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Fermions localized on solitons in flat and curved space-time

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and N Sawado

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Outline

- Warming up: Fermions localized by kinks in 1+1 dim
- Fermions localized by baby Skyrmions in 2+1 dim
- Backreaction of the fermions
- Fermionic zero mode localized on the non-Abelian monopole
- Self-gravitating non-Abelian monopole coupled to fermions
- Self-gravitating Skyrmion coupled to fermions
- Summary and outlook

Fermions localization on the kink in 1+1 dim

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - \frac{1}{2}\left(\phi^2 - 1\right)^2$$

R.Jackiw and C.Rebbi
Phys. Rev. D13 3398 (1976)

- **Field equations:**

$$i\gamma^\mu \partial_\mu \psi = g\phi\psi\,; \qquad \partial_\mu \partial^\mu \phi = 2\phi(1-\phi^2) - g\bar{\psi}\psi$$

Fixed background (g≪1):

$$\psi = e^{-i\epsilon t}\begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} \qquad \int dx\, |\bar{\psi}\psi| = 1$$

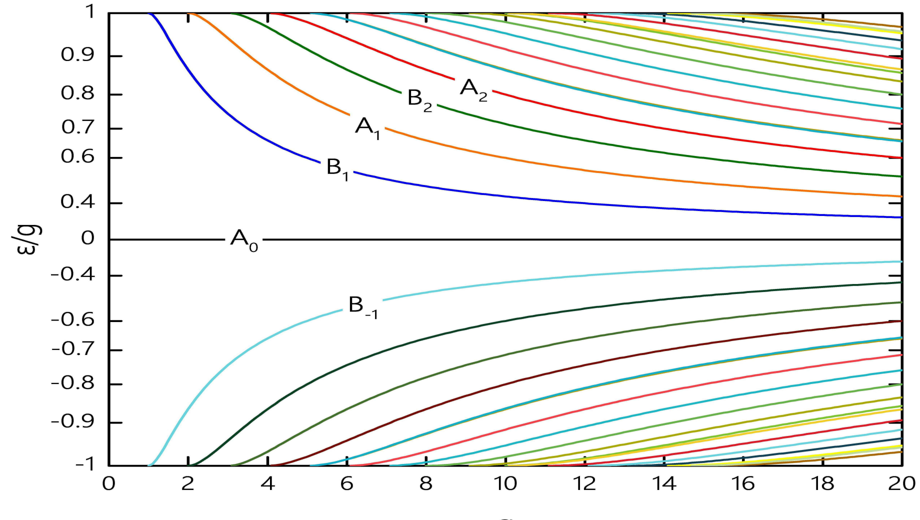
$$\phi_K = \tanh x \qquad |\epsilon| \le g$$

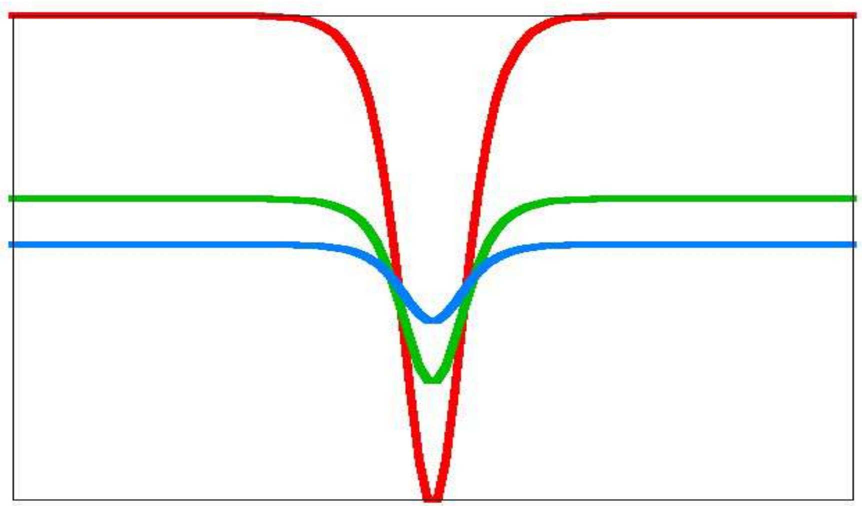
$$(\partial_x + g\tanh x)v_1 = -\epsilon v_2$$
$$(\partial_x - g\tanh x)v_2 = \epsilon v_1$$

$\Rightarrow$

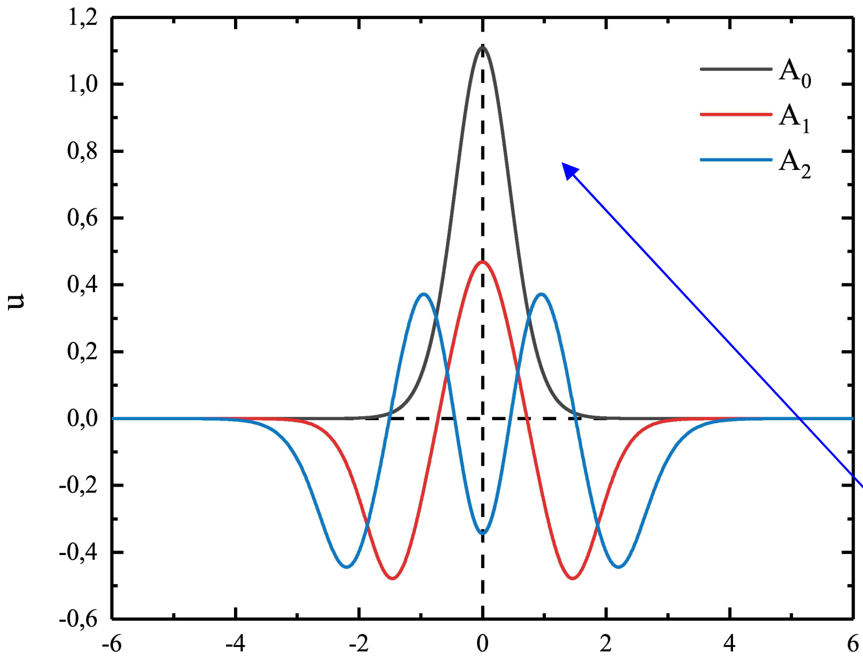
$$\left(-\partial_x^2 + U_\pm(x)\right)v_{1,2} = \epsilon^2 v_{1,2}$$

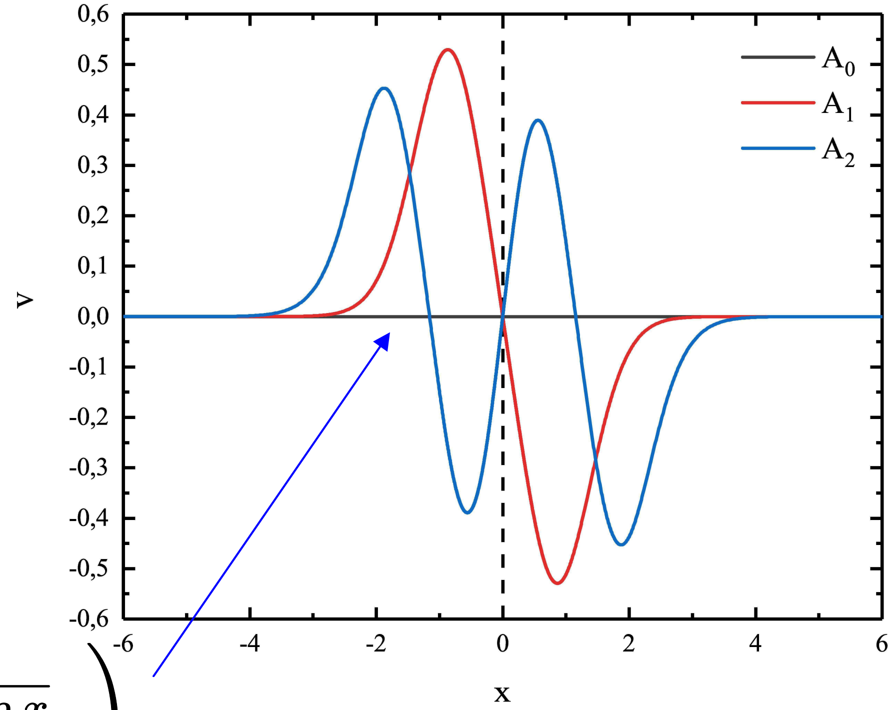
$$U_\pm(x) = g^2 - g(g \pm 1)\mathrm{sech}^2 x$$





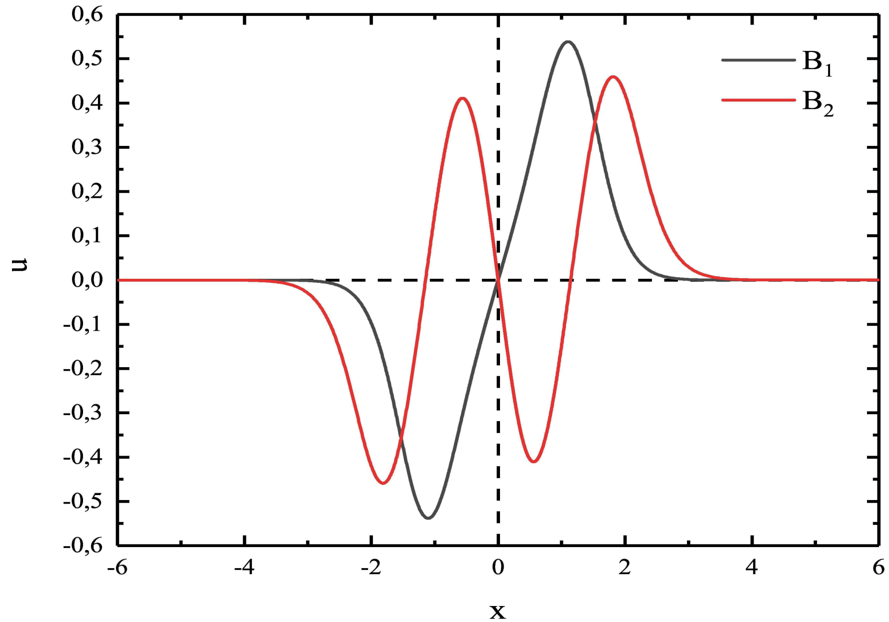
Fermionic modes

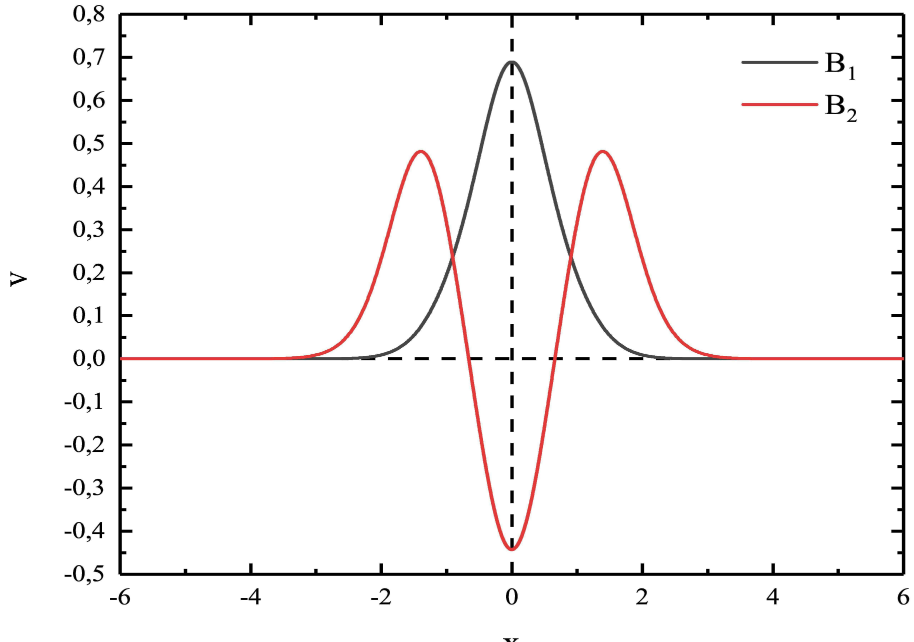




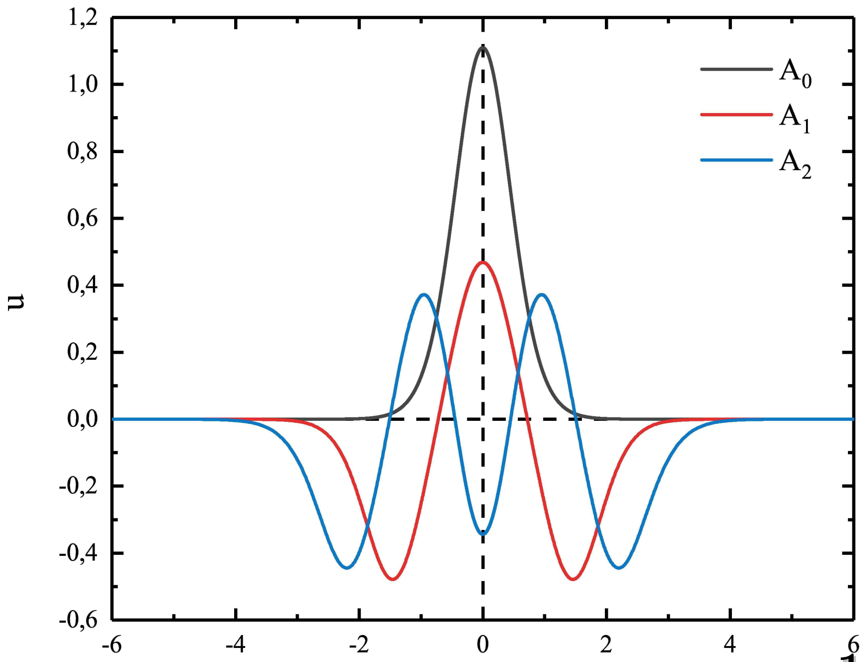
A$_0$ mode

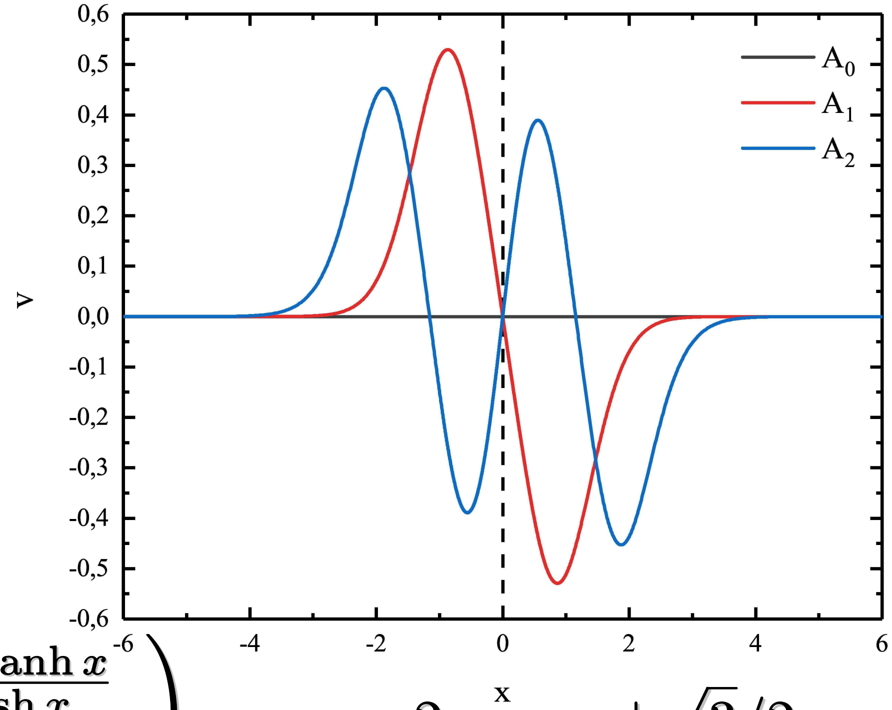
$$\psi_0 = \frac{1}{2}\begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$





