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# NLO Quarkonium Production in the FKS Subtraction Scheme

Groupement de Recherche QCD 2024

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**Lukas Simon**

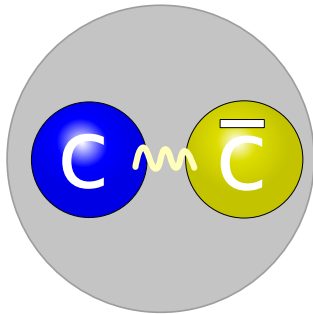
Laboratoire de Physique Théorique et Hautes Énergies  
Sorbonne Université et CNRS

based on arXiv:2402.19221

in collaboration with Ajjath A. H. and H.-S. Shao

May 29, 2024

# Quarkonium Production



# Quarkonium production in NRQCD factorisation

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$$d\sigma(AB \rightarrow H + X) = \sum_n \left( \sum_{a,b,X} \int dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) d\hat{\sigma}(ab \rightarrow Q\bar{Q}'[n] + X) \right) \langle \mathcal{O}_n^H \rangle$$

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partonic  
cross section

## Partonic cross section

- ▶ short distance production of a  $Q\bar{Q}'$  pair in a colour representation  $C$ , with spin  $S$  and orbital angular momentum state  $L$
- ▶ spectroscopic notation of Fock states:  $n = {}^{2S+1}L_J^{[C]}$

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$$d\sigma(AB \rightarrow H + X) = \sum_n \left( \sum_{a,b,X} \int dx_a dx_b \underbrace{f_{a/A}(x_a) f_{b/B}(x_b)}_{\text{PDF}} \underbrace{d\hat{\sigma}(ab \rightarrow Q\bar{Q}'[n] + X)}_{\text{partonic cross section}} \right) \langle \mathcal{O}_n^H \rangle$$

## Parton Distribution Function (PDF)

- ▶ parton distribution functions of partons  $a$  and  $b$  in the initial hadrons  $A$  and  $B$

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## Long Distance Matrix Element (LDME)

- ▶ hadronisation of the heavy quark pair into the physical quarkonium state  $H$

# Infrared Singularities

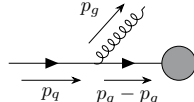


But my toes are getting wrinkly.

A character with white hair and a yellow shirt is speaking. The text next to him reads "But my toes are getting wrinkly."

None

# Infrared singularities


$$|\mathcal{M}_J|^2 = \left| \text{Diagram} \right|^2 \sim \frac{1}{(p_q - p_g)^2} = \frac{1}{E_q E_g (1 - \beta_q \cos \theta_{qg})}$$

with

$$\beta_q = \frac{|\vec{p}_q|}{E_q} \xrightarrow{m_q \rightarrow 0} 1$$

$$\int |\mathcal{M}_J|^2 F_J d\Pi^{(4)} \sim \int_0^{E^{\max}} dE_g \int_{-1}^1 d\cos \theta_{qg} \frac{1}{E_q E_g (1 - \beta_q \cos \theta_{qg})} \rightarrow \infty \begin{cases} \text{for } E_g \rightarrow 0 \\ \text{for } \theta_{qg} \rightarrow 0 \\ \wedge \beta_q \rightarrow 1 \end{cases}$$

## Infrared singularities:

- ▶ **soft** singularity ( $E_g \rightarrow 0$ )
- ▶ **collinear** singularity ( $\theta_{qg} \rightarrow 0 \wedge \beta_q \rightarrow 1$ )
- ▶ **soft-collinear** singularity ( $E_g \rightarrow 0$  and  $\theta_{qg} \rightarrow 0 \wedge \beta_q \rightarrow 1$ )



# Subtraction schemes

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$$\lim_{d \rightarrow 4} \int |\mathcal{M}_J|^2 F_J d\Pi^{(d)} \rightarrow \infty$$

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$$\lim_{d \rightarrow 4} \int |\mathcal{M}_J|^2 F_J d\Pi^{(d)} \rightarrow \infty$$

