}(r) = \frac{\alpha_s}{2\pi}\phi_{n-1}(r^{\not{k}})\frac{J^{n_L^{(B)}}}{\mathcal{N}(r^{\not{k}})}\mathcal{G}(r^{\not{k}})\Bigg[\sum_{k=n_I}^{n_L^{(B)}+n_H}\sum_{l=k}^{n_L^{(B)}+n_H}\bar{\mathcal{E}}(\{1,1\},\{k_k,k_l\})\mathcal{M}_{kl}(r^{\not{k}})$$
$$+\sum_{k=n_I}^{n_L^{(B)}+n_H}\left(\bar{\mathcal{E}}(\{1,1\},\{k_k,K\})\frac{k_{k,\mu}}{K\cdot k_k} - \bar{\mathcal{E}}_\mu(\{1,2\},\{k_k,K\})\right)\mathcal{M}^\mu_{k[18]}(r^{\not{k}},r_1^{\not{k}})$$
$$+\left(\frac{1}{2N_c}\frac{8}{m_Q m_{\bar{Q}'}}\left(\frac{1}{\bar{\epsilon}}+\log\frac{\mu^2}{\mu^2_{\mathrm{NRQCD}}}\right) - \frac{2\epsilon-2}{K^2}\bar{\mathcal{E}}(\{1,1\},\{K,K\})C_{\mathrm{eff}}(Q\bar{Q}'_{[18]})\right)\mathcal{M}(r_1^{\not{k}})\Bigg]$$

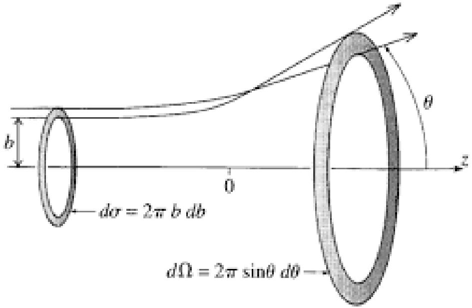
where the eikonal integrals

$$\bar{\mathcal{E}}^{\alpha_1\ldots\alpha_{n_1-1}\beta_1\ldots\beta_{n_2-1}}(\{n_1,n_2\},\{k_k,k_l\}) = 8\pi^2\mu^{2\epsilon}k_k\cdot k_l\int\frac{d^{3-2\epsilon}\mathbf{k}_i}{(2\pi)^{3-2\epsilon}2k_i^0}\frac{k_i^{\alpha_1}\ldots k_i^{\alpha_{n_1-1}}k_i^{\beta_1}\ldots k_i^{\beta_{n_2-1}}}{(k_k\cdot k_i)^{n_1}(k_l\cdot k_i)^{n_2}}\theta(\xi_{\mathrm{cut}}-\xi_i)$$

can be solved analytically.

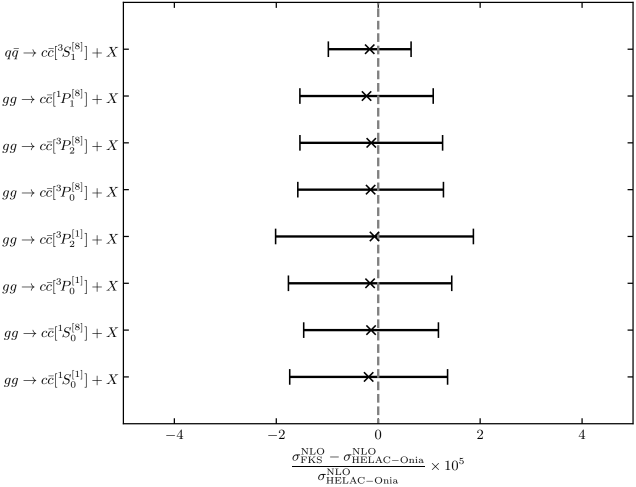
- singularities appear explicitly as poles in ϵ

Cross Sections



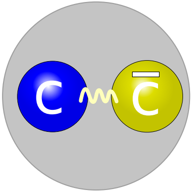
Inclusive NLO cross sections

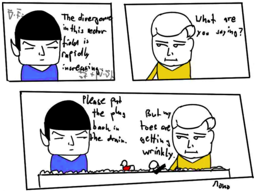
Validation of $2 \to 1$ processes

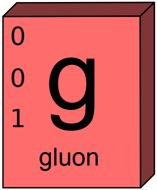


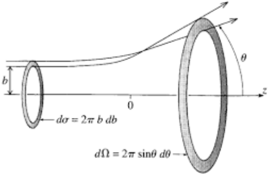
- inclusive cross sections of FKS approach (**fully numerical**) can be compared to `HELAC-Onia` [Shao, 2016] (**analytic phase-space integration**)

Summary


c
c̄


The divergence in this vector field is rapidly increasing.
What are you saying?
Please put the plug back in the drain.
But my toes are getting wrinkly.