Fermionic modes

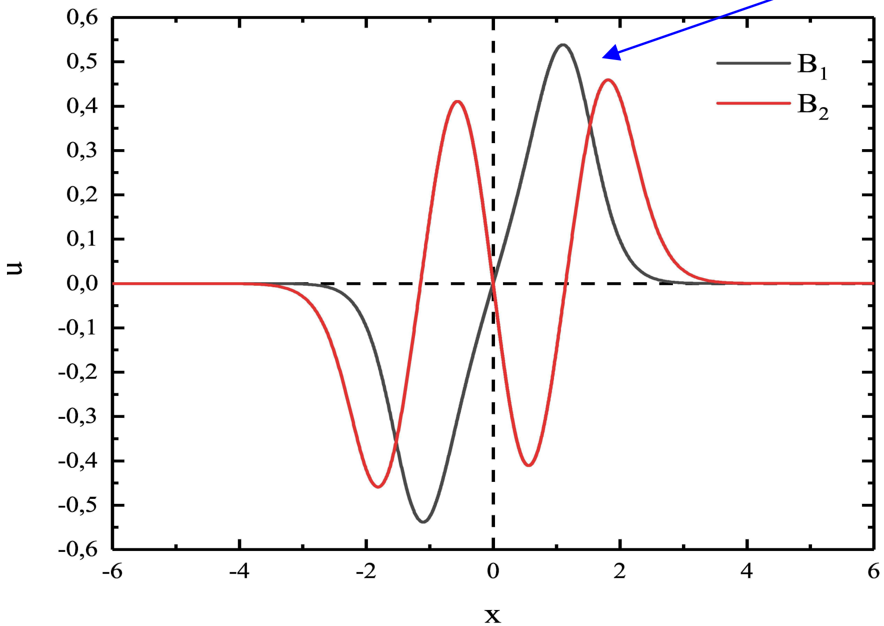


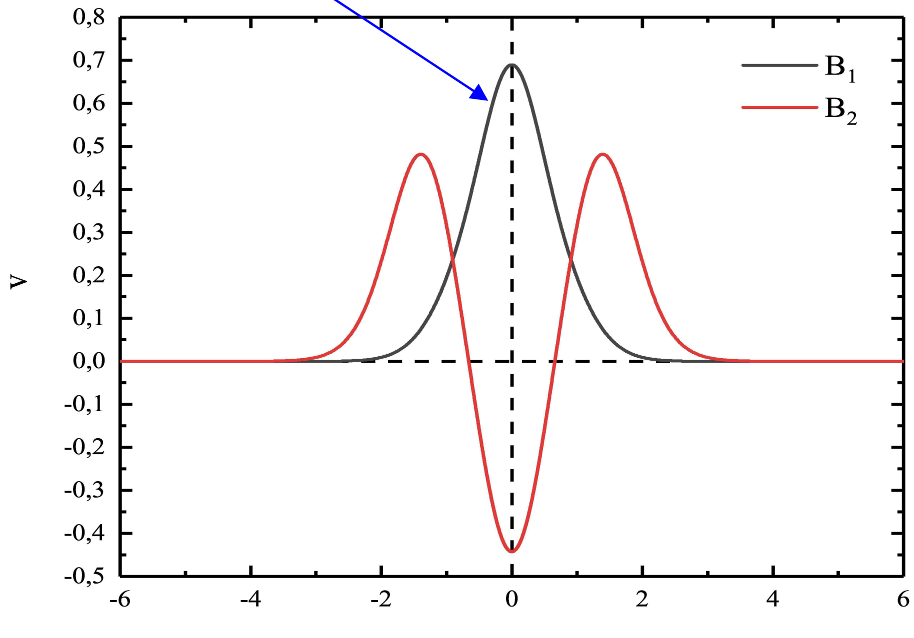


B$_1$ mode

$$\psi_1 = \frac{1}{2}\begin{pmatrix} \frac{\sqrt{3}\tanh x}{\cosh x} \\ \frac{1}{\cosh^2 x} \end{pmatrix}$$

$$g = 2, \quad \varepsilon = \pm\sqrt{3}/2$$





N=1 SUSY kink

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - \frac{1}{2}(\phi^2 - 1)^2$$

⇓

$$L_{N=1} = (\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + F^2 + 2FW - W'\bar{\psi}\psi$$

F - auxiliary field: $F = -W$

⇓

$$L_{N=1} = (\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - W'\bar{\psi}\psi - W^2$$

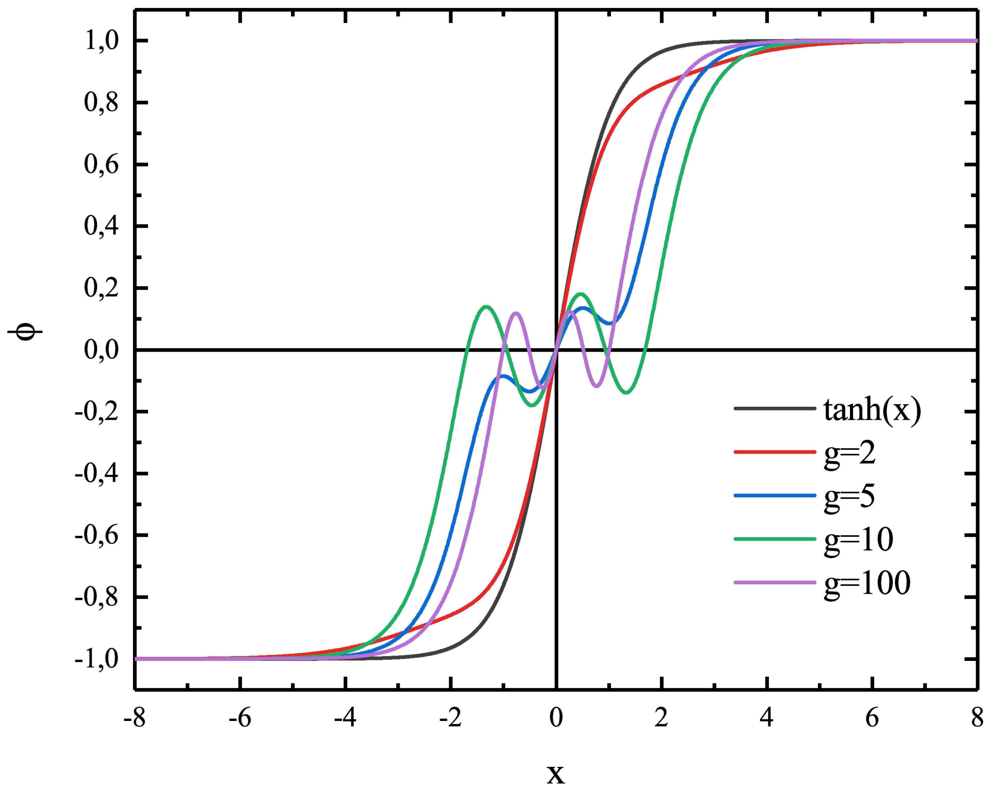
$$W[\phi] = \frac{1}{\sqrt{2}}(\phi^2 - 1)$$

SUSY transformations: $\delta\phi = \eta\psi; \qquad \delta\psi = \eta\left(\gamma^\mu \partial_\mu \phi - W\right)$

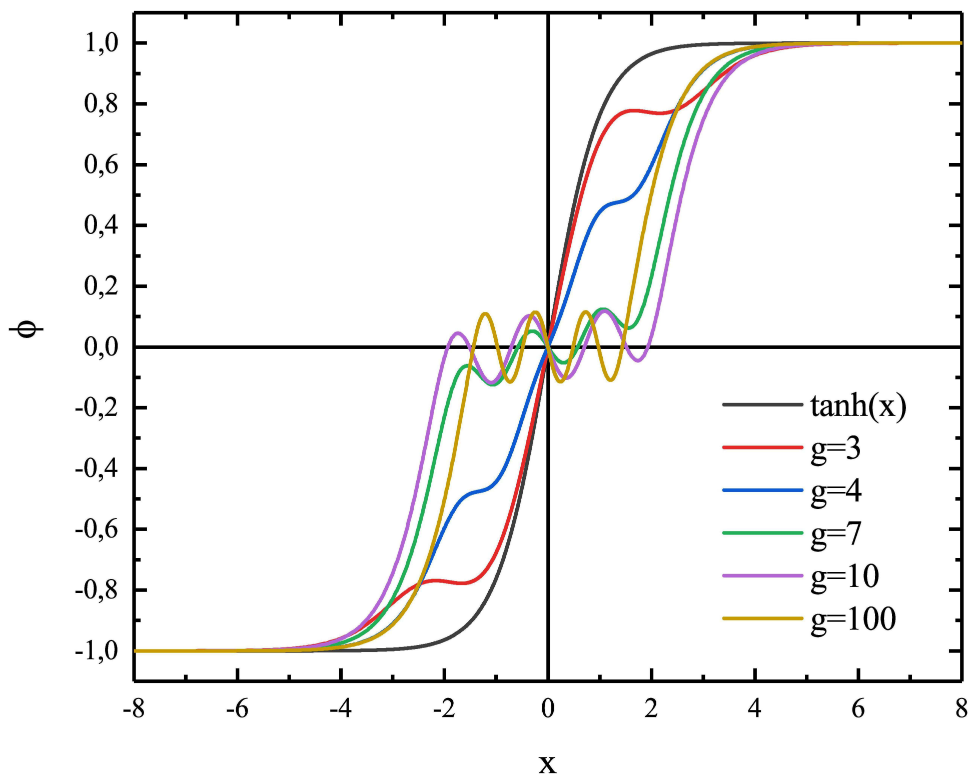
Fermionic zero mode of the kink:
Grassmann-valued deformation of the bosonic field

$$\psi_0 = \frac{1}{2}\begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$

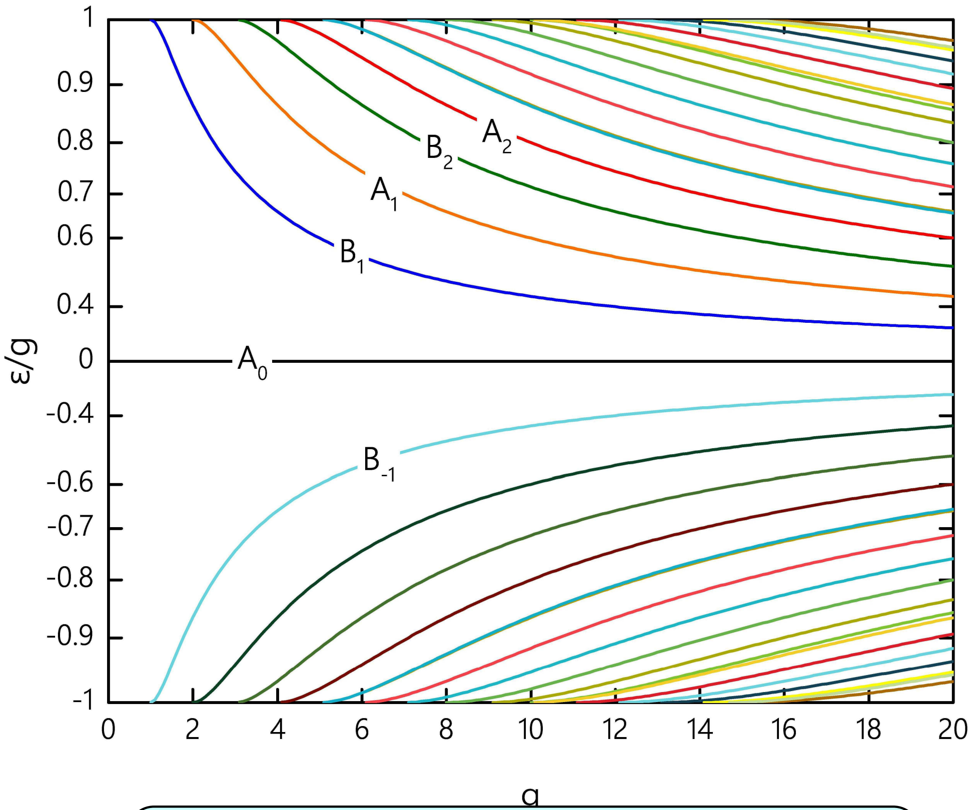
Backreaction of the fermions

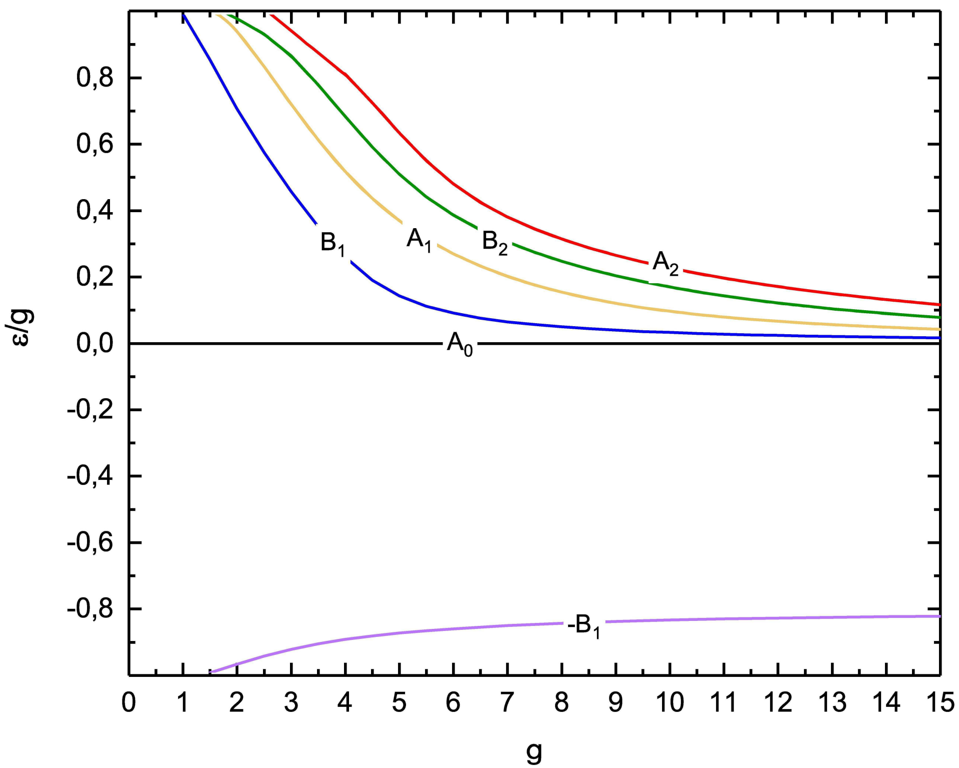


Kink + A_1 mode



Kink + B_1 mode





without backreaction

with backreaction

Symmetry on a fixed background: $x \to x, \quad \phi \to -\phi, \quad uv \to uv, \quad v \to u, \quad u \to v$

Backreaction breaks the symmetry of the spectral flow

Kinks bounded by fermions

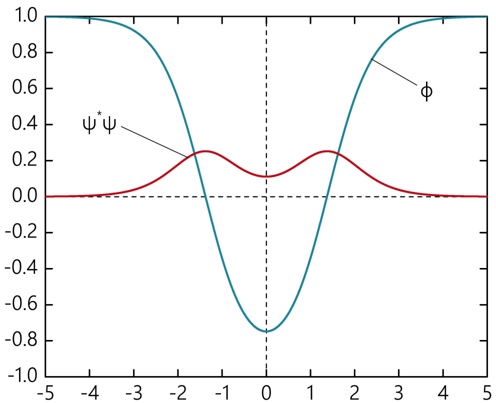
$$L = \frac{1}{2}(\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - U(\phi)$$

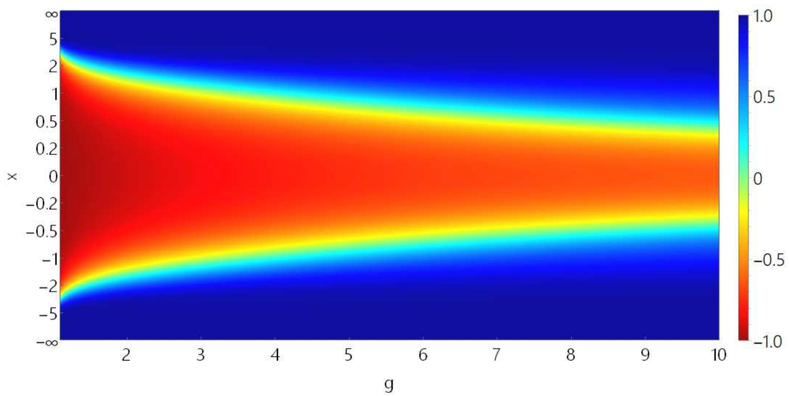
- **SG model:** $U(\phi) = 1 - \cos\phi$
- **ϕ^4 model:** $U(\phi) = \frac{1}{2}\left(1-\phi^2\right)^2$