## Subtraction function

find a suitable subtraction function  $S$  which

- ▶ reproduces the matrix element in the unresolved limit
- ▶ is simple to integrate over the unresolved phase space

$$\lim_{\text{IR poles}} S = \lim_{\text{IR poles}} |\mathcal{M}_J|^2 F_J$$

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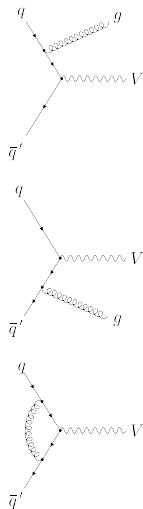
$$\lim_{\text{IR poles}} S = \lim_{\text{IR poles}} |\mathcal{M}_J|^2 F_J$$

$$\lim_{d \rightarrow 4} \int |\mathcal{M}_J|^2 F_J d\Pi^{(d)} = \underbrace{\int (|\mathcal{M}_J|^2 F_J - S) d\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d \rightarrow 4} \int S d\Pi^{(d)}}_{\text{analytic solution}}$$

# Infrared-safe cross section at NLO

The NLO cross section can be split in three parts

$$d\sigma^{\text{NLO}} = \underbrace{d\sigma^{\text{R}}}_{\text{contains IR poles from phase-space integration}} + \overbrace{d\sigma^{\text{V}} + d\sigma^{\text{PDF}}}_{\text{contain explicit poles}}$$



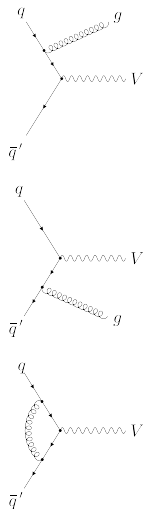
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Subtraction method makes **IR poles explicit**:

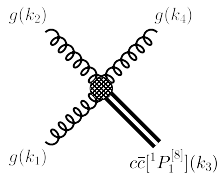
- ▶ poles cancel among the four parts
- ▶ analytic formula for finite remainder
- ▶ phase-space integration can be performed numerically



# FKS subtraction scheme

Frixione-Kunszt-Signer

$$d\sigma(r) = d\sigma(r)$$



# FKS subtraction scheme

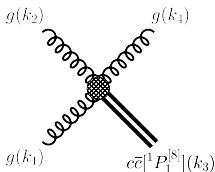
Frixione-Kunszt-Signer

$$d\sigma(r) = (1 - \mathcal{S}_4) d\sigma(r)$$

$$+ \mathcal{S}_4 d\sigma(r)$$

1. regulate soft singularities:  $\mathcal{S}_4 = \lim_{E_4 \rightarrow 0}$

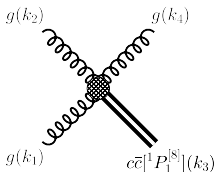
- ▶ soft gluon can be emitted from both initial state gluons **and** from the  $Q\bar{Q}'$  pair



# FKS subtraction scheme

Frixione-Kunszt-Signer

$$\begin{aligned}d\sigma(r) &= (1 - \mathcal{S}_4) (1 - (\mathcal{C}_{41} + \mathcal{C}_{42})) d\sigma(r) \\ &+ (1 - \mathcal{S}_4)(\mathcal{C}_{41} + \mathcal{C}_{42}) d\sigma(r) \\ &+ \mathcal{S}_4 d\sigma(r)\end{aligned}$$



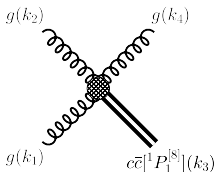
1. regulate soft singularities:  $\mathcal{S}_4 = \lim_{E_4 \rightarrow 0}$ 
  - ▶ soft gluon can be emitted from both initial state gluons **and** from the  $Q\bar{Q}'$  pair
2. regulate collinear singularities:  $\mathcal{C}_{4i} = \lim_{\theta_{i4} \rightarrow 0}$ 
  - ▶ collinear gluon can be emitted from both initial state gluons, but **not** from the  $Q\bar{Q}'$  pair



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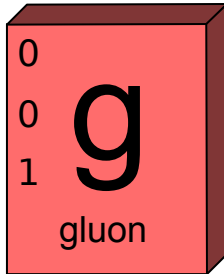


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## FKS for quarkonium production

- ▶ **new soft subtraction terms** w.r.t. production of elementary particles
- ▶ **collinear subtraction terms** are **unchanged**

