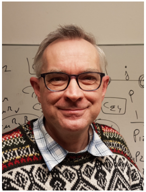
Interactions of Three Pions

CNRS GDR QCD AG 2024, Tours

Mattias Sjö, CPT Marseille

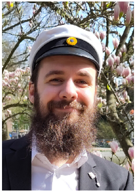
28th May 2024

The collaboration



Hans Bijnens,
Lund U.

Tomáš Husek,
Birmingham U.



Mattias Sjö,
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MIT

Jorge Baeza-Ballesteros,
U. de València

Background

Motivation

- **High-multiplicity** scattering (e.g., qq $\to$ *N*g, $N > 2$) interesting in perturbative QCD
  - Color structures/stripping
  - MHV amplitudes
  - etc.
- Analogous problems/solutions at **low energy** (ChPT)
  - Work of Trnka, Kampf, Bartsch, **Sjö**,...
- **But what use does it have?**
  - Extremely limited energy headroom
  - Irrelevant for scattering (who studies KK $\to 4\pi$?)
  - Irrelevant for decay ($\eta \to 5\pi$ is forbidden)

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Resonances with 3-body decays

$\omega(782)$ DECAY MODES

	Mode	Fraction (Γ_i/Γ)
Γ_1	$\pi^+\pi^-\pi^0$	(89.2 ± 0.7) %

$a_1(1260)$ DECAY MODES

	Mode	Fraction (Γ_i/Γ)
Γ_1	3π	seen

$N(1440)$ DECAY MODES

The following branching fractions are our estimates, not fits

	Mode	Fraction (Γ_i/Γ)
Γ_1	$N\pi$	55–75 %
Γ_2	$N\eta$	<1 %
Γ_3	$N\pi\pi$	17–50 %

n-body quantization condition

$n \to n$ scattering

Box of size L

n particles

$$\det\left[F_n^{-1} + \mathcal{K}_n\right] = 0$$

Finite-volume spectrum

n-body quantization condition

$n \to n$ scattering

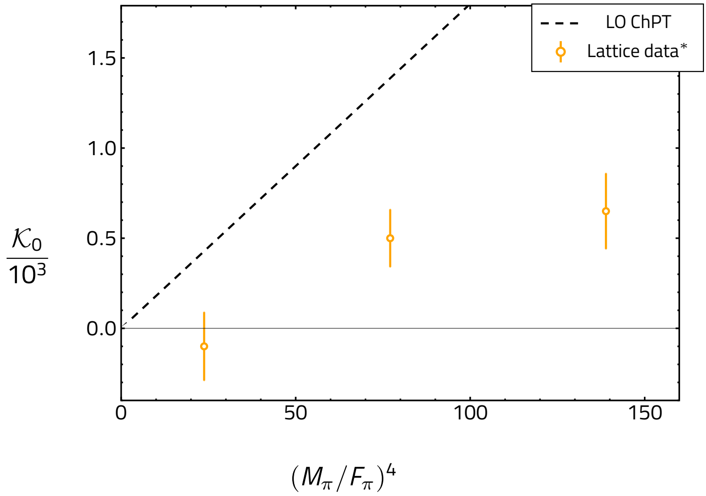
Box of size L

n particles

$$\underbrace{\det\left[F_n^{-1} + \mathcal{K}_n\right] = 0}$$

Finite-volume spectrum

The tension that was



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144[hep-lat]

The K-matrix formalism

Hansen & Sharpe, "*Relativistic, model independent, three-particle quantization condition*" *Phys.Rev.D*, 1408.5933[hep-lat]

Hansen & Sharpe, "*Lattice QCD and Three-particle Decays of Resonances*" *Ann.Rev.Nucl.Part.Sci.*, 1901.00483[hep-lat]

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3$ through **integral equation**

$$\mathcal{K}_{\mathrm{df},3}$$

Divergence-free
(no poles or cuts)

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Related to $\mathcal{M}_3^{\rm NLO}$ through **integral equation**

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Fixed **power counting** gives **algebraic** relation

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The stratergy

- Calculate $\mathcal{M}_3^{\mathrm{NLO}}$
- Discard imaginary part
- Subtract divergences

The $3\pi \to 3\pi$ amplitude

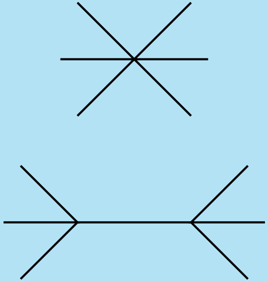
Bijnens & Husek, "*Six-pion amplitude*"
Phys.Rev.D, 2107.06291[hep-ph]