Kinks (decoupled limit g=0): $\phi_{SG} = 4\arctan e^x, \quad \phi_{\phi^4} = \tanh x$

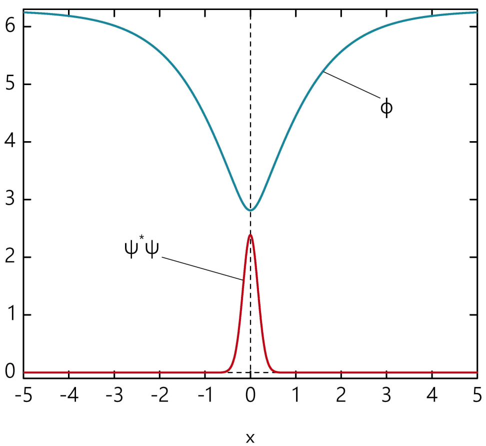
Bounded KK pair

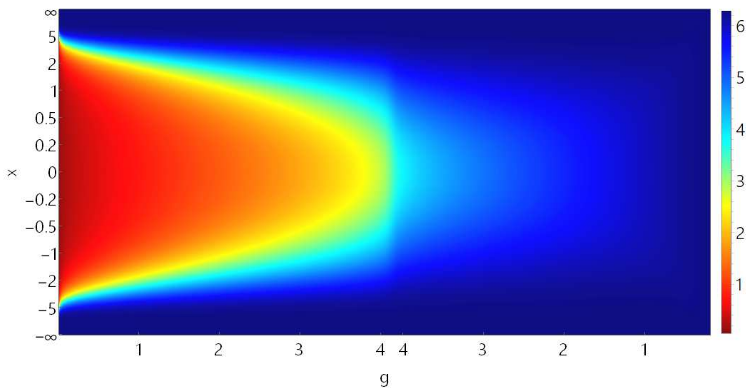
- **ϕ^4 model:**





- **SG model:**





Fermion-Skyrmion system in 2+1 dim

$$\mathcal{L} = \mathcal{L}_{Sk} + \mathcal{L}_f$$

$$\mathcal{L}_{Sk} = \frac{\kappa_2}{2}\left(\partial_\mu \vec{\phi}\right)^2 - \frac{\kappa_4}{4}\left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}\right)^2 - \kappa_0 U(\phi)$$

$$\mathcal{L}_f = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - g\vec{\tau}\cdot\vec{\phi} - m\right)\Psi$$

- **Rescaling:**

$$r \to \sqrt{\frac{\kappa_4}{\kappa_2}}r, \quad \Psi \to \sqrt{\frac{\kappa_2}{\kappa_4}}\Psi, \quad \kappa_0 \to \frac{\kappa_2^2}{\kappa_4}\kappa_0, \quad g \to \sqrt{\frac{\kappa_2}{\kappa_4}}g, \quad m \to \sqrt{\frac{\kappa_2}{\kappa_4}}m$$

$$\sqrt{\kappa_2\kappa_4} = 1\,, \quad \kappa_0\,, \quad m\,, \quad g$$

Stationary modes: $\Psi = \psi(r,\theta)e^{-i\varepsilon t}$

- **Fermionic density:** $\rho = \bar{\psi}\hat{\gamma}_3\psi = \psi^\dagger\psi, \qquad \int d^2x\ \psi^\dagger\psi = 1$

- **Hamiltonian:**

$$H = \int d^2x\ \psi^\dagger \mathcal{H}\psi + \int d^2x\left(\frac{1}{2}\left(\partial_k\vec{\phi}\right)^2 + \frac{1}{4}\left(\partial_k\vec{\phi}\times\partial_n\vec{\phi}\right)^2 + \kappa_0\ U\right)$$

- **Field equations:**

$$\mathcal{H}\psi \equiv \hat{\gamma}_3 \left(-i\hat{\gamma}_k\partial_k + g\vec{\tau}\cdot\vec{\phi} + m\right)\psi = \varepsilon\psi$$

$$\partial_\mu j^\mu = \kappa_0\ \vec{\phi}_\infty \times \vec{\phi} + g\vec{\phi} \times \left(\psi^\dagger\vec{\tau}\psi\right)$$

Scalar current

$$j_\mu = \vec{\phi} \times \partial_\mu\vec{\phi} + \partial_\nu\vec{\phi}\left(\partial^\nu\vec{\phi}\cdot(\vec{\phi}\times\partial_\mu\vec{\phi})\right)$$

Spin-Isospin fermions

- **Rotationally invariant configuration:**

Q=n:

$$\phi^1 = \sin f(r)\cos n\theta;$$
$$\phi^1 = \sin f(r)\sin n\theta;$$
$$\phi^3 = \cos f(r)$$

$$\psi^{(i)} = \mathcal{N}^{(i)}\begin{pmatrix} v_1(r)e^{il\theta} \\ iv_2(r)e^{i(l+n)\theta} \\ u_1(r)e^{i(l+1)\theta} \\ iu_2(r)e^{i(l+n+1)\theta}\end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} g\cos f + m & ge^{-in\theta}\sin f & -e^{-i\theta}\left(\partial_r - \frac{i\partial_\theta}{r}\right) & 0 \\ ge^{in\theta}\sin f & -g\cos f + m & 0 & -e^{-i\theta}\left(\partial_r - \frac{i\partial_\theta}{r}\right) \\ e^{i\theta}\left(\partial_r + \frac{i\partial_\theta}{r}\right) & 0 & -g\cos f - m & -ge^{-in\theta}\sin f \\ 0 & e^{i\theta}\left(\partial_r + \frac{i\partial_\theta}{r}\right) & -ge^{in\theta}\sin f & g\cos f - m\end{pmatrix}$$

- **Generalized angular momentum:**

$$J_k = -i\nabla_k + \frac{\gamma_k}{2}\otimes\mathbb{I} + \mathbb{I}\otimes\frac{\tau_k}{2}$$

$$J_3 = -i\frac{\partial}{\partial\theta} + \frac{\hat{\gamma}_3}{2} + n\frac{\tau_3}{2}$$

$$[\mathcal{H}, J_3] = 0\,, \qquad J_3\psi = \kappa\psi; \qquad \kappa = \frac{1}{2}(1+n+2l)$$

Ground state: $\kappa = 0$ → $l = -\frac{1+n}{2}\,, \qquad l = 0, \quad n = -1$

- **Asymptotic expansion:**

$r \to \infty$

$$\vec{\phi} \approx \vec{\phi}_\infty + \delta\vec{\phi} \quad \rightarrow \quad \begin{cases} (-i\hat{\gamma}_k\partial_k + g\tau_3 + m)\psi = 0\,, \\ (\Delta - \kappa_0)\delta\vec{\phi} = 0\,. \end{cases}$$

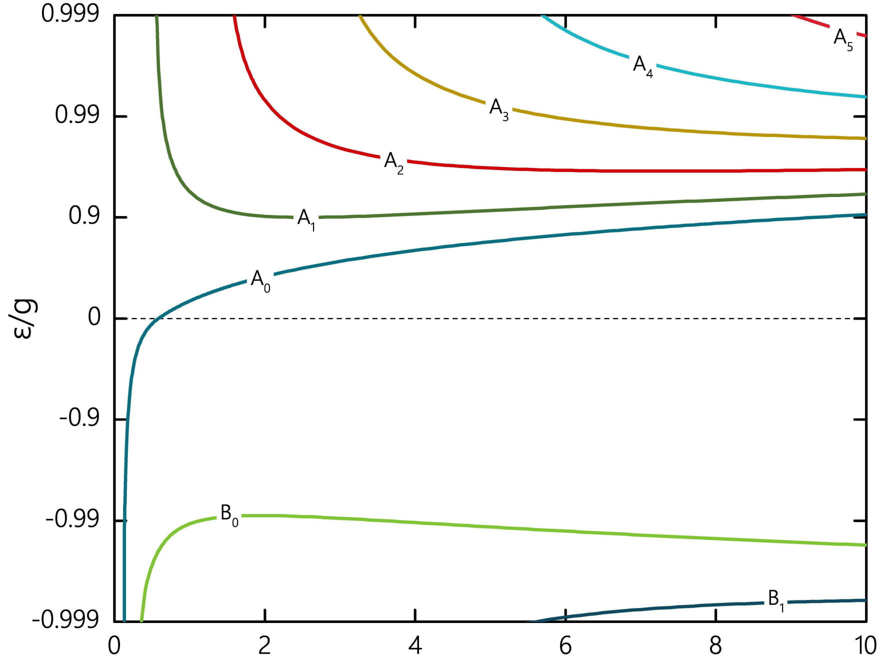
Pair of orthogonal scalar dipoles

$$\delta\vec{\phi} \sim K_n(\sqrt{\kappa_0}r)\begin{pmatrix} \cos(n\theta - \chi) \\ \sin(n\theta - \chi) \\ 0 \end{pmatrix}$$

$$\begin{cases} (\Delta - 4(g\pm m)^2)u_{1,2} = \varepsilon^2 u_{1,2} \\ (\Delta - 4(g\pm m)^2)v_{1,2} = \varepsilon^2 v_{1,2} \end{cases} \quad \rightarrow \quad \begin{array}{l} v_1 \sim e^{il(\varphi-\chi)}K_l(\sqrt{4(g+m)^2-\varepsilon^2}\,r) \\ v_2 \sim e^{i(l+n)(\varphi-\chi)}K_{l+n}(\sqrt{4(g-m)^2-\varepsilon^2}\,r) \end{array}$$

Asymptotic fermionic field: a pair of orthogonal 2^l-poles, together with a pair of collinear 2^{l+n}-poles

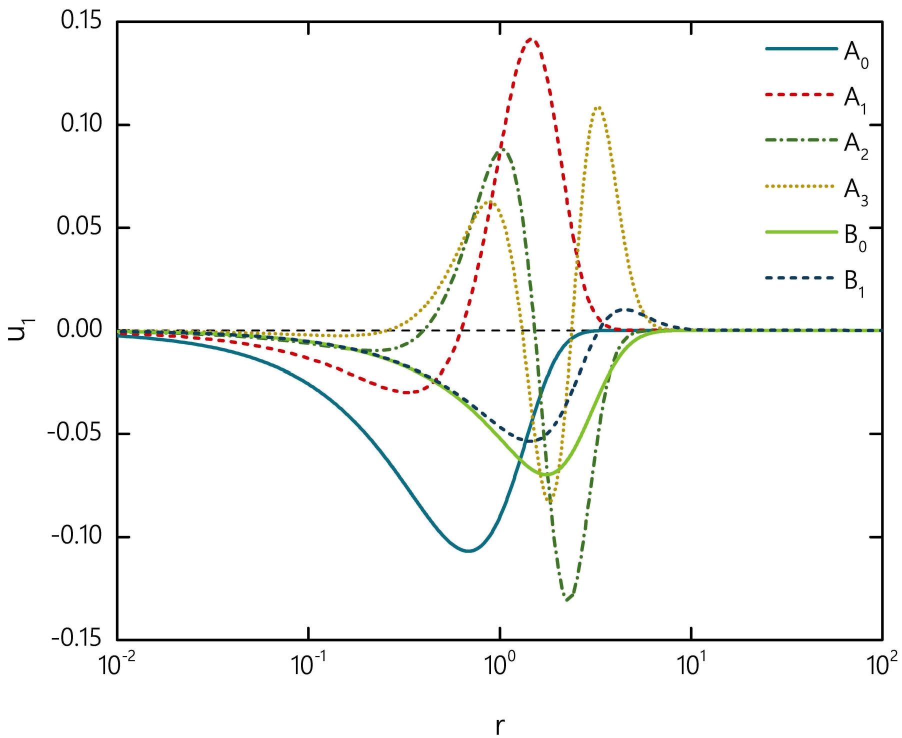
Fermions couped to a baby Skyrmion

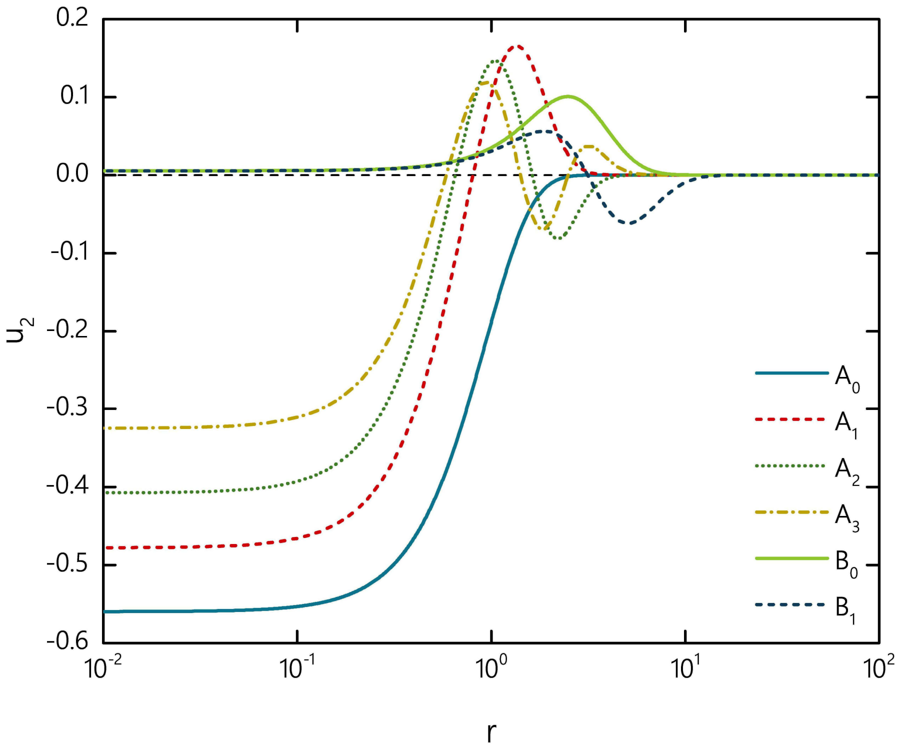


Massless localized fermions:

$$m = 0, \quad u_2 = v_1, \quad u_1 = -v_2$$

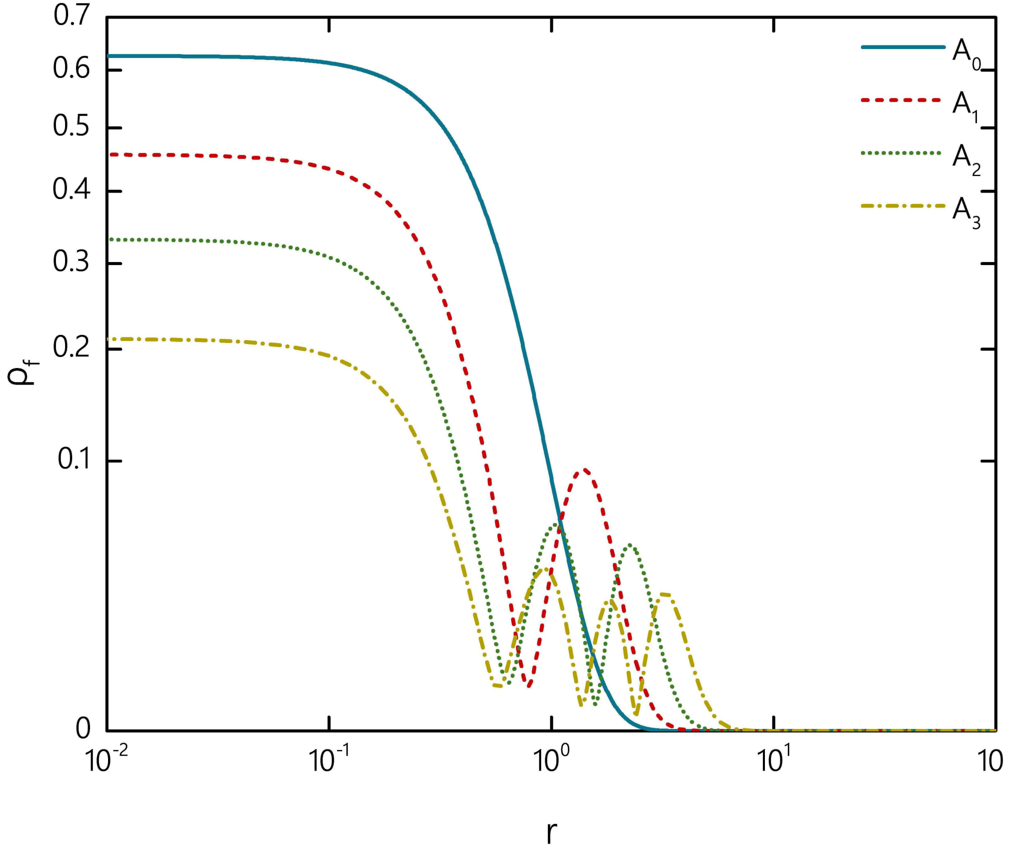
- Solutions found numerically by the shooting method;
- Spectral flow is in agreement with the index theorem
- There are two types of the modes, the solutions are characterized by the number of nodes