# Soft Gluon Emission



# Color-singlet spin-singlet P-wave

Soft limit of the amplitude

$$\begin{aligned} \lim_{k_g \rightarrow 0} & \text{Diagram} = \text{Eik} \left( \text{Diagram} \right) \otimes \text{Diagram} \\ & + \text{Diagram} \otimes \text{Eik} \left( \text{Diagram} \right) \end{aligned}$$

The diagram on the left shows a quarkonium production process in the soft limit. It features a central black circle representing the quarkonium state, with four external lines: two incoming gluons (green and blue) and two outgoing gluons (red and blue). The amplitude is labeled  $c\bar{c}[^1P_1^{(1)}]$ .

The first term on the right is the eikonal approximation of the soft gluon emission, where the soft gluon is attached to the quark lines. The second term is the eikonal approximation of the soft gluon emission, where the soft gluon is attached to the antiquark lines.

# Color-singlet spin-singlet P-wave

## Soft limit of the amplitude

$$\lim_{k_g \rightarrow 0} \text{Diagram} = \text{Eik} \left( \text{Diagram}_1 \right) \otimes \text{Diagram}_2 + \text{Diagram}_3 \otimes \text{Eik} \left( \text{Diagram}_4 \right)$$

$$\begin{aligned} \lim_{k_i \rightarrow 0} \mathcal{A}_{\{[1],0,1,1\}}^{(n+1)}(r) &= \sum_{j=n_I}^{n_L^{(R)}+n_H} g_s \frac{k_j \cdot \varepsilon_{\lambda_i}^*(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}_j) \mathcal{A}_{\{[1],0,1,1\}}^{(n)}(r^{\vec{\lambda}}) \\ &+ g_s \left[ \frac{\varepsilon_{\lambda_i}^*(K) \cdot \varepsilon_{\lambda_i}^*(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon_{\lambda_i}^*(k_i) k_i \cdot \varepsilon_{\lambda_i}^*(K)}{(K \cdot k_i)^2} \right] \vec{Q}_{\text{eff}}(Q\vec{Q}'_{[18]}) \mathcal{A}_{\{[8],0,0,0\}}^{(n)}(r^{\vec{\lambda}}) \end{aligned}$$

# Color-singlet spin-singlet P-wave

## Soft limit of the squared amplitude

$$\left| \lim_{k_g \rightarrow 0} \left( \text{diagram} \right) \right|^2 = \left| \text{Eik} \left( \text{diagram} \right) \otimes \text{diagram} \right|^2 + \left| \text{diagram} \otimes \text{Eik} \left( \text{diagram} \right) \right|^2$$

The diagrams are Feynman diagrams for a color-singlet spin-singlet P-wave process. The first diagram shows a quark-antiquark pair (black) with a gluon (green) and a ghost (red) exchange. The second diagram shows a quark-antiquark pair (black) with a gluon (green) and a ghost (red) exchange, but with different color flow. The third diagram shows a quark-antiquark pair (black) with a gluon (green) and a ghost (red) exchange, but with different color flow. The fourth diagram shows a quark-antiquark pair (black) with a gluon (green) and a ghost (red) exchange, but with different color flow.

$$\begin{aligned}
 \lim_{k_i \rightarrow 0} \mathcal{M}_{\{[1],0,1,1\}}(r) = g_s^2 & \left\{ \sum_{\substack{k,l=n_I \\ k,l \neq i, k \leq l}}^{n_L^{(R)} + n_H} \frac{k_k \cdot k_l}{k_k \cdot k_i k_l \cdot k_i} \mathcal{M}_{kl}(r^{\check{\chi}}) \right. \\
 & + \sum_{\substack{k=n_I \\ k \neq i}}^{n_L^{(R)} + n_H} \left[ \frac{k_{k,\mu}}{k_k \cdot k_i K \cdot k_i} - \frac{K \cdot k_k k_{i,\mu}}{k_k \cdot k_i (K \cdot k_i)^2} \right] \mathcal{M}_{k[18]}^\mu(r^{\check{\chi}}, r_1^{\check{\chi}}) \\
 & \left. - \frac{2\epsilon - 2}{(K \cdot k_i)^2} C_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{M}(r_1^{\check{\chi}}) \right\}
 \end{aligned}$$