Bijnens, Husek & **Sjö**, "*Six-meson amplitude in QCD-like theories*"
Phys.Rev.D, 2206.14212[hep-ph]

Bijnens, Kampf & **Sjö**, "*Higher-order tree-level amplitudes in the nonlinear sigma model*"
JHEP, 1909.13684[hep-th]

Leading order

Ancient current algebra result



Osborn (1969)

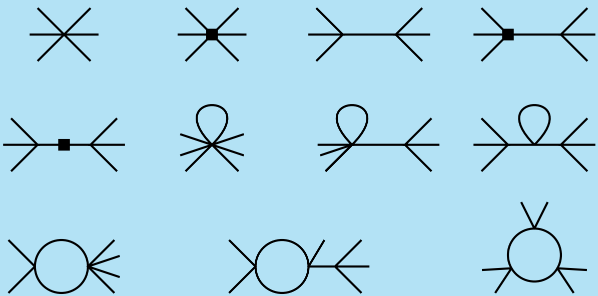
Susskind & Frye (1970)

NLO in CHPT

Vertices

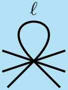
 = LO vertex  = NLO vertex

All the LO and NLO diagrams

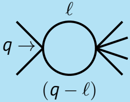


One-Loop Integrals

One- and two-propagator integrals

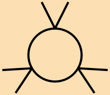


$$\sim \frac{1}{4-d} + \text{(finite)}$$



$$\sim \frac{1}{4-d} + \bar{J}(q^2) + \text{(finite)}$$

Three-propagator integral



$$\sim \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\{1, \ell^\mu, \ell^\mu\ell^\nu, \ell^\mu\ell^\nu\ell^\rho\}}{(\ell^2 - M^2)\left[(\ell - q_1)^2 - M^2\right]\left[(\ell + q_2)^2 - M^2\right]}$$

In principle reducible to $\bar{J}$ — **impractical** — redundant basis instead:

$$\{\bar{J}, C, C_{11}, C_{21}, C_3\}(p_1, \ldots, p_6)$$

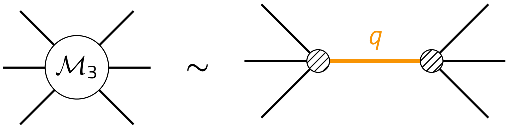
Simplifying the amplitude

$\mathcal{M}_3^{\mathrm{NLO}}$ is a function of...

- 6 particle flavors
- 9 kinematic invariants (8 in $d = 4$)
- 8 free parameters (5 with just pions)
- $\bar{J}(q_i, q_j)$ and 4 $C_X(p_i, p_j, p_k, p_l, p_m, p_n)$'s

$\sim$ **500 pages** in full $\rightarrow$ How to simplify?

Single-particle pole



Factorization

$$\mathcal{M}_3 = \sum_{\substack{\{ijk\} \\ \{lmn\}}} \frac{\mathcal{M}_2(p_i, p_j, p_k, +q) \times \mathcal{M}_2(p_k, p_l, p_n, -q)}{q^2 - M^2 + i\epsilon} + \text{(non-factorizable)}$$

Stripped amplitudes

The 4-point amplitude

$$\begin{aligned}\mathcal{M}^{abcd}(s,t) = \big[\langle \boldsymbol{abcd}\rangle + \langle dcba\rangle\big]\boldsymbol{B(s,t,u)} &+ \langle \boldsymbol{ab}\rangle\langle \boldsymbol{cd}\rangle \boldsymbol{C(s,t,u)}\\ + \big[\langle acdb\rangle + \langle bdca\rangle\big]B(t,u,s) &+ \langle ac\rangle\langle bd\rangle C(t,u,s)\\ + \big[\langle adbc\rangle + \langle cbda\rangle\big]B(u,s,t) &+ \langle ad\rangle\langle bc\rangle C(u,s,t)\end{aligned}$$

The ***stripped*** 4-point amplitude

$$B = \mathcal{M}_{\{4\}}\,, \qquad C = \mathcal{M}_{\{2,2\}}$$

Stripped amplitudes

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\ldots,a_6) = \langle a_1 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\ldots,a_6) = \langle a_1 a_2 a_3 \rangle \langle a_4 a_5 a_6 \rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle \langle a_5 a_6 \rangle$$

$$\mathcal{M}(p_1,a_1;p_2,a_2;\ldots) = \sum_R \sum_\sigma \mathcal{M}_R(\sigma[p_1,\ldots])\mathcal{F}_R(\sigma[a_1,\ldots])$$

Stripping

$\sigma \notin$ symmetries of $\mathcal{F}_R$

$\rightarrow$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$

$\rightarrow$ novel, non-unique!