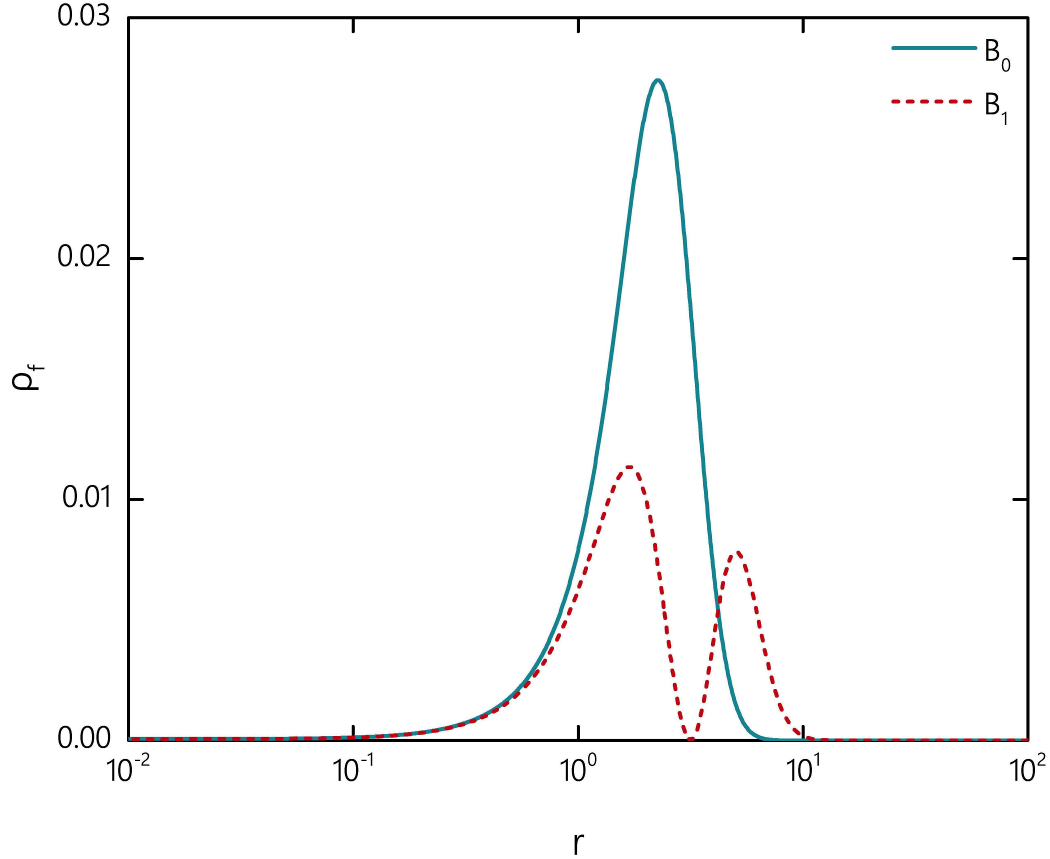


Fermionic density

$$m = 0, \quad g = 1, \quad \kappa_0 = 0.1$$

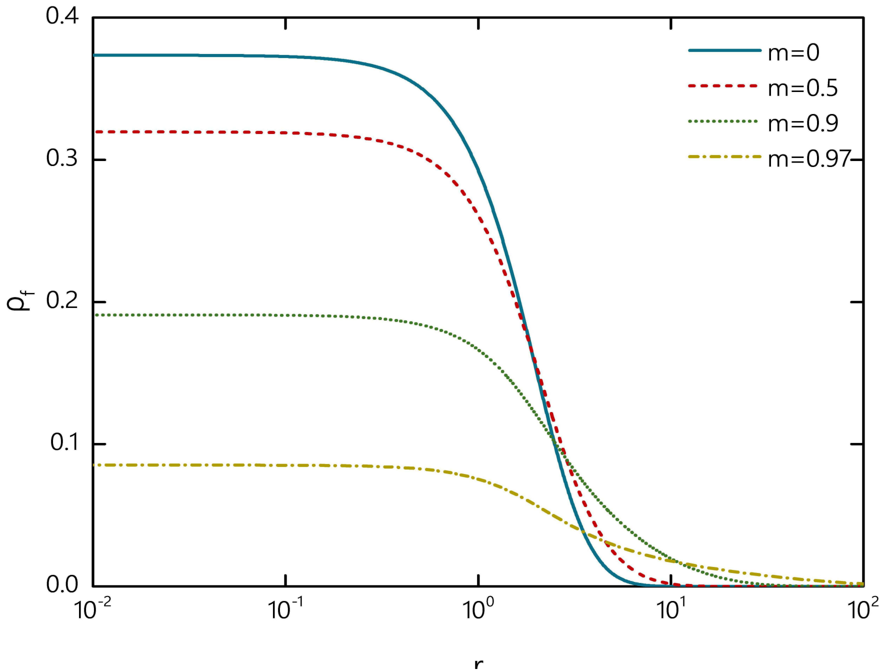


A-modes



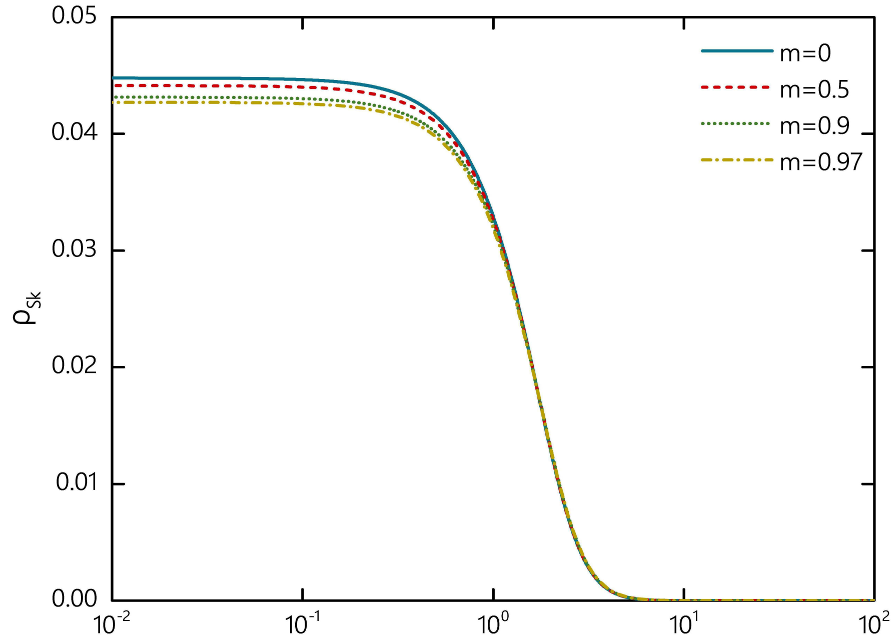
B-modes

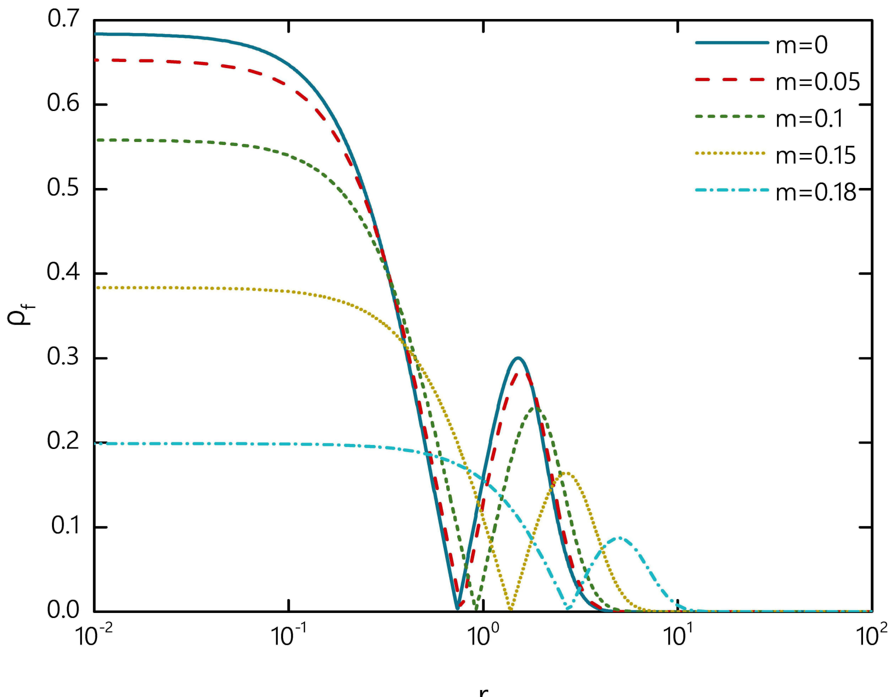
Fermion-Skyrmion system: vanishing potential



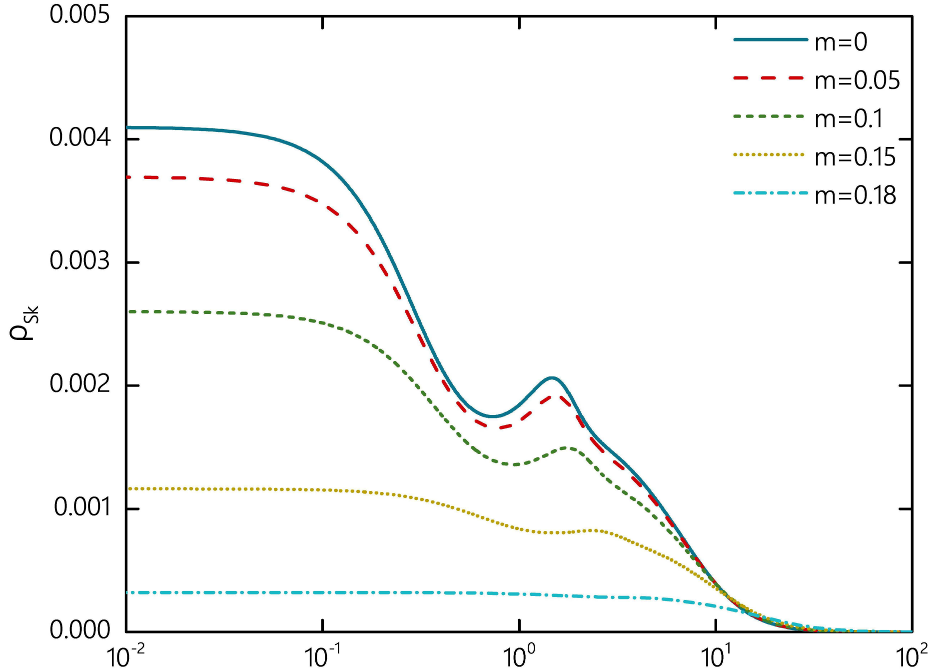
$g = 1,$
$\kappa_0 = 0$

A_0





A_1



Magnetic Skyrmions

- **Heisenberg model:** $H = \sum_{i<j=1} J_{ij} S_i S_j$ ⇒ $\frac{J}{2}\int d^2x\ \partial_n\vec{m}\cdot\partial_n\vec{m}$

classical nearest neighbour interaction

mean field approximation

- **Dzyaloshinskii-Moriya interaction:** $H_{DM} = \sum_{ij} D_{ij}(S_i \times S_j)$ ⇒ $D\int d^2x\ \vec{m}\cdot(\nabla\times\vec{m})$

- **Zeeman interaction:** $H_{ext} = \vec{B}\cdot\vec{m}$

Chiral magnetic Skyrmions:

$$E = \int d^2x\ \left(\frac{J}{2}(\nabla\vec{m})^2 + D\vec{m}\cdot(\nabla\times\vec{m}) - \vec{B}\cdot\vec{m}\right)$$

Magnetic Skyrmions

Field equation: $J\Delta\vec{m} - 2D\nabla\times\vec{m} + \vec{B} = 0$

$$\alpha = 2D/J, \quad \mu = |B|/J$$

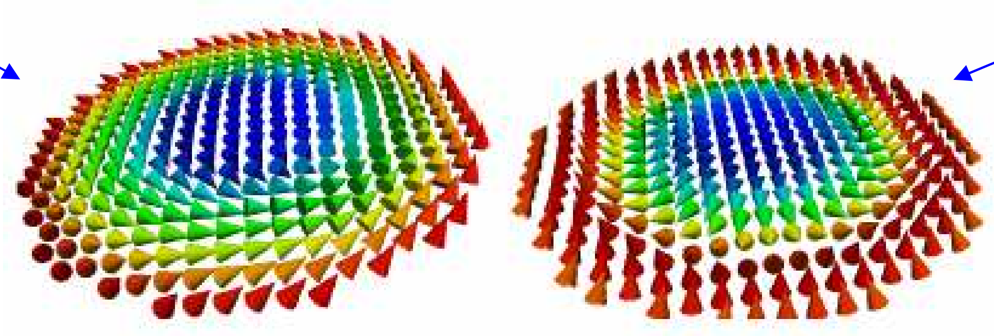
$$m_1 = \sin f(r)\cos(Q\varphi - \delta);$$
$$m_2 = \sin f(r)\sin(Q\varphi - \delta);$$
$$m_3 = \cos f(r)$$