# Color-singlet spin-singlet P-wave

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## Subtracted squared matrix element

$$\mathcal{M}_{\{[1],0,1,1\}}(r) - \sum_{i \in g} \lim_{k_i \rightarrow 0} \mathcal{M}_{\{[1],0,1,1\}}(r)$$

- ▶ free of soft singularities
- ▶ (numerical) integration in  $d = 4$  dimensions possible

# Color-singlet spin-singlet P-wave

Integrated soft counterterms

## Reminder: subtraction schemes

$$\lim_{d \rightarrow 4} \int |\mathcal{M}_J|^2 F_J d\Pi^{(d)} = \underbrace{\int (|\mathcal{M}_J|^2 F_J - S) d\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d \rightarrow 4} \int S d\Pi^{(d)}}_{\text{analytic solution}}$$

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### Analytic solution:

$$\begin{aligned} d\hat{\sigma}^{(S)}(r) = & \frac{\alpha_s}{2\pi} \phi_{n-1}(r^{\vec{\lambda}}) \frac{J^{n_L^{(B)}}}{\mathcal{N}(r^{\vec{\lambda}})} \mathcal{G}(r^{\vec{\lambda}}) \left[ \sum_{k=n_I}^{n_L^{(B)}+n_H} \sum_{l=k}^{n_L^{(B)}+n_H} \bar{\mathcal{E}}(\{1, 1\}, \{k_k, k_l\}) \mathcal{M}_{kl}(r^{\vec{\lambda}}) \right. \\ & + \sum_{k=n_I}^{n_L^{(B)}+n_H} \left( \bar{\mathcal{E}}(\{1, 1\}, \{k_k, K\}) \frac{k_{k,\mu}}{K \cdot k_k} - \bar{\mathcal{E}}_\mu(\{1, 2\}, \{k_k, K\}) \right) \mathcal{M}_{k[18]}^\mu(r^{\vec{\lambda}}, r_1^{\vec{\lambda}}) \\ & \left. + \left( \frac{1}{2N_c} \frac{8}{m_Q m_{Q'}} \left( \frac{1}{\bar{\epsilon}} + \log \frac{\mu^2}{\mu_{\text{NRQCD}}^2} \right) - \frac{2\epsilon - 2}{K^2} \bar{\mathcal{E}}(\{1, 1\}, \{K, K\}) C_{\text{eff}}(Q\bar{Q}_{[18]}) \right) \mathcal{M}(r_1^{\vec{\lambda}}) \right] \end{aligned}$$

where the eikonal integrals

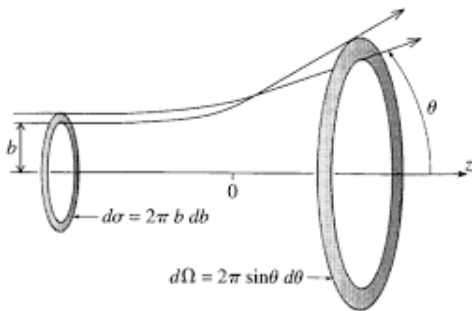
$$\bar{\mathcal{E}}^{\alpha_1 \dots \alpha_{n_1-1} \beta_1 \dots \beta_{n_2-1}}(\{n_1, n_2\}, \{k_k, k_l\}) = 8\pi^2 \mu^{2\epsilon} k_k \cdot k_l \int \frac{d^{3-2\epsilon} \mathbf{k}_i}{(2\pi)^{3-2\epsilon} 2k_i^0} \frac{k_i^{\alpha_1} \dots k_i^{\alpha_{n_1-1}} k_i^{\beta_1} \dots k_i^{\beta_{n_2-1}}}{(k_k \cdot k_i)^{n_1} (k_l \cdot k_i)^{n_2}} \theta(\xi_{\text{cut}} - \xi_i)$$

can be solved analytically.