$\mathcal{M}_3^{\mathrm{NLO}}$ still won't fit on a slide, but not far from it!

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Calculating the 3-pion K-matrix at NLO

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **Sjö** "*The isospin-3 three-particle K-matrix at NLO in ChPT*"

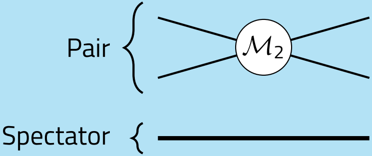
JHEP, 2303.13206[hep-ph]

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **Sjö** "*The three-pion K-matrix at NLO in ChPT*"

JHEP, 2401.14293[hep-ph]

Building blocks

3 particles, 2 scattering



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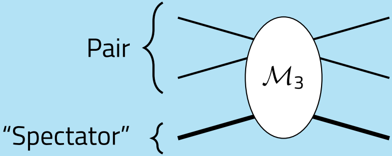
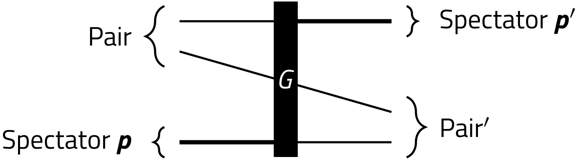


Spectator exchange



Properties of G

- Purely **on-shell**
- **Propagator-like** near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M^2 + i\epsilon}$$

- Smooth **cutoff** away from pole:
  - No UV problems...
  - ...but **non-analytic**
  - ...and **scheme-dependent**

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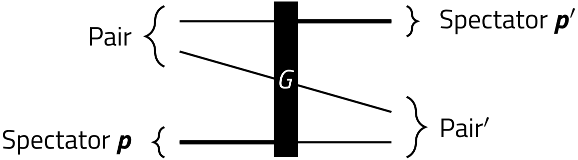

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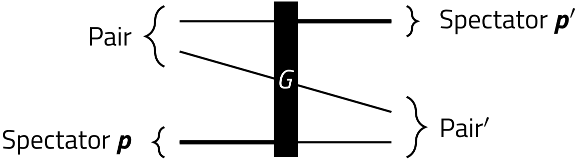

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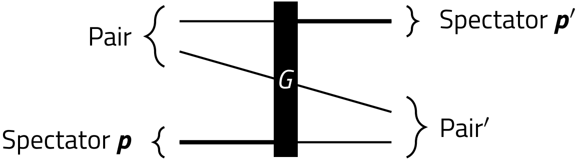

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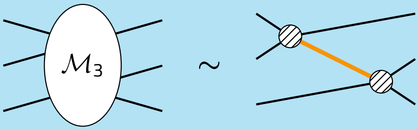
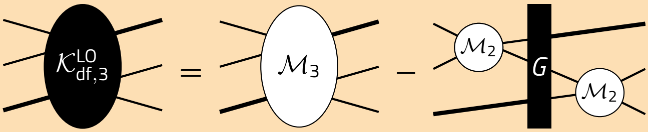
$\mathcal{K}_{\mathbf{df},3}$ at leading order

One-particle exchange (OPE) pole

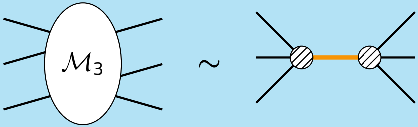


OPE subtraction



$\mathcal{K}_{\mathbf{df},3}$ at leading order

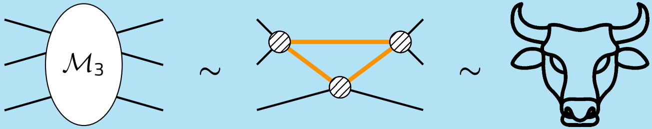
OPE in the s-channel



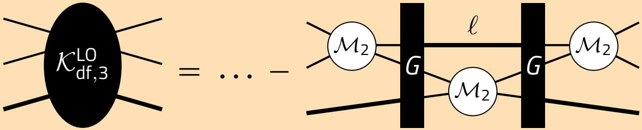
- Only present at **isospin 1**
- **No subtraction** needed since pole is sub-threshold