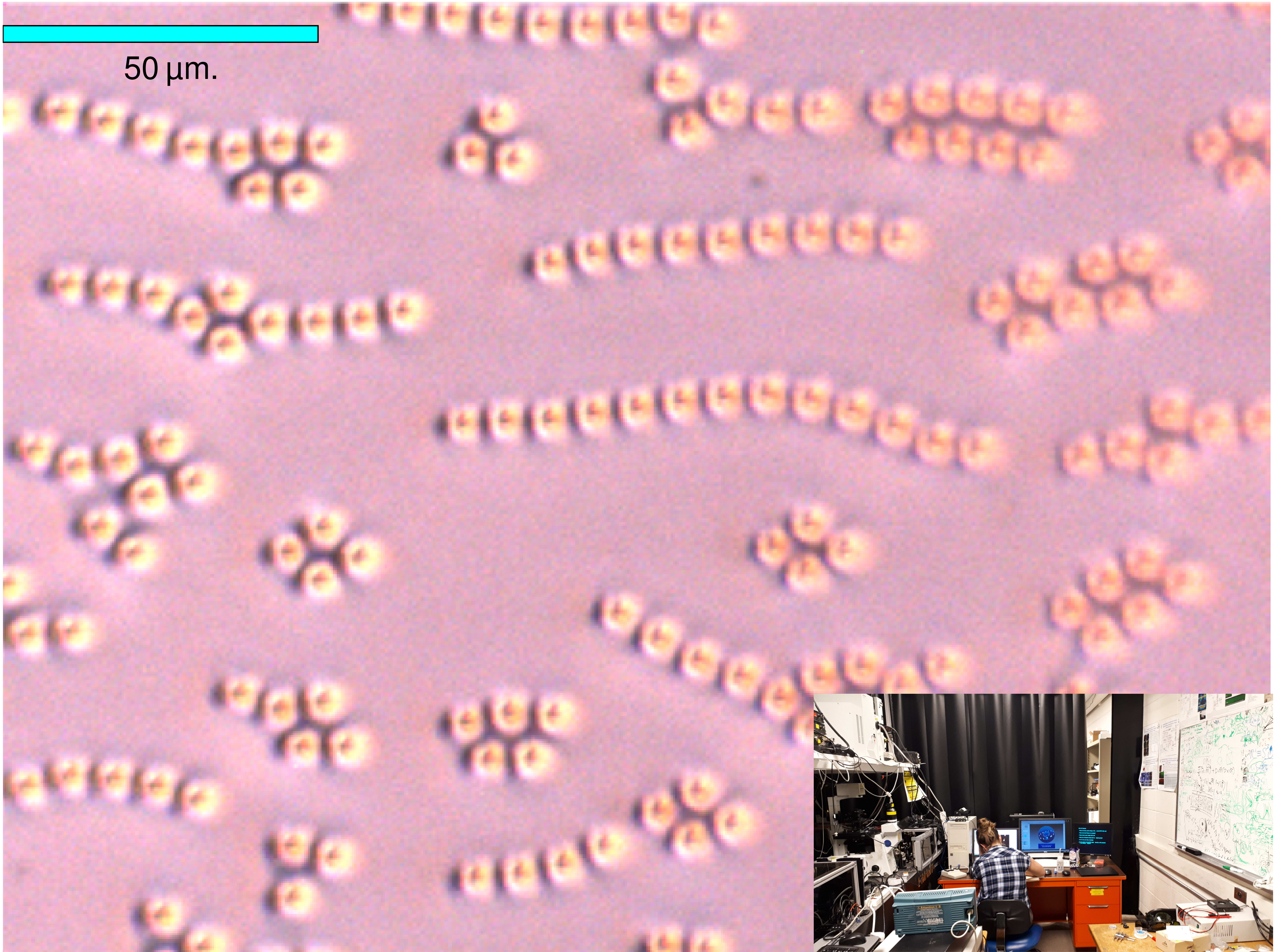
$$E = 2\pi\int rdr\Big\{\frac{1}{2}f'^2 + \frac{Q^2}{2r^2}\sin^2 f - \mu\cos f + \frac{\alpha}{Q-1}\sin(Q\pi)\sin(\delta+Q\pi)\left(f' + \frac{Q}{2r}\sin(2f)\right)\Big\}$$

$$\frac{\alpha}{Q-1}\sin(Q\pi)\sin(\delta+Q\pi) = \begin{cases}\alpha\sin\delta\ , & \text{if} \quad Q=1\\ 0\ , & \text{if} \quad Q\neq 1\,.\end{cases}$$

Bloch-type skyrmions: $\delta = \pm\ \pi/2$

Nėel-type skyrmions: $\delta = 0,\ \pi$




50 µm.

Fermion-magnetic Skyrmion system in 2+1 dim

$$\mathcal{H} = \mathcal{H}_{Sk} + \mathcal{H}_f$$

$$\mathcal{H}_{Sk} = \frac{J}{2}\left(\nabla\vec{\phi}\right)^2 + D\vec{\phi}\cdot\left(\nabla\times\vec{\phi}\right) - \vec{B}\cdot\vec{\phi}$$

$$\mathcal{H}_f = \Psi^\dagger\hat{\gamma}^3\left(-i\hat{\gamma}^k\partial_k + e\hat{\gamma}^k A_k + m + g\vec{\tau}\cdot\vec{\phi}\right)\Psi.$$

$$A_k = \frac{B}{2}(0, -y, x)$$

Stationary configuration: $\vec{\phi} = \vec{\phi}(r,\theta), \quad \Psi = \psi(r,\theta)e^{-i\varepsilon t}$

- **Field equations:**

$$\left\{\begin{array}{l} \Delta\vec{\phi} - 2\nabla\times\vec{\phi} + \vec{B} - g\psi^\dagger\hat{\gamma}_3\vec{\tau}\psi = 0 \\ \hat{\gamma}^3\left(-i\hat{\gamma}^k\partial_k + e\hat{\gamma}^k A_k + m + g\vec{\tau}\cdot\vec{\phi}\right)\psi = \varepsilon\psi \end{array}\right.$$

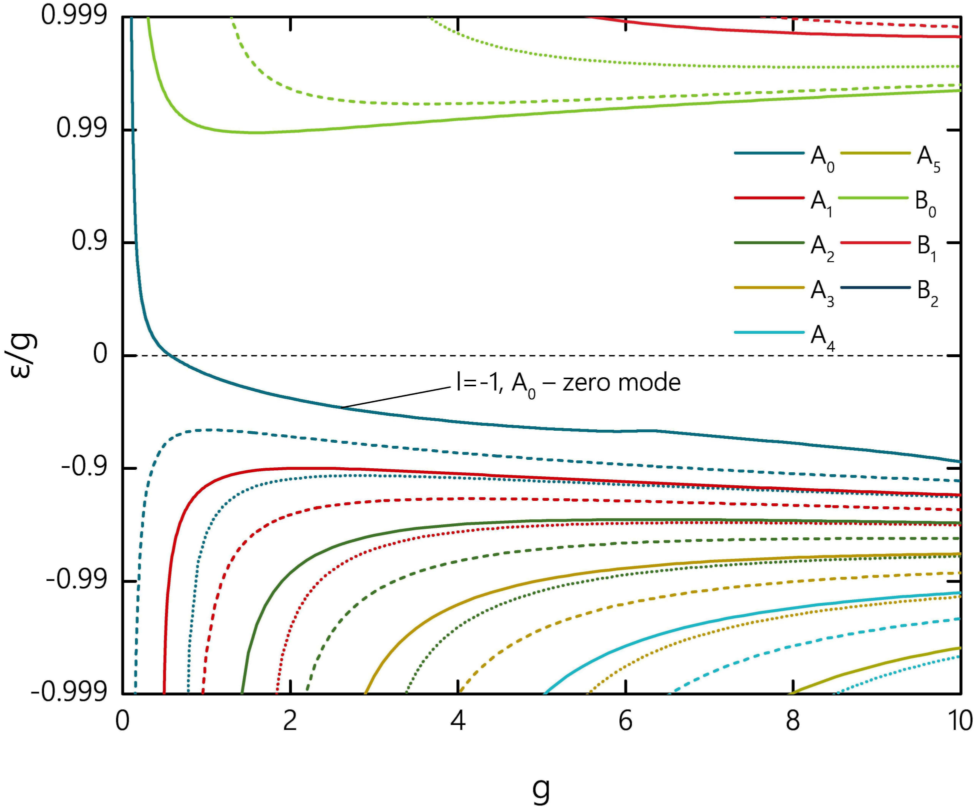
Spin-Isospin fermions

$$\psi^{(i)} = \mathcal{N}^{(i)}\begin{pmatrix} v_1(r)e^{il\varphi} \\ iv_2(r)e^{i(l+n)\varphi} \\ u_1(r)e^{i(l+1)\varphi} \\ iu_2(r)e^{i(l+n+1)\varphi} \end{pmatrix}$$

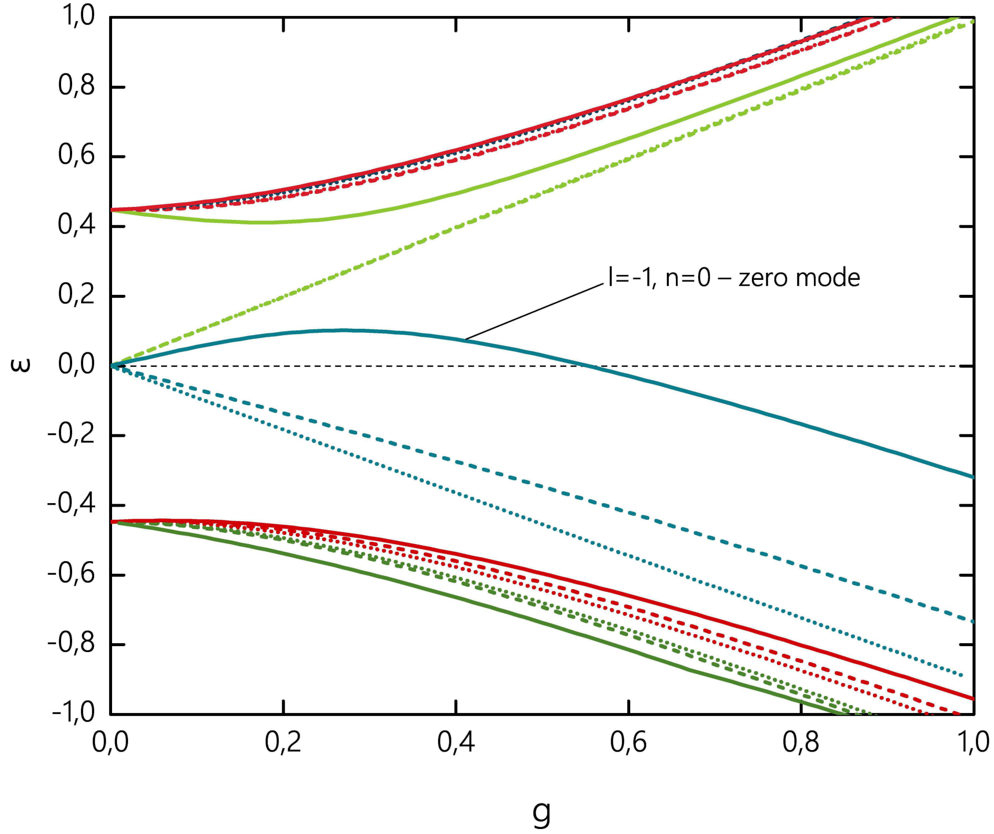
- **Generalized angular momentum:**

$$J_k = -i\nabla_k + \frac{\gamma_k}{2}\otimes\mathbb{I} + \mathbb{I}\otimes\frac{\tau_k}{2}$$