- ▶ singularities appear explicitly as poles in  $\epsilon$

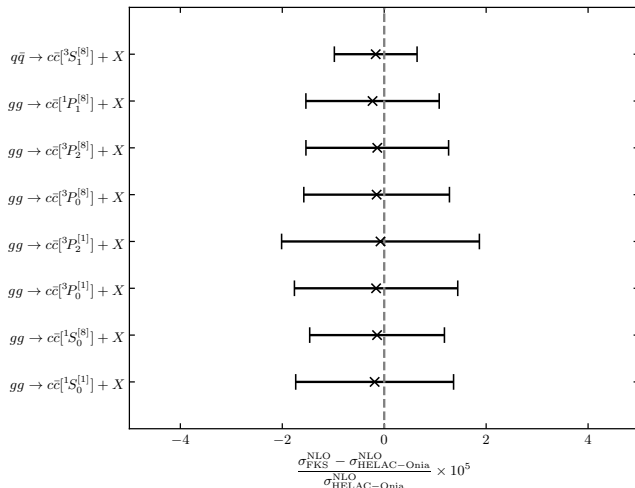


# Cross Sections



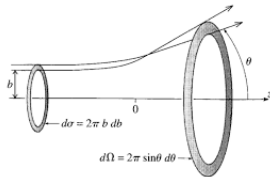
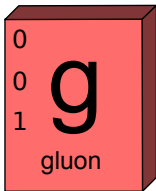
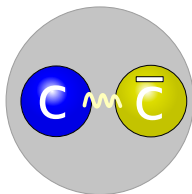
# Inclusive NLO cross sections

## Validation of $2 \rightarrow 1$ processes



- ▶ inclusive cross sections of FKS approach (**fully numerical**) can be compared to HELAC-Onia [Shao, 2016] (**analytic phase-space integration**)

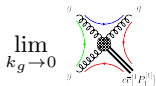
# Summary



# Summary

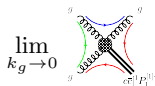
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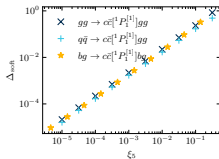
- ▶ fully differential NLO results require knowledge about soft gluon emission from quarkonia
  - ▶ S-wave quarkonia behave like “massive gluons”
  - ▶ P-wave quarkonia lead to more complex structures



# Summary

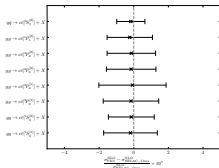
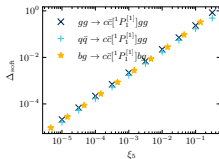
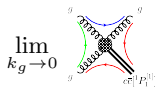
- ▶ fully differential NLO results require knowledge about soft gluon emission from quarkonia
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- ▶ local FKS subtraction term and the corresponding integrated counterterm were computed
  - ▶ eikonal limits are checked numerically
  - ▶  $\epsilon$ -poles of integrated counterterms cancel analytically against virtual corrections

$$\lim_{k_g \rightarrow 0} \text{diagram}$$




# Summary

- ▶ fully differential NLO results require knowledge about soft gluon emission from quarkonia
  - ▶ S-wave quarkonia behave like “massive gluons”
  - ▶ P-wave quarkonia lead to more complex structures
- ▶ local FKS subtraction term and the corresponding integrated counterterm were computed
  - ▶ eikonal limits are checked numerically
  - ▶  $\epsilon$ -poles of integrated counterterms cancel analytically against virtual corrections
- ▶ inclusive cross sections of FKS approach coincide with known results



# MadGraph5



# Quarkonia in MadGraph5

---

We aim to launch a version of MadGraph5\_aMC@NLO<sup>[Alwall et al. 2014]</sup> that allows for automated computations of cross section which include bound states:

1. LO cross sections for processes with an arbitrary number of quarkonia (S-wave and P-wave) and additional elementary particles

Example:

$$pp \rightarrow J/\psi + \eta_c + c\bar{c}g$$

Syntax:

```
MG_aMC>generate p p > J/psi etac c c~ g
```

or in an alternative notation

```
MG_aMC>generate p p > c.c~(1|3S11) c.c~(1|1S01) c c~ g
```

2. NLO cross sections with one quarkonium plus elementary particles
3. extension of FKS approach to multiple quarkonia final states



# Backup



# Amplitudes with quarkonia

Projection to Fock states

---

$$a(k_1)b(k_2) \rightarrow Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

# Amplitudes with quarkonia

## Projection to Fock states

$$a(k_1)b(k_2) \rightarrow Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

↓ colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3, c_4} \mathbb{P}_C \mathcal{A}(r)$$