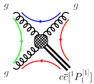

0
0
1
g
gluon


b
θ
z
0
$d\sigma = 2\pi\, b\, db$
$d\Omega = 2\pi \sin\theta\, d\theta$

Summary

- fully differential NLO results require knowledge about soft gluon emission from quarkonia
  - S-wave quarkonia behave like “massive gluons”
  - P-wave quarkonia lead to more complex structures

$$\lim_{k_g \to 0}$$



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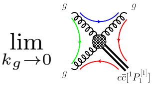
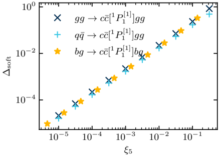
- local FKS subtraction term and the corresponding integrated counterterm were computed
  - eikonal limits are check numerically
  - ϵ-poles of integrated counterterms cancel analytically against virtual corrections





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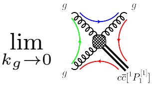
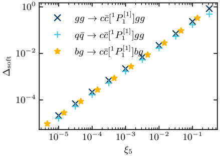
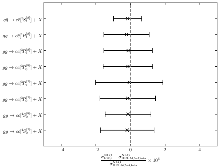
  - eikonal limits are check numerically
  - ϵ-poles of integrated counterterms cancel analytically against virtual corrections
- inclusive cross sections of FKS approach coincide with known results







MadGraph5

Quarkonia in MadGraph5

We aim to launch a version of MadGraph5_aMC@NLO[Alwall *et al.* 2014] that allows for automated computations of cross section which include bound states:

1. LO cross sections for processes with an arbitrary number of quarkonia (S-wave and P-wave) and additional elementary particles
   Example:
   $$pp \to J/\psi + \eta_c + c\bar{c}g$$
   Syntax:
   ```
   MG_aMC>generate p p > J/psi etac c c~ g
   ```
   or in an alternarive notation
   ```
   MG_aMC>generate p p > c.c~(1|3S11) c.c~(1|1S01) c c~ g
   ```
2. NLO cross sections with one quarkonium plus elementary particles
3. extension of FKS approach to multiple quarkonia final states

Backup

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\boxed{\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)}$$

Amplitudes with quarkonia

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↓ colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

Colour projector

$$\mathbb{P}_{C=1} = \frac{\delta^{c_3 c_4}}{\sqrt{N_c}}$$

$$\mathbb{P}_{C=8} = \sqrt{2}\, T^{c_{34}}_{c_4 c_3}$$

BACKUP

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$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

↓ spin projection

$$\mathcal{A}_{\{[C],S\}}(r) = \sum_{\lambda_Q,\lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

Spin projector

$$\mathbb{P}_{S=0} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\gamma_5 u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

$$\mathbb{P}_{S=1} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\not{\epsilon}^*_{\lambda_s}(K) u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

with

$$K^\mu = k_3^\mu + k_4^\mu$$

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

↓ colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

↓ spin projection

$$\mathcal{A}_{\{[C],S\}}(r) = \sum_{\lambda_Q,\lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

↓ orbital angular momentum projection

$$\mathcal{A}_{\{[C],S,L\}}(r) = \left[\left(\varepsilon^{\mu,*}_{\lambda_l}(K)\frac{d}{dq^\mu}\right)^L \mathcal{A}_{\{[C],S\}}(r)\right]_{q=0}$$

Orbital angular momentum

$$k_3^\mu = \frac{m_Q}{m_Q + m_{\bar{Q}'}}K^\mu + q^\mu$$

$$k_4^\mu = \frac{m_{\bar{Q}'}}{m_Q + m_{\bar{Q}'}}K^\mu - q^\mu$$

BACKUP

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \ldots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

↓ colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

↓ spin projection

$$\mathcal{A}_{\{[C],S\}}(r) = \sum_{\lambda_Q,\lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

↓ orbital angular momentum projection

$$\mathcal{A}_{\{[C],S,L\}}(r) = \left[\left(\varepsilon^{\mu,*}_{\lambda_l}(K)\frac{d}{dq^\mu}\right)^L \mathcal{A}_{\{[C],S\}}(r)\right]_{q=0}$$