e=0



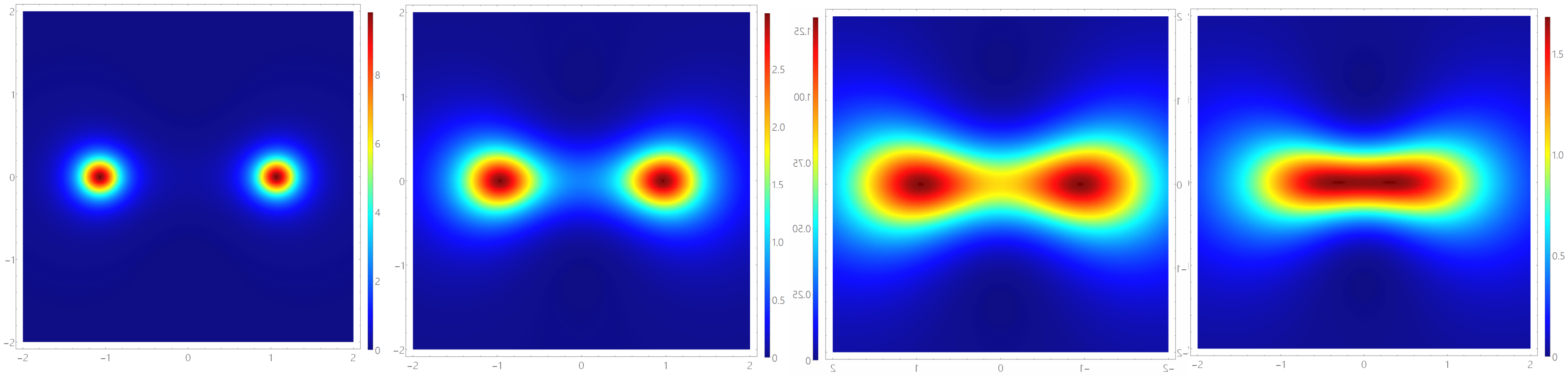
e=-1

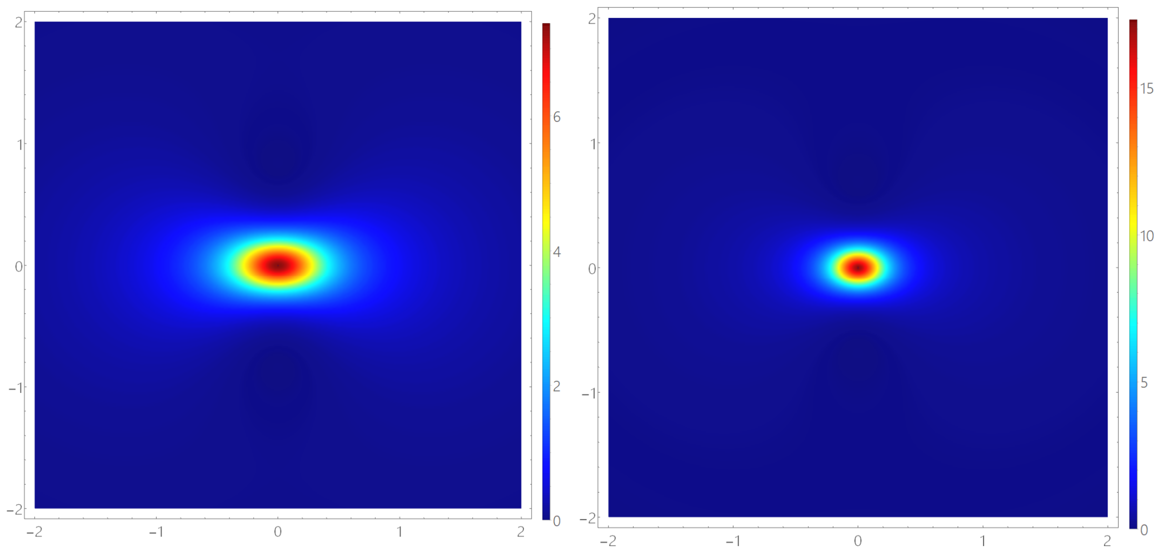


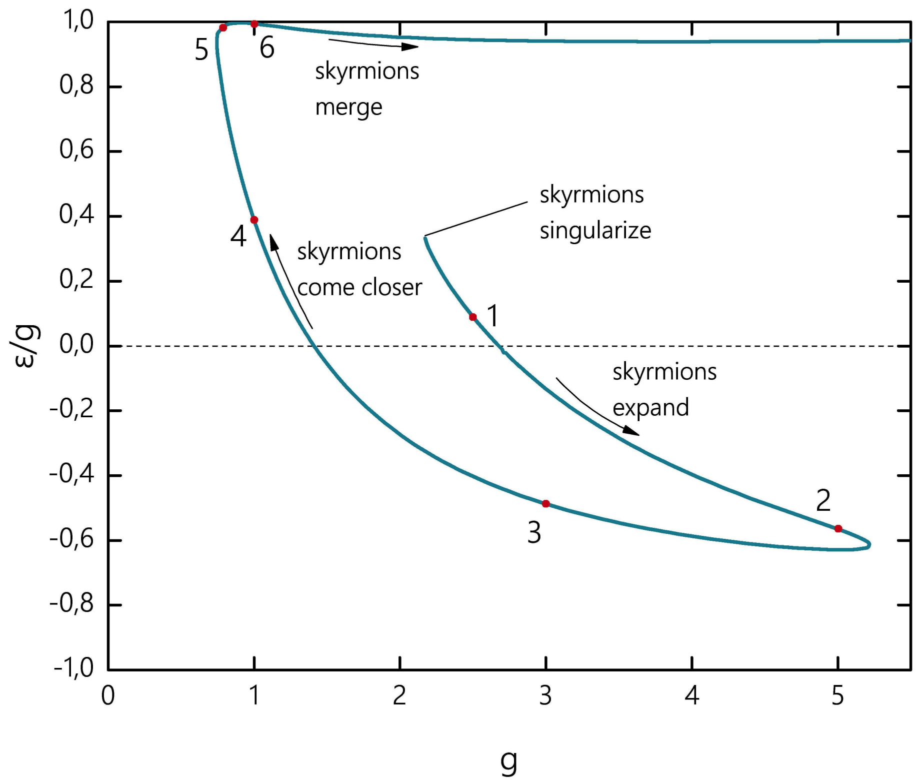
g=0: Landau levels

$$\varepsilon_k^2 = m^2 + B\left(|e|(2k+1) \pm e\right)$$

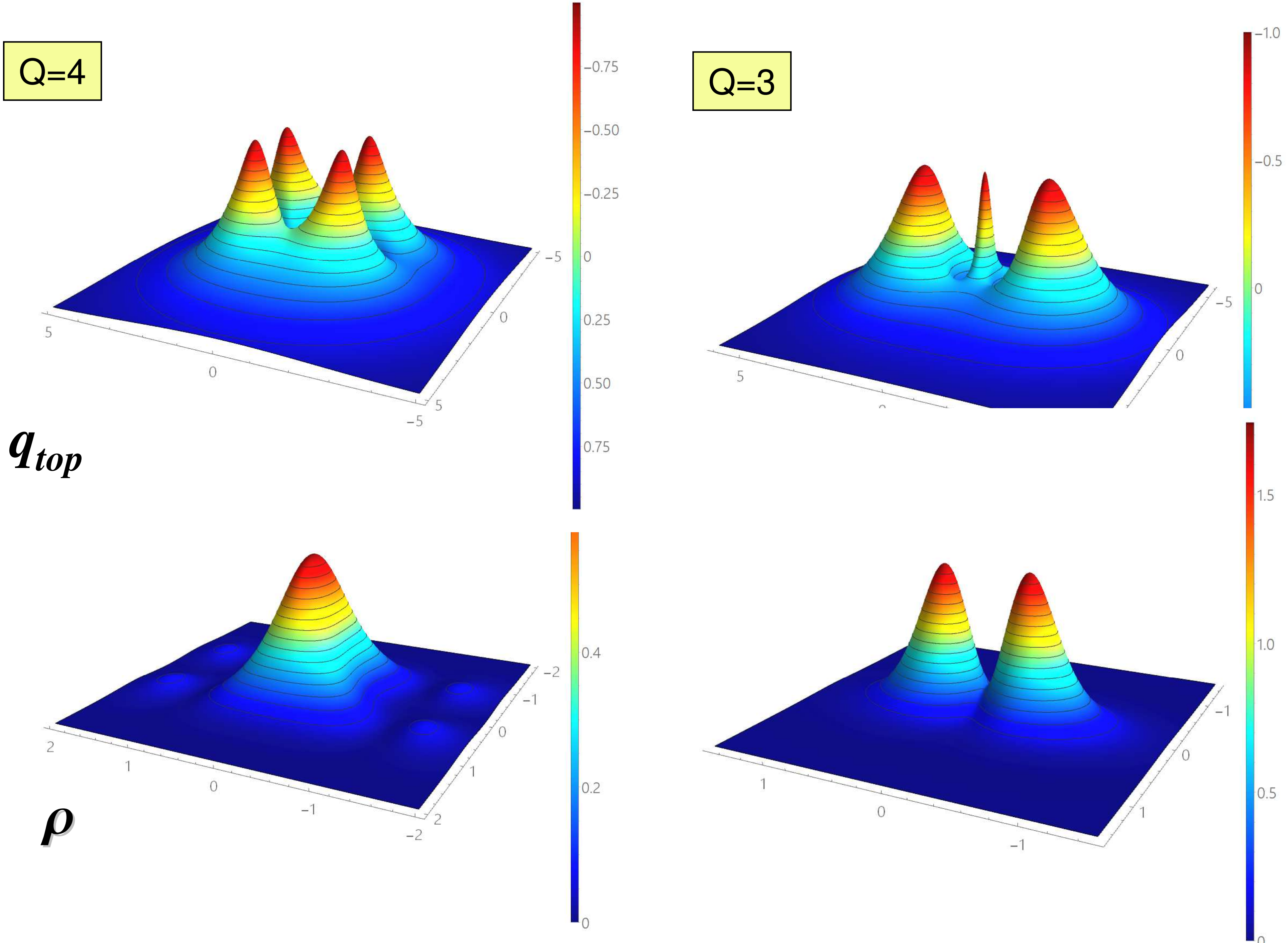
Chiral Fermions bounded by a fermionic mode







Magnetic Skyrmions bounded by a fermionic mode



Non-Abelian SU(2) monopole: fermionic zero mode

$$L_{YMH} = \frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) - \mathrm{Tr}\,(D_\mu\Phi)^2 + \lambda \mathrm{Tr}\,(\Phi^2 - a^2)^2 \qquad D_\mu = \partial_\mu - igA_\mu^a\frac{\sigma^a}{2}$$

- **'t Hooft–Polyakov monopole:** $\Phi : S^2_\infty \mapsto S^2_{vac}, \quad \Pi_2(S^2) = \mathbb{Z} \qquad \Phi = \phi^a\sigma^a$

$$\phi^a = \frac{r^a}{gr^2}H(r), \quad A_n^a = \varepsilon_{amn}\frac{r^m}{gr^2}[1 - W(r)], \quad A_0^a = 0$$

+ fermions:

$$L_{\rm sp} = \frac{i}{2}\left((\hat{\not{D}}\bar\psi)\psi - \bar\psi\hat{\not{D}}\psi\right) - m\bar\psi\psi - \frac{i}{2}h\bar\psi\gamma^5\phi\psi$$

$$\begin{cases} D_\nu F^{a\nu\mu} = -e\epsilon^{abc}\phi^b D^\mu\phi^c - \dfrac{e}{2}\bar\psi\gamma^\mu\sigma^a\psi\,, \\ D_\mu D^\mu\phi^a + \lambda\phi^a\left(\phi^2 - 1\right) + ih\bar\psi\gamma^5\sigma^a\psi = 0\,, \\ \imath\hat{\not{D}}\psi - i\dfrac{h}{2}\gamma^5\sigma^a\phi^a\psi - m\psi = 0 \end{cases}$$

- **spin-isospin fermions:**

$$\psi = e^{-i\omega t}\begin{pmatrix}\chi \\ \eta\end{pmatrix}$$

$$\int d^3x\psi^\dagger\psi = 1$$

$$\chi = \frac{u(r)}{\sqrt{2}}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \qquad \eta = i\frac{v(r)}{\sqrt{2}}\begin{pmatrix}\sin\theta e^{-i\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{i\varphi}\end{pmatrix}.$$

Two dimensionless parameters of the model:

$$\beta = \frac{M_s}{M_v}, \quad h = \frac{2M_f}{M_v}$$

- **m=0**

$$u' + u\left(\frac{1-W}{x} - \frac{h}{2}H\right) = 0, \quad v' + v\left(\frac{1+W}{x} + \frac{h}{2}H\right) = 0$$

- **BPS limit:** $\beta \to 0, \quad \hat{\phi}^a \underset{r\to\infty}{\to} \hat{r}^a$ ⇒ $W = \frac{x}{\sinh x}, \quad H = \coth x - \frac{1}{x}, \quad x = agr$

Generalized angular momentum: $\vec{J} = \vec{L} + \vec{S} + \vec{T} = \vec{L} + \vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{\tau}$

Spherical symmetry: $\vec{S} + \vec{T} = 0$

$$v = 0\,, \quad u \sim e^{-\int dx\left[\frac{1-W(x)}{x} - \frac{h}{2}H(x)\right]}$$

Fermionic zero mode (ω=0)

- **BPS limit:** $v = 0\,, \quad u = \frac{1}{\cosh^2(x/2)} \qquad (h = -2)$

Fermions+GR (Dirac stars)

H Weyl and V Fock (1929)

$$ds^2 = \eta_{ab}(\mathbf{e}^a_\mu dx^\mu)(\mathbf{e}^b_\nu dx^\nu) \qquad \gamma^\alpha = \mathbf{e}^\alpha_\mu \gamma^\mu$$

$$\mathcal{L}_{sp} = -i\frac{1}{2}\left(\gamma^\mu D_\mu \bar{\Psi}\Psi - \bar{\Psi}\gamma^\mu D_\mu \Psi\right) + \mu\bar{\Psi}\Psi$$

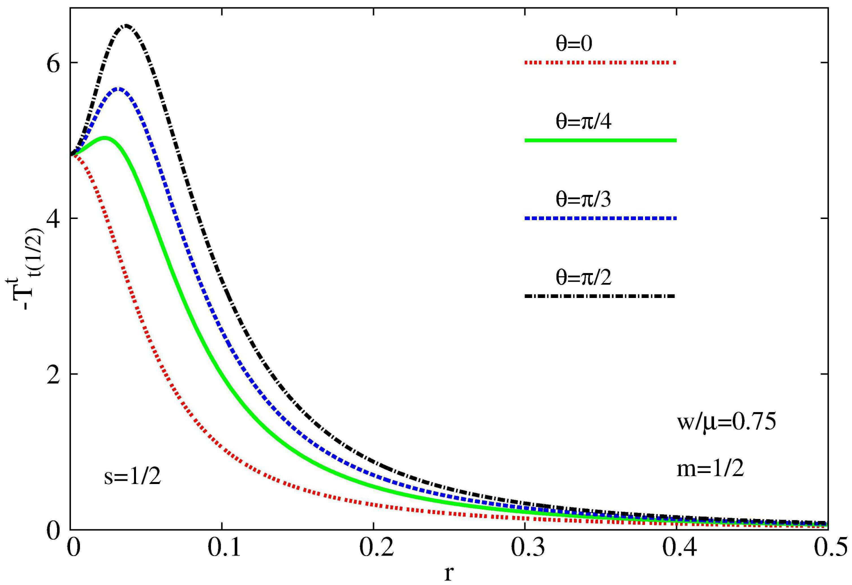
$$D_\mu \Psi = (\partial_\mu - \Gamma_\mu)\Psi$$

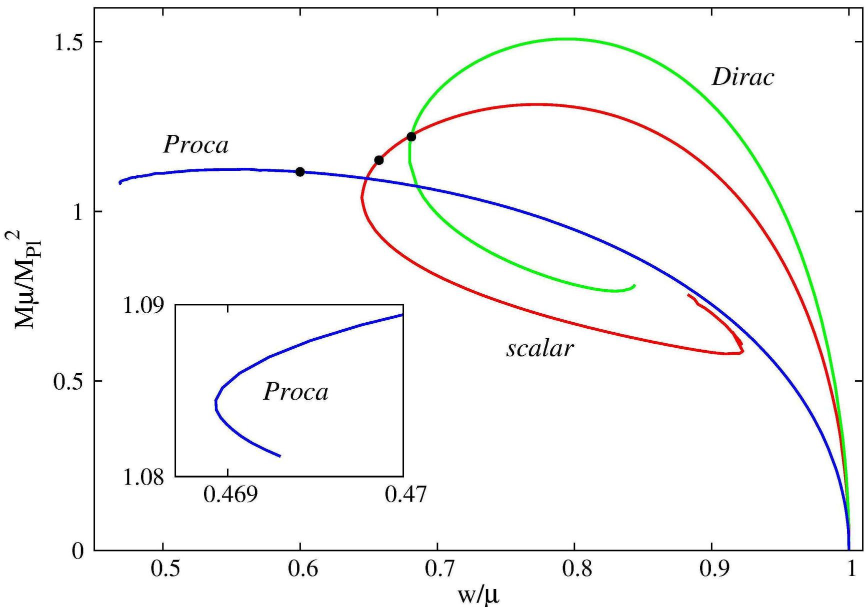
The spinor connection matrices

- **Fermionic current:** $j_\mu = \bar{\Psi}\gamma_\mu\Psi$
  - **Metric tetrad:** $\mathbf{e}^0_\mu dx^\mu = e^{F_0} dt\ , \quad \mathbf{e}^1_\mu dx^\mu = e^{F_1} dr\ ,$

$$\mathbf{e}^2_\mu dx^\mu = e^{F_1} r d\theta\ , \qquad \mathbf{e}^3_\mu dx^\mu = e^{F_2} r \sin\theta\left(d\varphi - \frac{W}{r}dt\right)$$

(Herdeiro, Perapechka, Radu & Ya S 2019)





Localized Fermions+GR

Self-gravitating fermions?

Assumptions

- **only single-particle normalizible state is considered**
- **second quantization of the fields is ignored**
- **gravity is treated purely classically**

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Einstein-Yang-Mills sphalerons and level crossing

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Non-Abelian self-gravtating monopole+fermions

$$S=\int d^4x\,\sqrt{-g}\left[\frac{R}{16\pi G}+\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})-\mathrm{Tr}(D_\mu\phi\;D^\mu\phi)+\lambda\mathrm{Tr}\left(\phi^2-a^2\right)^2+L_{\rm sp}\right.$$

$$L_{\rm sp}=\frac{i}{2}\left((\hat{\not{D}}\bar\psi)\psi-\bar\psi\hat{\not{D}}\psi\right)-\frac{i}{2}h\bar\psi\gamma^5\phi\psi,\qquad \hat D_\mu\psi=(\partial_\mu-\Gamma_\mu+ieA_\mu)\psi$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G\left[(T_{\mu\nu})_{YM}+(T_{\mu\nu})_\phi+(T_{\mu\nu})_s\right]$$

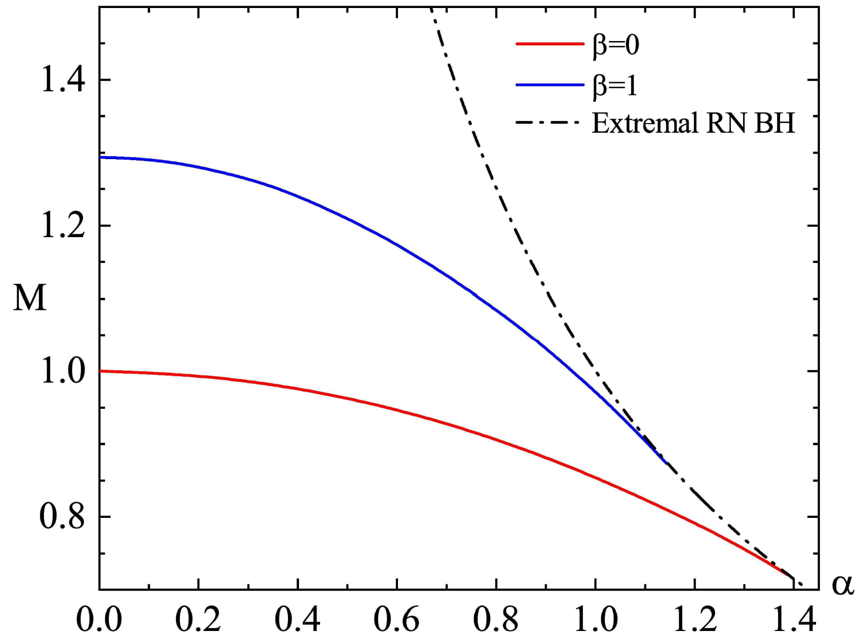
Three dimensionless parameters of the model: $(\alpha^2=4\pi Ga^2)$

$$\alpha=\sqrt{4\pi}\frac{M_v}{gM_{Pl}},\qquad \beta=\frac{M_s}{M_v},\qquad h=\frac{2M_f}{M_v}$$