### Colour projector

$$\mathbb{P}_{C=1} = \frac{\delta^{c_3 c_4}}{\sqrt{N_c}}$$

$$\mathbb{P}_{C=8} = \sqrt{2} T_{c_4 c_3}^{c_3 c_4}$$

# Amplitudes with quarkonia

## Projection to Fock states

$$a(k_1)b(k_2) \rightarrow Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3, c_4} \mathbb{P}_C \mathcal{A}(r)$$

spin projection

$$\mathcal{A}_{\{[C], [S]\}}(r) = \sum_{\lambda_Q, \lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

## Spin projector

$$\mathbb{P}_{S=0} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\gamma_5 u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

$$\mathbb{P}_{S=1} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\not{K}_{\lambda_S}^* u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

with

$$K^\mu = k_3^\mu + k_4^\mu$$

# Amplitudes with quarkonia

## Projection to Fock states

$$a(k_1)b(k_2) \rightarrow Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3, c_4} \mathbb{P}_C \mathcal{A}(r)$$

spin projection

$$\mathcal{A}_{\{[C], [S]\}}(r) = \sum_{\lambda_Q, \lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

orbital angular momentum projection

$$\mathcal{A}_{\{[C], [S], [L]\}}(r) = \left[ \left( \varepsilon_{\lambda_i}^{\mu, *}(K) \frac{d}{dq^\mu} \right)^L \mathcal{A}_{\{[C], [S]\}}(r) \right]_{q=0}$$

Orbital angular momentum

$$k_3^\mu = \frac{m_Q}{m_Q + m_{\bar{Q}'}} K^\mu + q^\mu$$

$$k_4^\mu = \frac{m_{\bar{Q}'}}{m_Q + m_{\bar{Q}'}} K^\mu - q^\mu$$

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$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3, c_4} \mathbb{P}_C \mathcal{A}(r)$$

spin projection

$$\mathcal{A}_{\{[C], S\}}(r) = \sum_{\lambda_Q, \lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

orbital angular momentum projection

$$\mathcal{A}_{\{[C], S, L\}}(r) = \left[ \left( \varepsilon_{\lambda_l}^{\mu, *}(K) \frac{d}{dq^\mu} \right)^L \mathcal{A}_{\{[C], S\}}(r) \right]_{q=0}$$

total angular momentum projection

$$\mathcal{A}_{\{[C], S, L, J\}}(r) = \sum_{\lambda_s, \lambda_l} \underbrace{\langle J, \lambda_j | L, \lambda_l; S, \lambda_s \rangle}_{\text{Clebsch-Gordan coefficient}} \mathcal{A}_{\{[C], S, L\}}(r)$$

# Color-octet spin-singlet P-wave

## Soft limit of the amplitude

$$\begin{aligned}
 \lim_{k_g \rightarrow 0} & \left( \begin{array}{c} g \\ \text{Diagram 1} \\ g \end{array} + \begin{array}{c} g \\ \text{Diagram 2} \\ g \end{array} \right) = \text{Eik} \left( \begin{array}{c} g \\ \text{Diagram 3} \\ g \end{array} \right) \otimes \begin{array}{c} g \\ \text{Diagram 4} \\ g \end{array} \sigma_{\bar{c}c}^{[1]P_1^{[8]}} \\
 & + \begin{array}{c} g \\ \text{Diagram 5} \\ g \end{array} \sigma_{\bar{c}c}^{[1]P_1^{[8]}} \otimes \text{Eik} \left( \begin{array}{c} g \\ \text{Diagram 6} \\ g \end{array} \right) \\
 & + \begin{array}{c} g \\ \text{Diagram 7} \\ g \end{array} \sigma_{\bar{c}c}^{[1]S_0^{[1]}} \otimes \text{Eik} \left( \begin{array}{c} g \\ \text{Diagram 8} \\ g \end{array} \right) \\
 & + \begin{array}{c} g \\ \text{Diagram 9} \\ g \end{array} \sigma_{\bar{c}c}^{[1]S_0^{[8]}} \otimes \text{Eik} \left( \begin{array}{c} g \\ \text{Diagram 10} \\ g \end{array} \right)
 \end{aligned}$$

The diagrams are Feynman diagrams for the production of a color-octet spin-singlet P-wave quarkonium state. Diagrams 1 and 2 are the full tree-level diagrams. Diagrams 3-10 represent the soft-gluon limit, where the gluon is emitted from the quark lines, and the amplitude is factorized into an eikonal factor and a hard part.