$\mathcal{K}_{\mathrm{df},3}$ at next-to-leading order

Bull's head cut



Bull's head subtraction



The bull's head

The bull's head integral is **awful**:

- Triangle loop ⇒ complicated, pole-ridden integrand
- On-shell ⇒ no loop momentum shift
- Non-analytic ⇒ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

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Threshold expansion

Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2$$ (**system** above-threshold-ness)

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2$$ (**pair** above-threshold-ness)

$$\tilde{t}_{ij} \propto (p_i - p_j')^2$$ (**spectator** above-threshold-ness)

Compound parameters

$$\Delta_\mathrm{A} = \sum(\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_\mathrm{B} = \sum \tilde{t}_{ij}^2 - \Delta^2$$

Maximum isospin threshold expansion

$$\mathcal{K}_{\mathrm{df},3}^{[I=3]} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_\mathrm{A}\Delta_\mathrm{A} + \mathcal{K}_\mathrm{B}\Delta_\mathrm{B} + \mathcal{O}(\Delta^3)$$

Threshold expansion

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Non-maximal isospin

$I = 3$ — Singlet

$I = 2$ — Doublet

$I = 1$ — Singlet, Doublet

$I = 0$ — Antisymmetric singlet

Minimum isospin threshold expansion

$$\mathcal{K}_{\text{df},3}^{[I=0]} = \mathcal{K}_0^{\text{AS}} \sum \epsilon_{ijk}\epsilon_{lmn}\tilde{t}_{il}\tilde{t}_{jm} + \mathcal{O}(\Delta^3)$$

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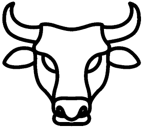
Semi-analytic evaluation



Expand → $\sum \int \mathrm{d}x\ [\text{End-point divergent}](x)$

↓ **Hadamard partie finie**

$$\sum \int \mathrm{d}x\ [\text{Finite}](x)$$

↓

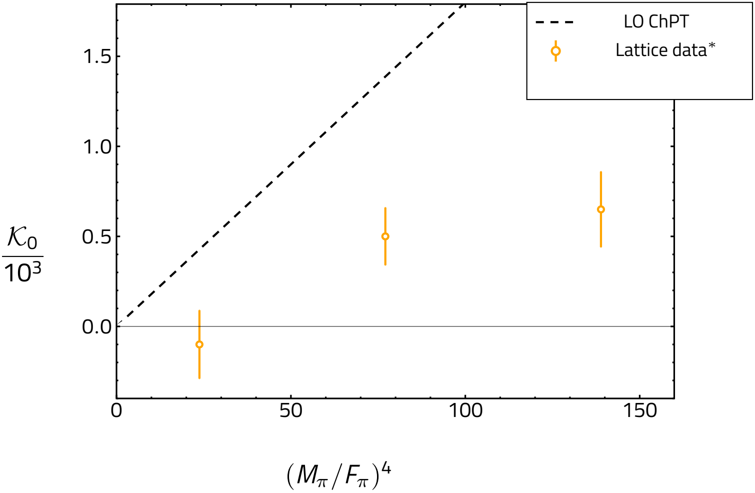
$$\underbrace{\sum \int \mathrm{d}x\ [\text{Simple}](x)}_{\text{analytic}} + \underbrace{\int \mathrm{d}x\ [\text{Small}](x)}_{\substack{\text{numeric}\\ \text{cutoff-dependent}}}$$

Costin & Friedman, "*Foundational aspects of singular integrals*"

J.Functional Analysis, 1401.7045[math.FA]

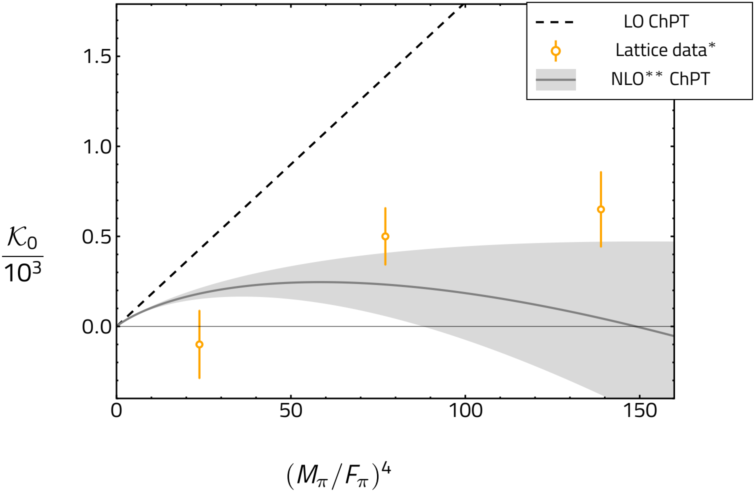
Results

Resolving the tension



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
"*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, `2021.06144 [hep-lat]`