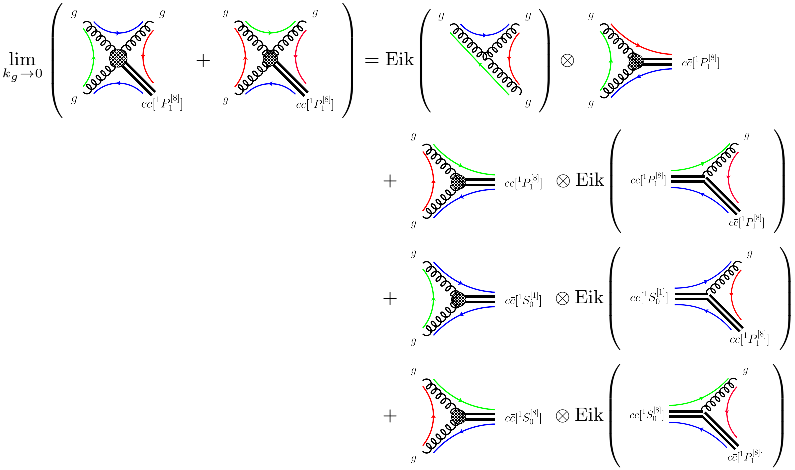
↓ total angular momentum projection

$$\mathcal{A}_{\{[C],S,L,J\}}(r) = \sum_{\lambda_s,\lambda_l} \underbrace{\langle J,\lambda_j|L,\lambda_l;S,\lambda_s\rangle}_{\text{Clebsch–Gordan coefficient}} \mathcal{A}_{\{[C],S,L\}}(r)$$

BACKUP

Color-octet spin-singlet P-wave

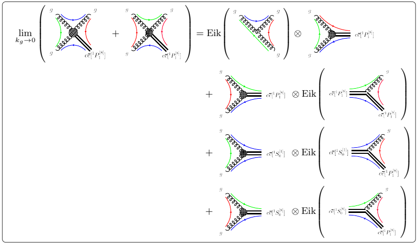
Soft limit of the amplitude



BACKUP

Color-octet spin-singlet P-wave

Soft limit of the amplitude



$$\lim_{k_i \to 0} \mathcal{A}_{\{[8],0,1,1\}}(r) = \sum_{\substack{j=n_I \\ j \neq i}}^{n_L^{(R)}+n_H} g_s \frac{k_j \cdot \varepsilon^*_{\lambda_i}(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}) \mathcal{A}_{\{[8],0,1,1\}}(r^{\not{i}})$$

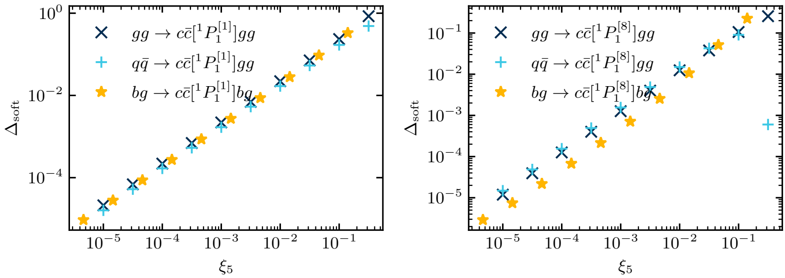
$$+ g_s \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} \vec{Q}(Q\bar{Q}'[{}^1P_1^{[8]}]) \mathcal{A}_{\{[8],0,1,1\}}(r^{\not{i}})$$

$$+ g_s \left[\frac{\varepsilon^*_{\lambda_l}(K) \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i) k_i \cdot \varepsilon^*_{\lambda_l}(K)}{(K \cdot k_i)^2} \right]$$

$$\times \left[\vec{Q}_{\text{eff}}(Q\bar{Q}'_{[81]}) \mathcal{A}_{\{[1],0,0,0\}}(r^{\not{i}}) + \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[88]}) \mathcal{A}_{\{[8],0,0,0\}}(r^{\not{i}}) \right]$$

BACKUP

Local soft subtraction terms



$$\Delta_{\rm soft} = \left| \frac{\mathcal{M}(r) - \lim_{k_5 \to 0} \mathcal{M}(r)}{\mathcal{M}(r)} \right| \propto \xi_5 + \mathcal{O}(\xi_5^2)\,, \quad \text{with} \quad E_5 = \xi_5 \frac{\sqrt{s}}{2}$$