# Color-octet spin-singlet P-wave

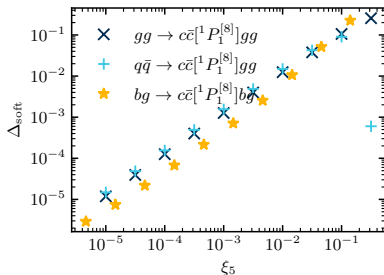
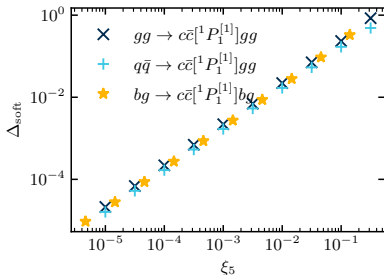
## Soft limit of the amplitude

$$\lim_{k_q \rightarrow 0} \left( \text{Diagram 1} + \text{Diagram 2} \right) = \text{Eik} \left( \text{Diagram 3} \right) \otimes \text{Diagram 4} + \text{Diagram 5} \otimes \text{Eik} \left( \text{Diagram 6} \right) + \text{Diagram 7} \otimes \text{Eik} \left( \text{Diagram 8} \right) + \text{Diagram 9} \otimes \text{Eik} \left( \text{Diagram 10} \right)$$

$$\begin{aligned} \lim_{k_i \rightarrow 0} \mathcal{A}_{\{[8],0,1,1\}}(r) &= \sum_{\substack{j=n_I \\ j \neq i}}^{n_L^{(R)} + n_H} g_s \frac{k_j \cdot \varepsilon_{\lambda_i}^*(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}) \mathcal{A}_{\{[8],0,1,1\}}(r^{\check{\lambda}}) \\ &+ g_s \frac{K \cdot \varepsilon_{\lambda_i}^*(k_i)}{K \cdot k_i} \vec{Q}(Q\bar{Q}'[{}^1P_1^{[8]}]) \mathcal{A}_{\{[8],0,1,1\}}(r^{\check{\lambda}}) \\ &+ g_s \left[ \frac{\varepsilon_{\lambda_l}^*(K) \cdot \varepsilon_{\lambda_i}^*(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon_{\lambda_i}^*(k_i) k_i \cdot \varepsilon_{\lambda_l}^*(K)}{(K \cdot k_i)^2} \right] \\ &\times \left[ \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[81]}) \mathcal{A}_{\{[1],0,0,0\}}(r^{\check{\lambda}}) + \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[88]}) \mathcal{A}_{\{[8],0,0,0\}}(r^{\check{\lambda}}) \right] \end{aligned}$$



# Local soft subtraction terms



$$\Delta_{\text{soft}} = \left| \frac{\mathcal{M}(r) - \lim_{k_5 \rightarrow 0} \mathcal{M}(r)}{\mathcal{M}(r)} \right| \propto \xi_5 + \mathcal{O}(\xi_5^2), \quad \text{with} \quad E_5 = \xi_5 \frac{\sqrt{s}}{2}$$

## Reminder: subtraction schemes

$$\lim_{d \rightarrow 4} \int |\mathcal{M}_J|^2 F_J d\Pi^{(d)} = \underbrace{\int (|\mathcal{M}_J|^2 F_J - S) d\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d \rightarrow 4} \int S d\Pi^{(d)}}_{\text{analytic solution}}$$

# Subtraction schemes

## Concept

$$\begin{aligned} I &= \int_0^1 dx \frac{1}{x^{1+\epsilon}} F(x) \\ &= \int_0^1 dx \frac{1}{x^{1+\epsilon}} [F(x) - F(0)] + F(0) \int_0^1 dx \frac{1}{x^{1+\epsilon}} \end{aligned}$$