Resolving the tension



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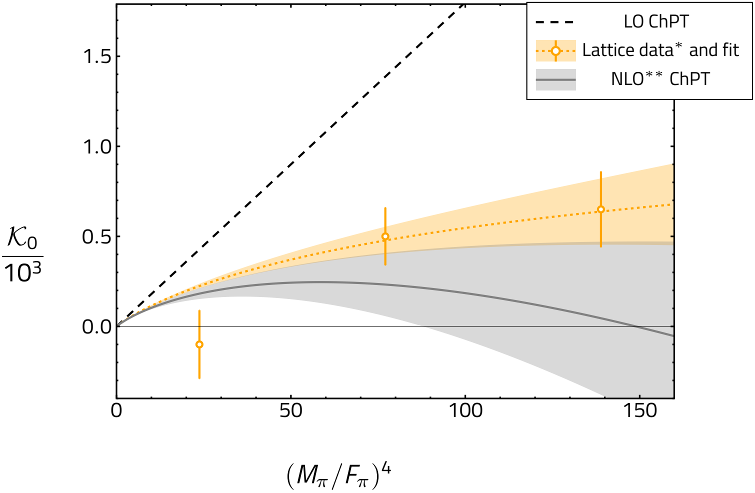

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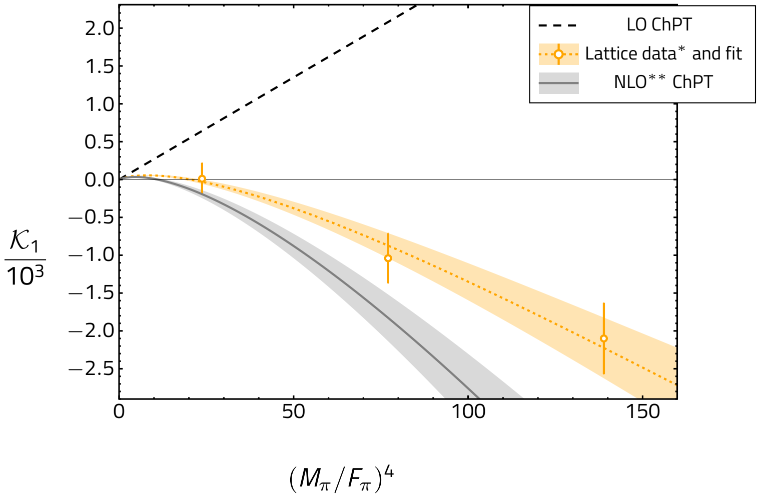
Ditto: Subleading order



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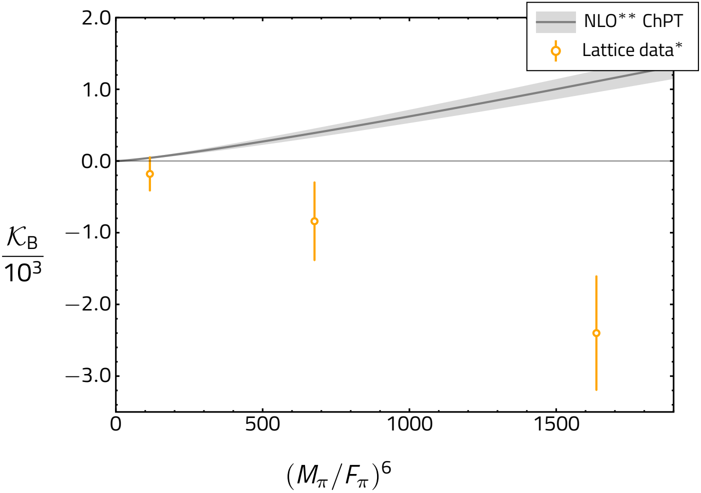
Some tension remains