- **decoupled limit *h*=0**

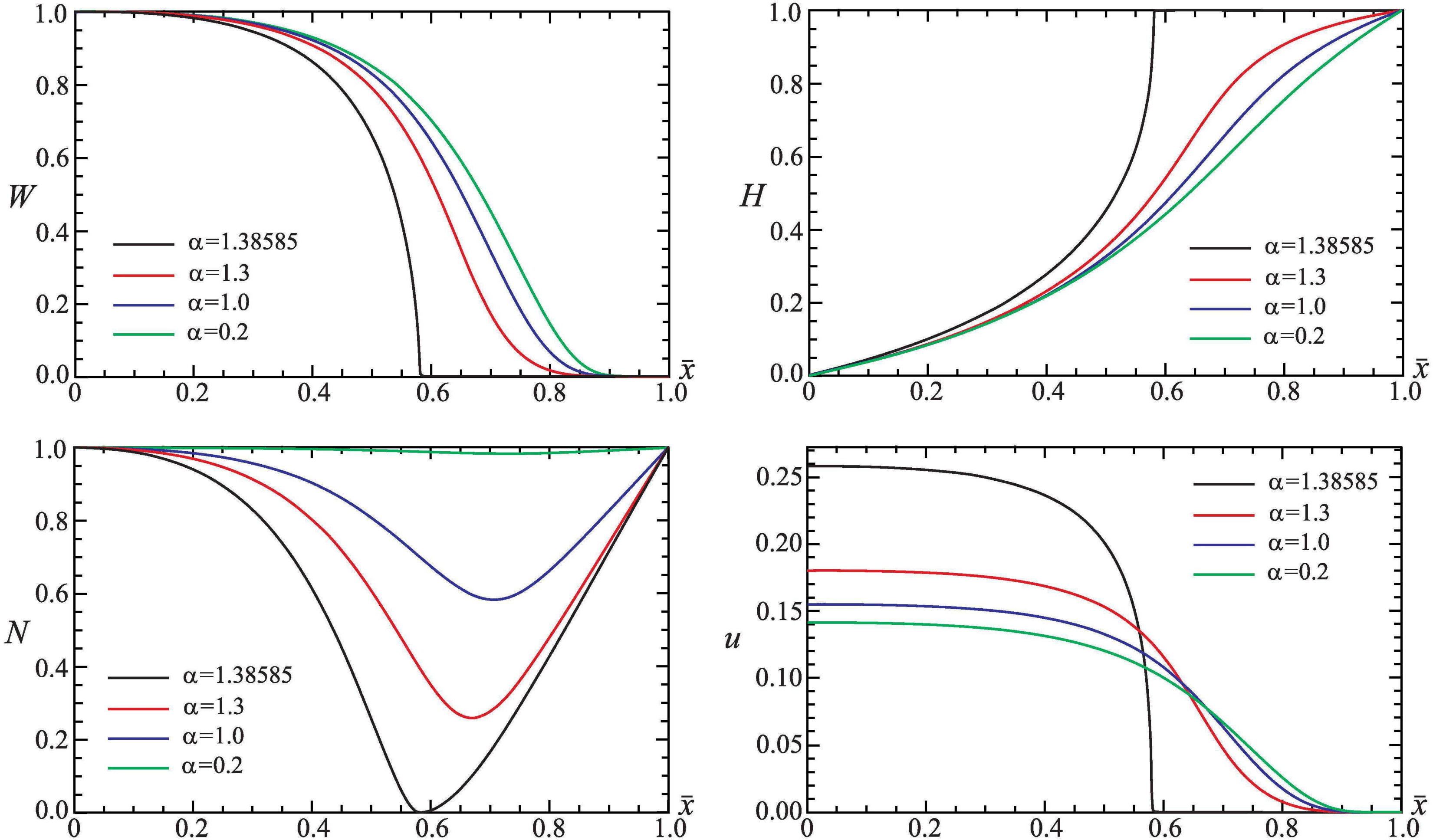
self - gravitating 't Hooft–Polyakov monopole

Breitenlohner, Forgacs, Maison (1992),
Lee, Nair, Weinberg (1992)



Spherical symmetry:

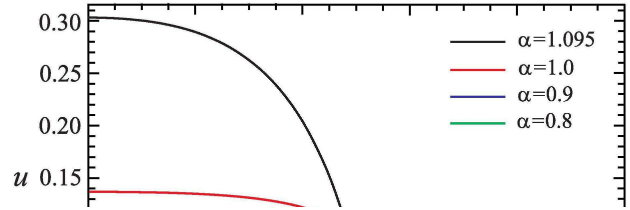
$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



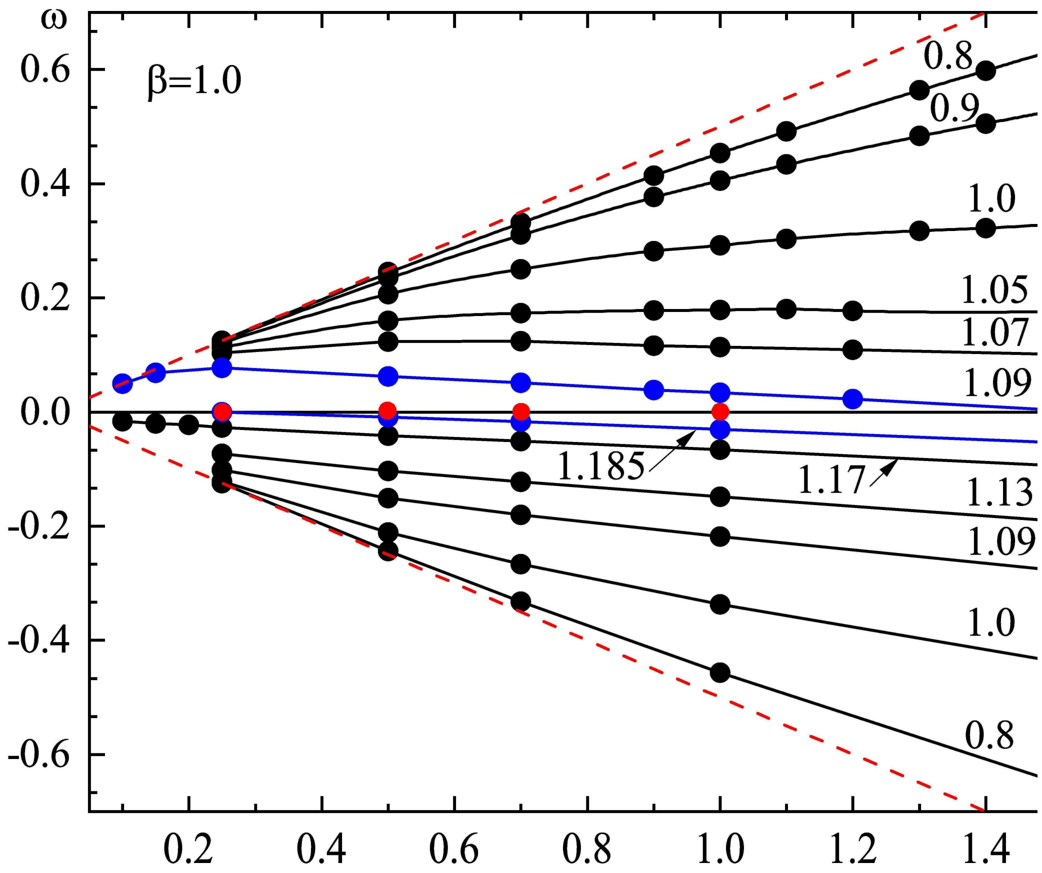
β=0, h=-1, ω=0

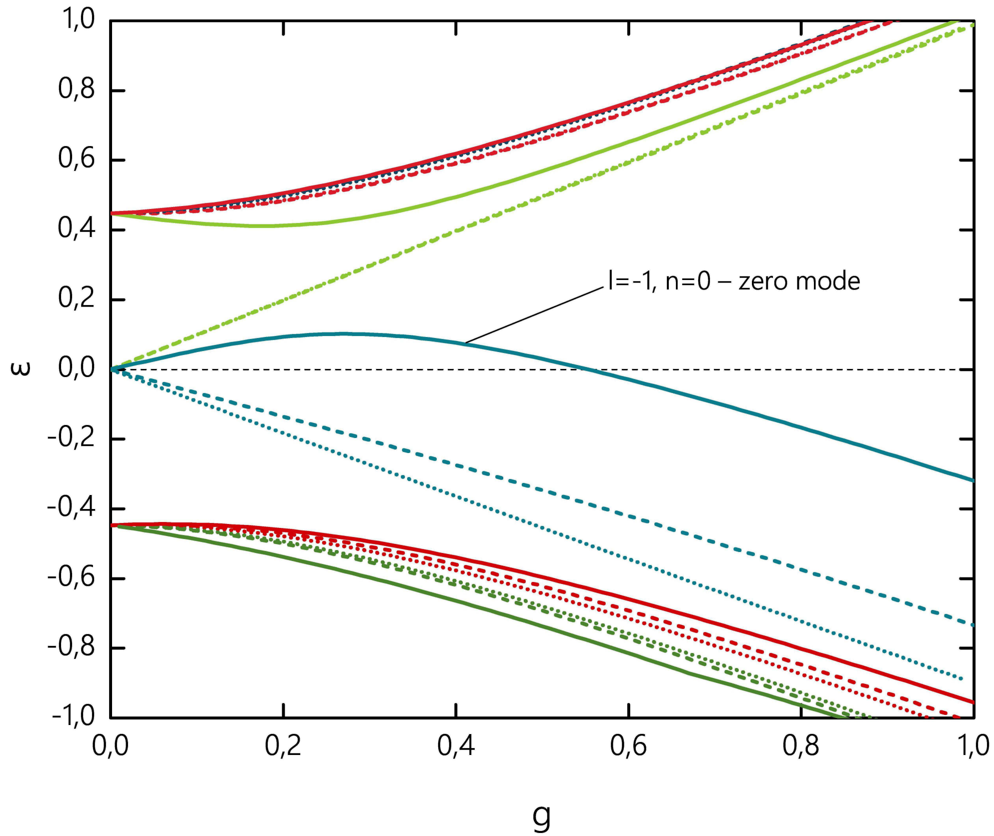
No fermion hair for RN BH

β=1, h=1

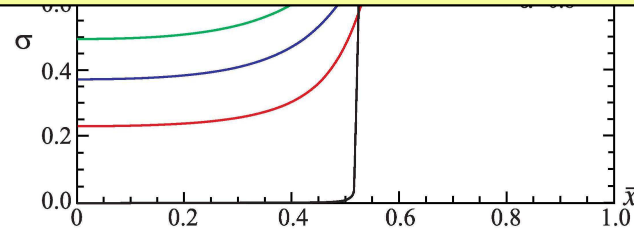


Non-zero modes: $|\omega| < |h/2|$





U(1) gauged baby Skyrmions:
h=0 ⟶ Landau levels



No fermion hair for RN BH
(possible loophole: axially symmetric modes)

Gravitating Skyrmions

$$S = \int \left\{ \frac{R}{\alpha^2} + \mathcal{L}_{Sk} \right\} \sqrt{-g}\, d^4x$$

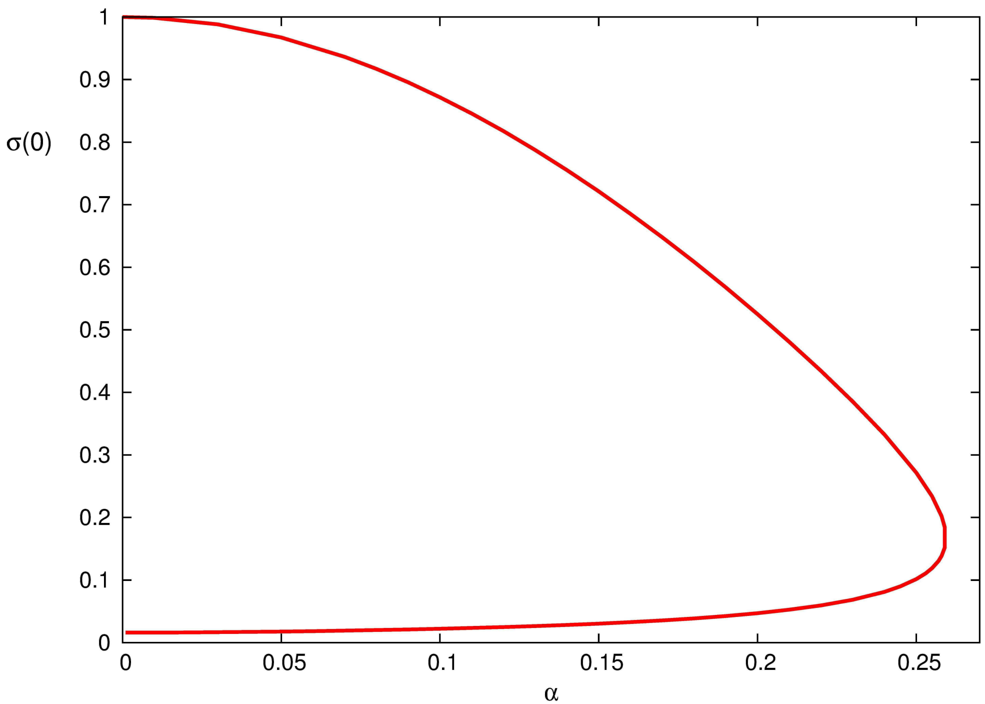
- **The Skyrme field:** $U(\vec{r}, t) \xrightarrow[r \to \infty]{} \mathbb{I}$

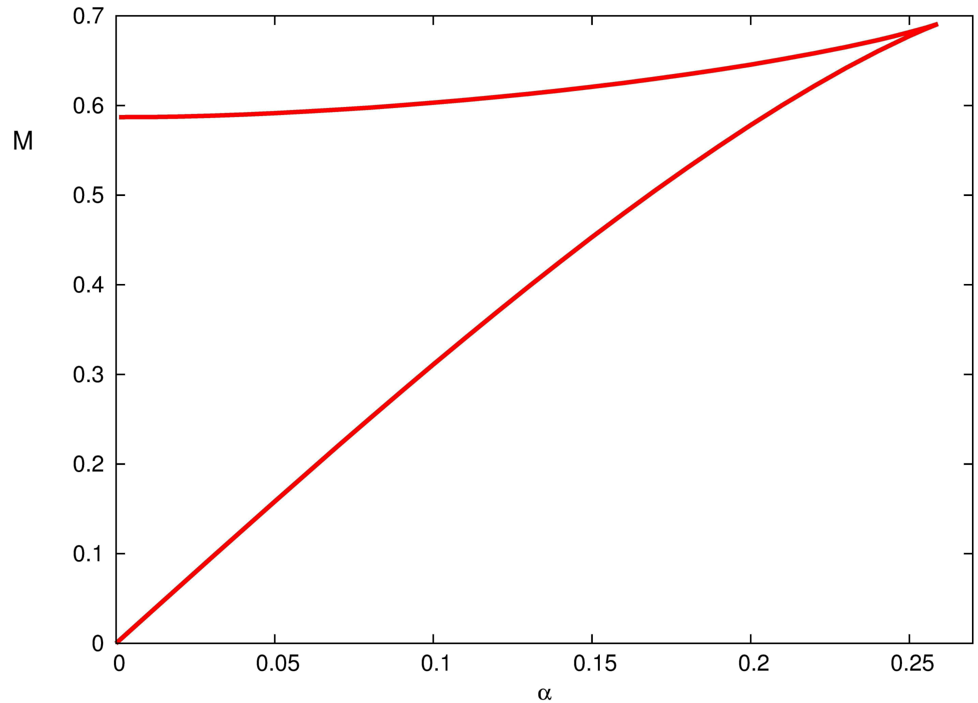
  $U: \; S^3 \to S^3$

Spherical symmetry:

$$ds^2 = -\sigma^2(r) N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\mathcal{L}_{Sk} = \frac{1}{2} \mathrm{Tr}\ \left(\partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{4} \mathrm{Tr}\ \left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right) + m^2 \mathrm{Tr}\ \left(U - \mathbb{I} \right)$$





Self-gravtating skyrmion+fermions

Yet another hedgehog $U(r) = \phi_0 + \phi^a \cdot \sigma^a = \cos F(r) + i\hat{n}^a \cdot \sigma^a \sin F(r)$

$$L_{Sk} = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\partial_\mu \phi^a \partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^b \partial^\nu \phi^b) - m^2(1 - \phi_0)$$