## Subtraction function $F(0)$

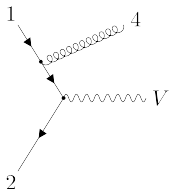
- ▶  $\frac{1}{x^{1+\epsilon}} [F(x) - F(0)]$  is free of divergences (numerical integration)
- ▶  $F(0) \int_0^1 dx \frac{1}{x^{1+\epsilon}}$  divergent but simple to integrate analytically

$$I = \underbrace{\int_0^1 dx \frac{1}{x} [F(x) - F(0)]}_{\text{free of divergences}} - \underbrace{\frac{1}{\epsilon} F(0)}_{\text{analytic solution}} + \mathcal{O}(\epsilon)$$

# FKS subtraction

## Local soft subtraction function

$$\langle F_{\text{LM}}(1, 2 | 4) \rangle = \langle (1 - \mathcal{S}_4) F_{\text{LM}}(1, 2 | 4) \rangle + \langle \mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) \rangle$$



**Step 1:** Find suitable subtraction functions.

- ▶  $\mathcal{S}_4$  extracts the leading soft singularity
- ▶ Eikonal approximation

$$\mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) = 2C_F g_s^2 \frac{p_1 \cdot p_2}{(p_1 \cdot p_4)(p_2 \cdot p_4)} F_{\text{LM}}(1, 2)$$

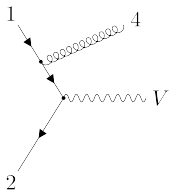
Free of soft divergences

$$F_{\text{LM}}(1, 2 | 4) - 2C_F g_s^2 \frac{p_1 \cdot p_2}{(p_1 \cdot p_4)(p_2 \cdot p_4)} F_{\text{LM}}(1, 2)$$

# FKS subtraction

## Integrated soft subtraction function

$$\langle F_{\text{LM}}(1, 2 | 4) \rangle = \langle (1 - \mathcal{S}_4) F_{\text{LM}}(1, 2 | 4) \rangle + \langle \mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) \rangle$$



**Step 2:** Analytic solution of the subtraction functions.

$$\begin{aligned} \langle \mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) \rangle &= \int [dg_4] \mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) \\ &= \int \frac{d^{d-1} p_4}{(2\pi)^d 2E_4} \boxed{2C_F g_s^2 \frac{p_1 \cdot p_2}{(p_1 \cdot p_4)(p_2 \cdot p_4)}} F_{\text{LM}}(1, 2) \\ &= \mathcal{F}(1, 2; \epsilon) F_{\text{LM}}(1, 2) \end{aligned}$$

## Analytic solution

- ▶ singularities are manifest in  $\mathcal{F}(1, 2; \epsilon)$

# FKS subtraction

## Finite remainder

$$\begin{aligned}\langle F_{\text{LM}}(1, 2 | 4) \rangle &= \langle (1 - \mathcal{S}_4)(1 - (\mathcal{C}_{41} + \mathcal{C}_{42}))F_{\text{LM}}(1, 2 | 4) \rangle \\ &\quad + \langle (1 - \mathcal{S}_4)(\mathcal{C}_{41} + \mathcal{C}_{42})F_{\text{LM}}(1, 2 | 4) \rangle \\ &\quad + \langle \mathcal{S}_4 F_{\text{LM}}(1, 2 | 4) \rangle\end{aligned}$$

**Step 3:** Combination of real and virtual contributions.

$$\begin{aligned}d\sigma^{\text{NLO}} &= F_{\text{LV}}^{\text{fin}}(1, 2) + \langle (1 - \mathcal{S}_4)(1 - (\mathcal{C}_{41} + \mathcal{C}_{42}))F_{\text{LM}}(1, 2 | 4) \rangle \\ &\quad + \frac{\alpha_s}{2\pi} \int dz \mathcal{P}_{qq}(z) \frac{F_{\text{LM}}(z \cdot 1, 2) + F_{\text{LM}}(1, z \cdot 2)}{z} \\ &\quad + \frac{\alpha_s}{2\pi} \frac{2\pi^2}{3} C_F F_{\text{LM}}(1, 2)\end{aligned}$$

## Finite remainder

- ▶ can be integrated (numerically) in four dimension