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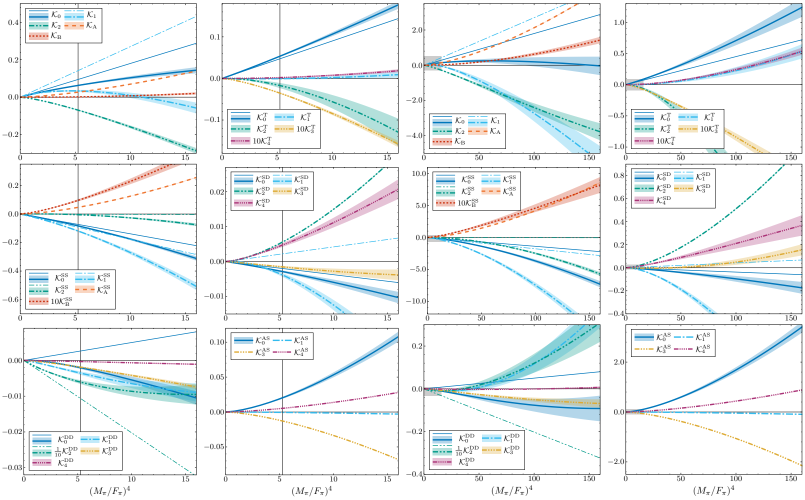
Awaiting more lattice results...



Summary & Outlook

Summary

- All three-pion channels covered
- Main tension resolved
  (where lattice data are available)
- What's next?

Outlook

- Next step: **pion/kaon** systems
  - Lattice data exist:
    Draper, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
    "*Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD*"
    JHEP, 2302.13587[hep-lat]
  - ChPT amplitude WIP
- Ultimate goal: **meson/nucleon** systems
  - Groundwork being laid:
    Draper, Hansen, Romero-López & Sharpe,
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- Also interesting: analogous $K \to 3\pi$ quantity
  - Formalism in place:
    Hansen, Romero-López & Sharpe,
    "*Decay amplitudes to three hadrons from finite-volume matrix elements*"
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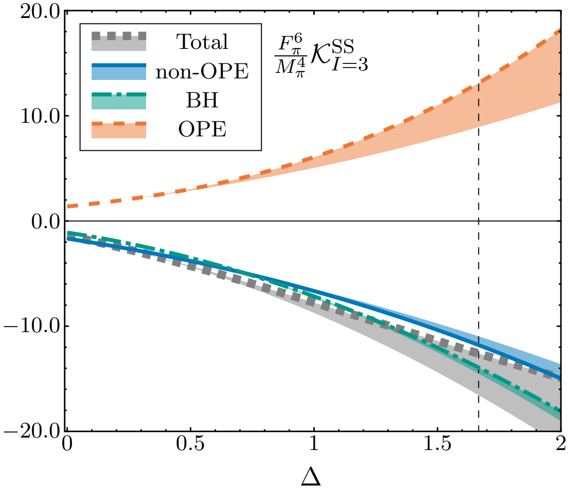
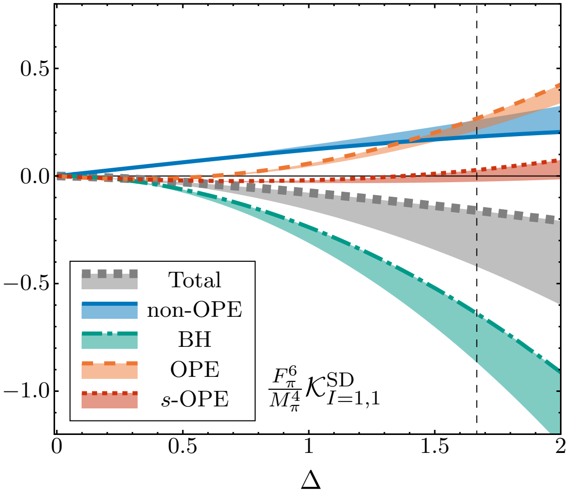
Backup slides

Convergence

The threshold expansion works



...better than it has to



Does ChPT converge?