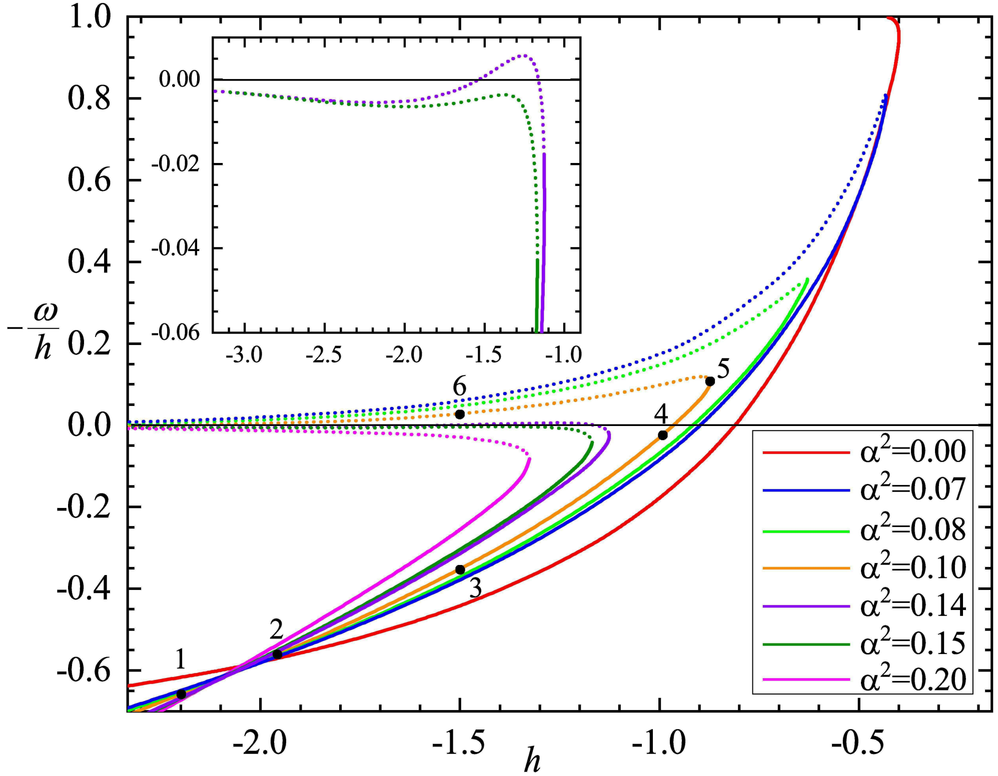
$$L_{\text{sp}} = \frac{i}{2}\left((\hat{\not{D}}\bar{\psi})\psi - \bar{\psi}\hat{\not{D}}\psi\right) - h\bar{\psi}[\phi_0 + i\gamma_5(\phi^a \cdot \sigma^a)]\psi, \qquad \hat{D}_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi$$

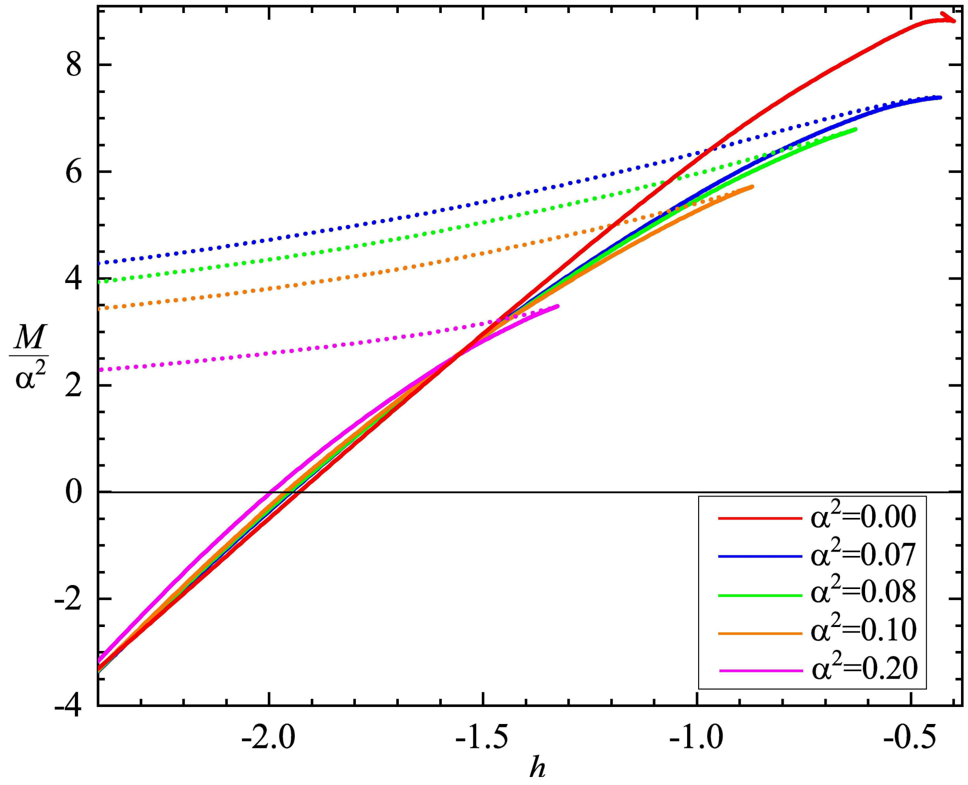
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left[(T_{\mu\nu})_{Sk} + (T_{\mu\nu})_s\right]$$

$$T^{\mu\nu}_{\text{Sk}} = 2\left[\partial^\mu \phi_a \, \partial^\nu \phi^a - \left(\partial^{[\mu} \phi^a \, \partial^{\alpha]} \phi^b\right)\left(\partial^{[\nu} \phi_a \partial_{\alpha]} \phi_b\right)\right] - g^{\mu\nu}\left[(\partial_\alpha \phi_a)^2 - \frac{1}{2}\left(\partial_{[\alpha} \phi_a \, \partial_{\beta]} \phi_b\right)^2\right]$$

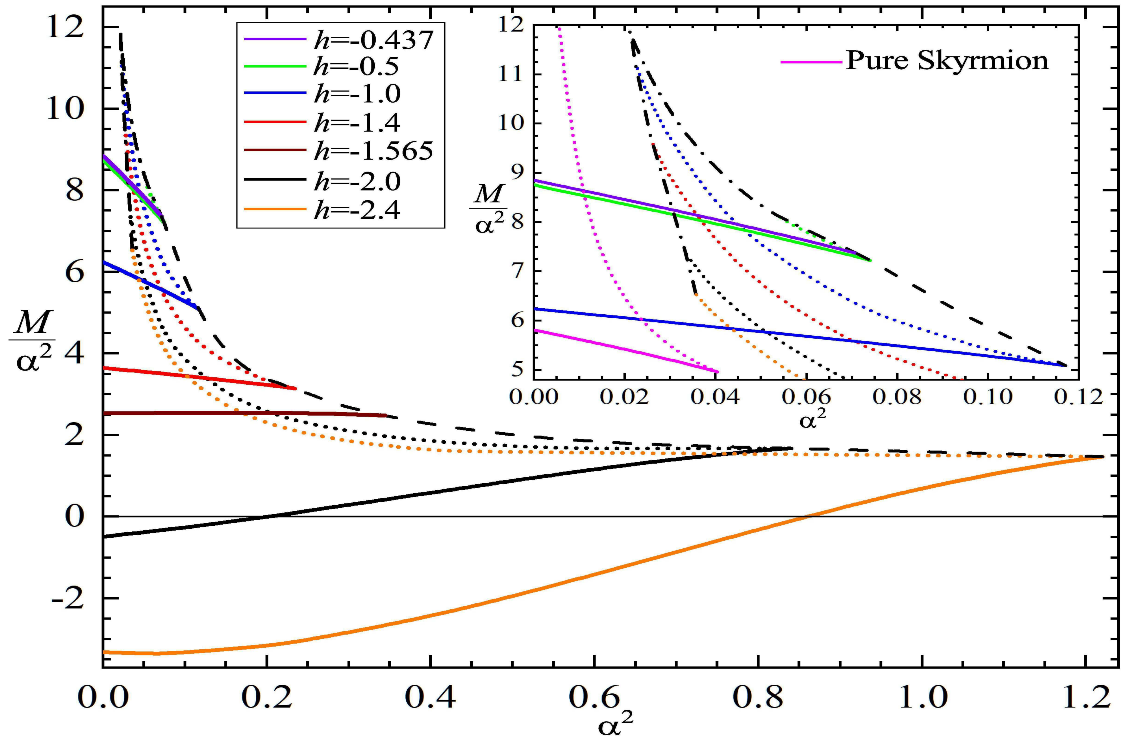
$$T^{\mu\nu}_{\text{sp}} = \frac{i}{4}\left[\bar{\psi}\gamma^\mu(\hat{D}^\nu \psi) + \bar{\psi}\gamma^\nu(\hat{D}^\mu \psi) - (\hat{D}^\mu \bar{\psi})\gamma^\nu \psi - (\hat{D}^\nu \bar{\psi})\gamma^\mu \psi\right] - g^{\mu\nu}\mathcal{L}_{\text{s}}$$

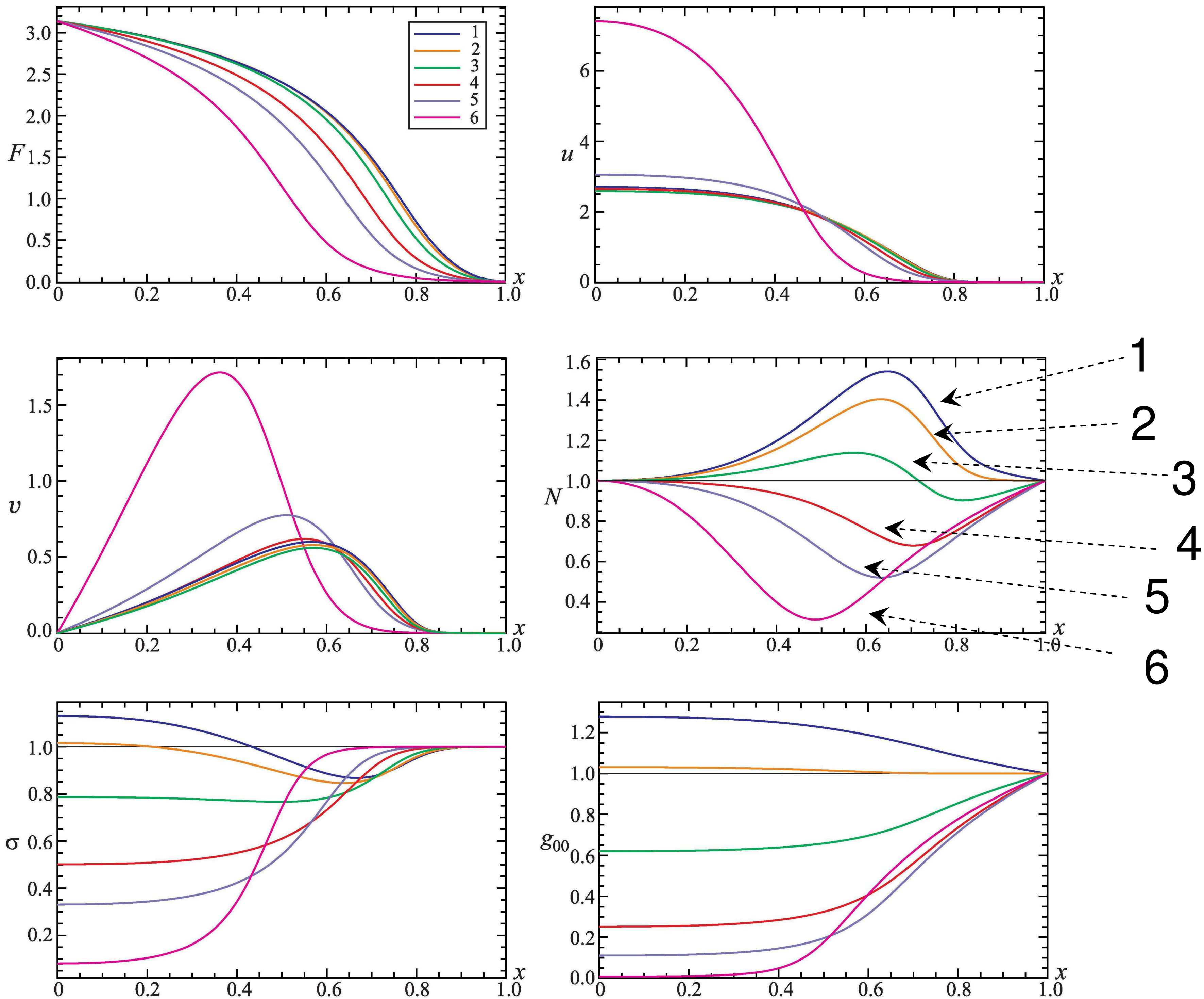
$$\psi = e^{-i\omega t}\begin{pmatrix}\chi \\ \eta\end{pmatrix} \quad \text{with} \quad \chi = \frac{u(r)}{\sqrt{2}}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \quad \eta = i\frac{v(r)}{\sqrt{2}}\begin{pmatrix}\sin\theta e^{-i\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{i\varphi}\end{pmatrix}$$





$$\alpha^2 = 4\pi G f_\pi^2, \quad h \to h/(a_0 f_\pi)$$





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Violation of the energy conditions

null and weak energy conditions:

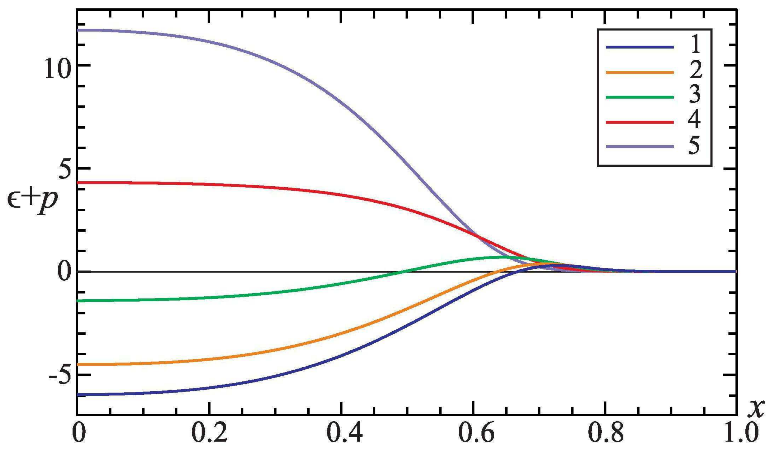
$$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad \text{and} \quad T_{\mu\nu}V^\mu V^\nu \geq 0$$

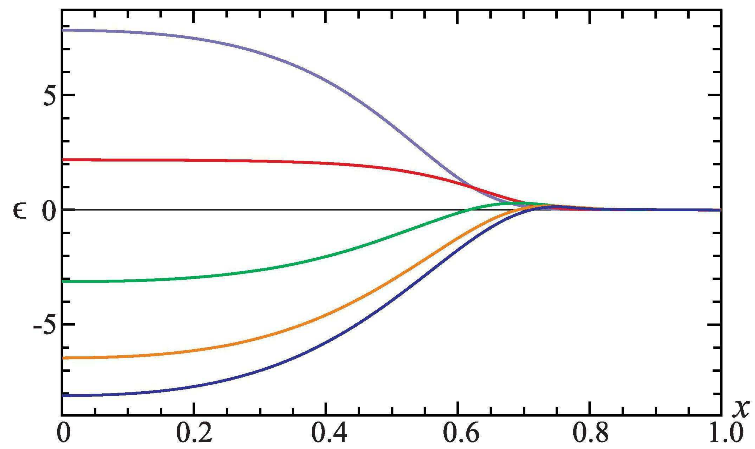
$$g_{\mu\nu}k^\mu k^\nu = 0, \qquad g_{\mu\nu}V^\mu V^\nu > 0$$

Light-like vector ***timelike vector***

The null/weak energy conditions for gravitating Skyrmion-fermion system:

$$\epsilon + p \equiv T_0^0 - T_1^1 \geq 0, \quad \epsilon = T_0^0 > 0$$





Summary

- Backreaction of the localized fermions may strongly affect the solitons inself, it breack the symmetry of the solutions.
- Localization of the fermions produces additional channels of interaction between the solitons, it may bound solitons with repulsive scalar interactions
- Dynamics of the solitons with localized fermionic modes?
- There are spinning Dirac stars, they possess non-zero angular momentum J=nQ with half-integer n
- The fermion zero mode localized on the gravitating monopole is fully absorbed into the interior of the forming RN black hole
- Localization of the backreacting fermionic mode on a self-gravitating Skyrmion violates energy conditions, configuration may possess a negative ADM mass
- No-go for BHs with fermionic hairs in asymptotically flat 3+1 dim?
- Other examples of violation of energy conditions related to self-gravitating fermions localized on a soliton?

Thank you!