Reminder: subtraction schemes

$$\lim_{d\to 4} \int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} = \underbrace{\int \left(|\mathcal{M}_J|^2 F_J - S\right) \mathrm{d}\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d\to 4} \int S\, \mathrm{d}\Pi^{(d)}}_{\text{analytic solution}}$$

BACKUP

Subtraction schemes

Concept

$$I = \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}} F(x)$$
$$= \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}} [F(x) - F(0)] + F(0) \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}}$$

Subtraction function $F(0)$

- $\frac{1}{x^{1+\epsilon}} [F(x) - F(0)]$ is free of divergences (numerical integration)
- $F(0) \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}}$ divergent but simple to integrate analytically

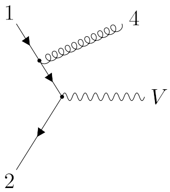
$$I = \underbrace{\int_0^1 \mathrm{d}x \frac{1}{x} [F(x) - F(0)]}_{\text{free of divergences}} - \underbrace{\frac{1}{\epsilon} F(0)}_{\text{analytic solution}} + \mathcal{O}(\epsilon)$$

BACKUP

FKS subtraction

Local soft subtraction function

$$\langle F_{\rm LM}(1,2\,|\,4)\rangle = \langle (1-\mathcal{S}_4) F_{\rm LM}(1,2\,|\,4)\rangle + \langle \mathcal{S}_4 F_{\rm LM}(1,2\,|\,4)\rangle$$



Step 1: Find suitable subtraction functions.

- $\mathcal{S}_4$ extracts the leading soft singularity
- Eikonal approximation

$$\mathcal{S}_4 F_{\rm LM}(1,2\,|\,4) = 2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)} F_{\rm LM}(1,2)$$

Free of soft divergences

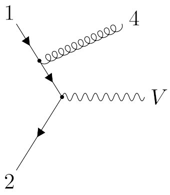
$$F_{\rm LM}(1,2\,|\,4) - 2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)} F_{\rm LM}(1,2)$$

BACKUP

FKS subtraction

Integrated soft subtraction function

$$\langle F_{\rm LM}(1,2\,|\,4)\rangle = \langle (1-\mathcal{S}_4) F_{\rm LM}(1,2\,|\,4)\rangle + \langle \mathcal{S}_4 F_{\rm LM}(1,2\,|\,4)\rangle$$



Step 2: Analytic solution of the subtraction functions.

$$\langle \mathcal{S}_4 F_{\rm LM}(1,2\,|\,4)\rangle = \int [\mathrm{d}g_4]\mathcal{S}_4 F_{\rm LM}(1,2\,|\,4)$$
$$= \int \frac{\mathrm{d}^{d-1}p_4}{(2\pi)^d 2E_4} \boxed{2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)}} F_{\rm LM}(1,2)$$
$$= \mathcal{F}(1,2;\epsilon) F_{\rm LM}(1,2)$$

Analytic solution

- singularities are manifest in $\mathcal{F}(1,2;\epsilon)$

FKS subtraction

Finite remainder

$$
\begin{aligned}
\langle F_{\mathrm{LM}}(1,2\,|\,4)\rangle = {} & \langle (1-\mathcal{S}_4)(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))F_{\mathrm{LM}}(1,2\,|\,4)\rangle \\
& + \langle (1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})F_{\mathrm{LM}}(1,2\,|\,4)\rangle \\
& + \langle \mathcal{S}_4 F_{\mathrm{LM}}(1,2\,|\,4)\rangle
\end{aligned}
$$

Step 3: Combination of real and virtual contributions.

$$
\begin{aligned}
\mathrm{d}\sigma^{\mathrm{NLO}} = {} & F_{\mathrm{LV}}^{\mathrm{fin}}(1,2) + \big\langle (1-\mathcal{S}_4)(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))F_{\mathrm{LM}}(1,2\,|\,4)\big\rangle \\
& + \frac{\alpha_s}{2\pi}\int \mathrm{d}z\,\mathcal{P}_{qq}(z)\frac{F_{\mathrm{LM}}(z\cdot 1,2)+F_{\mathrm{LM}}(1,z\cdot 2)}{z} \\
& + \frac{\alpha_s}{2\pi}\frac{2\pi^2}{3}C_F F_{\mathrm{LM}}(1,2)
\end{aligned}
$$

Finite remainder

- can be integrated (numerically) in four dimension