- Large LO-NLO difference is troubling...
- ...but LO is very constrained
  $\Rightarrow$ **qualitative** difference expected
- Adding NNLO: **extremely difficult**:
  - Two-loop 6-point amplitude
  - Integral relation between $\mathcal{M}_3$ and $\mathcal{K}_{\text{df},3}$

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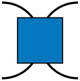
Derivation of the formalism

(in)finite-volume 2-pt amplitude

$$\mathcal{M}_{2(,L)} \equiv$$



- Infinite volume: **integral** over internal momenta
- Finite volume: **sum** over internal momenta
- Differ only with **on-shell** internal momenta
  (up to exponentially suppressed terms)
- Assume $>$2 on-shell particles not possible

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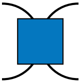
$\mathcal{M}_{2(,L)} \equiv$ 

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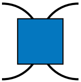
$\mathcal{M}_{2(,L)} \equiv$ 

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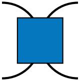
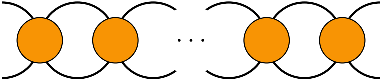
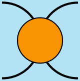
...in terms of kernels

$\mathcal{M}_{2(,L)} \equiv$  $= \sum$ 

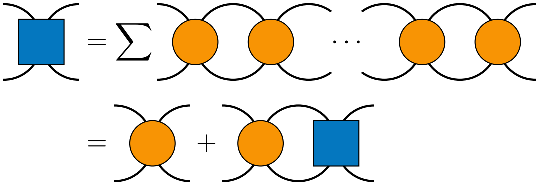
Bethe-Salpeter kernel

$B_2 \equiv$ 

- sum of all 2-particle irreducible diagrams
- is **the same** in both finite and infinite volume
  (up to exponentially suppressed terms)

Recurrence relation

$\mathcal{M}_{2(,L)} \equiv$



Infix notation

- Infinite-volume **integral**: $\mathcal{M}_2 = B_2 + B_2 \otimes \mathcal{M}_2$
- Finite-volume **sum**: $\mathcal{M}_{2,L} = B_2 + B_2 \otimes_L \mathcal{M}_{2,L}$

The F-matrix

$$\otimes_L = \otimes + \boldsymbol{F_2^{i\epsilon}} \,,$$

- purely **geometric** (no dependence on field content)
- purely **on-shell**

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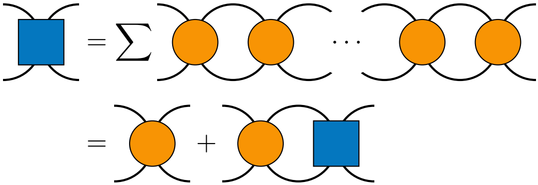

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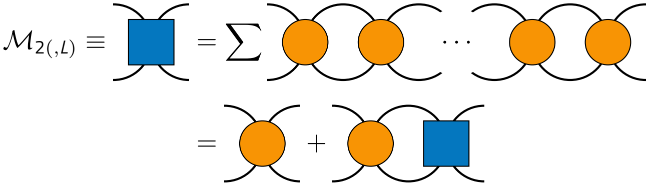

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Resummation

$$\mathcal{M}_{2,L} = B_2 + B_2[\otimes + F_2^{i\epsilon}]\mathcal{M}_{2,L}$$

$$= \sum_{a=0}^{\infty} B_2 \Big([\otimes + F_2^{i\epsilon}] B_2 \Big)^a$$

$$= \sum_{a=0}^{\infty} \Bigg(\underbrace{\sum_{b=0}^{\infty} B_2 [\otimes B_2]^b} \Bigg) \Bigg(F_2^{i\epsilon} \underbrace{\sum_{b=0}^{\infty} B_2 [\otimes B_2]^b} \Bigg)^a$$

$$= \mathcal{M}_2 \sum_{a=0}^{\infty} [F_2^{i\epsilon} \mathcal{M}_2]^a$$

$$= \mathcal{M}_2 \left[1 - F_2^{i\epsilon} \mathcal{M}_2 \right]^{-1}$$

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2-particle quantization condition

- **Spectrum = poles** of $\mathcal{M}_{2,L} = \mathcal{M}_2 \left[1 - F_2^{i\epsilon} \mathcal{M}_2\right]^{-1}$
- Equivalently, $\det\left[F_2^{i\epsilon} - \mathcal{M}_2^{-1}\right] = 0$
- Transfer the 2-particle cut:

$$\mathcal{M}_2^{-1} \equiv \boldsymbol{\mathcal{K}_2^{-1}} - \frac{i}{16\pi E^*}\sqrt{\frac{1}{4}E^{*2} - M^2}$$

$$F_2^{i\epsilon} \equiv \boldsymbol{F_2} \quad + \frac{i}{16\pi E^*}\sqrt{\frac{1}{4}E^{*2} - M^2}$$

Lüscher's quantization condition

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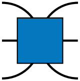
On to 3 particles!

$\mathcal{M}_{3(,L)} \equiv$  $= \ldots$

Many more possibilities, some not too complicated:

- Chain of B_3: $\ldots +$ $+ \ldots$

  Like before, but now with $\boldsymbol{F_3^{i\epsilon}}$

- Chain of B_2: $\ldots +$ $+ \ldots$

  Sum into $\mathcal{M}_2$, absorb into $F_3^{i\epsilon}$ (no longer purely geometric)

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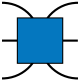
$\mathcal{M}_{3(,L)} \equiv$  $= \dots$

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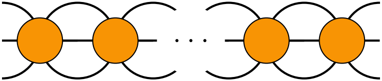
- Chain of B_3: $\dots +$  $+ \dots$

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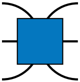
$\mathcal{M}_{3(,L)} \equiv$  $= \ldots$

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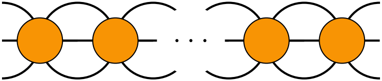
- Chain of B_3: $\ldots +$  $+ \ldots$

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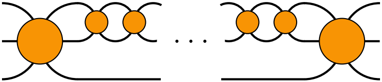
- Chain of B_2: $\ldots +$  $+ \ldots$

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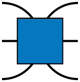
$\mathcal{M}_{3(,L)} \equiv$  $= \ldots$

Many more possibilities, some **very complicated**:

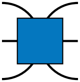
- Alternating $\mathcal{M}_2$'s: $\ldots +$ $+ \ldots$

  New sum-integral difference matrix $\boldsymbol{G}_\infty$ (more on it later)

- ...with loops: $\ldots +$ $+ \ldots$

  On-shell loop momenta **remain to be integrated**

On to 3 particles!

$\mathcal{M}_{3(,L)} \equiv$  $= \ldots$

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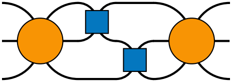
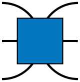
- Alternating $\mathcal{M}_2$'s: $\ldots +$  $+ \ldots$

  New sum-integral difference matrix $\boldsymbol{G}_\infty$ (more on it later)

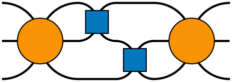
- ...with loops: $\ldots +$ $+ \ldots$

  On-shell loop momenta **remain to be integrated**

On to 3 particles!

$\mathcal{M}_{3(,L)} \equiv$  $= \ldots$

Many more possibilities, some **very complicated**:

- Alternating $\mathcal{M}_2$'s: $\ldots +$  $+ \ldots$

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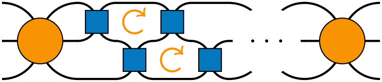
- ...with loops: $\ldots +$  $+ \ldots$

  On-shell loop momenta **remain to be integrated**

On to 3 particles!

The ensuing resummation is **horrendous**. But, in the end,

3-particle quantization condition

$$\det\left[F_3^{-1} + \mathcal{K}_3\right] = 0$$

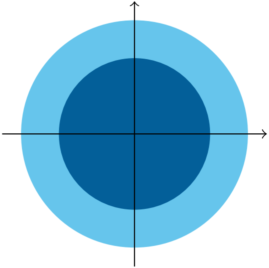
...with a few different approaches to the details.

More on the bull's head

Divide & conquer the bull's head

The integral

$$\int \frac{\mathrm{d}^3\boldsymbol{r}}{2\omega_r}\,\frac{[\text{Non-analytic}]}{[\text{Complicated}]}$$

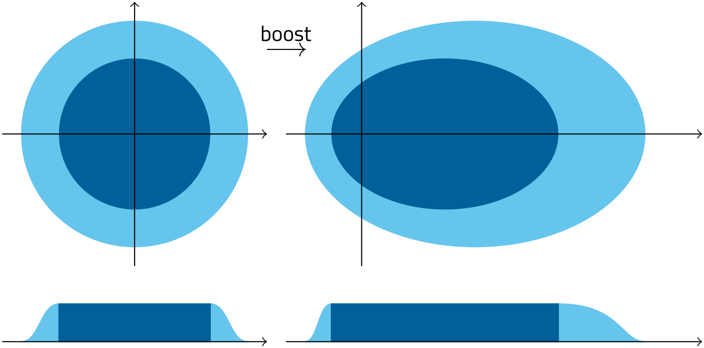


- analytic, **has poles**
- **non-analytic**, smooth

Divide & conquer the bull's head

The integral

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{[\text{Complicated angular dependence}]}{[\text{Much simpler}]}$$



Divide & conquer the bull's head

The integral

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{[\text{Simple}] - [\text{Numerics-friendly}]}{[\text{Much simpler}]}$$