Initial data for Minkowski stability with arbitrary decay

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General relativity in vacuum

- **General relativity (1915):** theory of gravitation, spacetime is a 4d Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ where $\mathbf{g}$ solves the **Einstein vacuum equations**

$$\mathrm{Ricci}(\mathbf{g}) = 0.$$

- **Exact stationnary vacuum solutions:**
  - Minkowski $\mathbf{m} = -\mathrm{d}t^2 + e$,
  - Black holes: Schwarzschild, Kerr (with rotation).

Initial value problem in vacuum

- **Cauchy data:**
  - (Σ, g) a 3d Riemannian manifold,
  - π a symmetric 2-tensor on Σ.
- **Cauchy formulation of EVE:** find $(\mathcal{M}, \mathbf{g})$ with $\mathrm{Ricci}(\mathbf{g}) = 0$ such that
  - Σ is a spacelike hypersurface in $\mathcal{M}$,
  - (g, π) are the first and second fundamental form of $\Sigma \subset \mathcal{M}$.

Theorem (Choquet-Bruhat 1952)

Cauchy problem locally solvable $\iff$ (Σ, g, π) *solve constraint*

$$\begin{cases} R(g) + \frac{1}{2}(\mathrm{tr}_g \pi)^2 - |\pi|_g^2 = 0, \\ \mathrm{div}_g \pi = 0. \end{cases} \qquad \text{(C)}$$

Stability of Minkowski

Theorem

Let $0 < \delta < 1$. If the data are of the form

- (Bieri 2010, Shen 2023)

$$(g, \pi) = \left(e + O\left(\varepsilon r^{-\delta}\right), O\left(\varepsilon r^{-1-\delta}\right)\right),$$

- (Christodoulou–Klainerman 1993, Lindblad–Rodnianski 2010)

$$(g, \pi) = \left(e + \frac{2m}{r} + O\left(\varepsilon r^{-1-\delta}\right), O\left(\varepsilon r^{-2-\delta}\right)\right),$$

- (Klainerman–Nicolò 2003, Shen 2023) *(exterior problem)*

$$(g, \pi) = \left(e + \frac{2m}{r} + O\left(\varepsilon r^{-q-\delta}\right), O\left(\varepsilon r^{-q-\delta-1}\right)\right), \quad q \geq 3,$$

then the resulting spacetime is geodesically complete and converges to Minkowski.

First version of the theorem

Theorem (Fang–Szeftel–T 2023)

Let $q \in \mathbb{N}^*$, $0 < \delta < 1$. *Let* $(\check{g}, \check{\pi})$ *TT tensors in a cone with*

$$|\check{g}| + r|\check{\pi}| \leq \varepsilon r^{-q-\delta}.$$

There exists a solution of (C) *on* $\mathbb{R}^3$ *of the form*

$$\begin{cases} g = g_{\vec{p}} + \check{g} + 4\check{u}e + O\left(r^{-q-\delta-1}\right), \\ \pi = \pi_{\vec{p}} + \check{\pi} + L_e\check{X} + O\left(r^{-q-\delta-2}\right), \end{cases}$$

for large r, *where* $\vec{p}$ *is a black hole parameter and*

$$|\check{u}| + r\left|L_e\check{X}\right| \lesssim \varepsilon^2 r^{-q-\delta}.$$

The constraint equations

$$\Phi(g,\pi) := \left(R(g) + \frac{1}{2}(\mathrm{tr}_g \pi)^2 - |\pi|_g^2, \mathrm{div}_g \pi \right) = 0 \qquad \text{(C)}$$

- **Underdetermined $\Longrightarrow$ conformal ansatz near Minkowski:**

$$\begin{cases} g = (1+\breve{u})^4 (e+\breve{g}), \\ \pi = \breve{\pi} + L_e \breve{X}, \end{cases}$$

with $(\breve{g}, \breve{\pi})$ TT tensors (traceless and divergence free) and $L_e \breve{X} = \nabla \otimes \breve{X} - \left(\mathrm{div} \breve{X} \right) e$.

$$4_{\text{(equations)}} + 4_{\text{(two TT tensors)}} + 4_{\text{(diffeo invariance)}} = 12_{\text{(unknowns for } (g,\pi))}.$$

- **Elliptic system for $\left(\breve{u}, \breve{X}\right)$:**

$$\text{(C)} \iff \Delta \left(\breve{u}, \breve{X}\right) = \Phi\left(e+\breve{g}, \breve{\pi}\right) + \text{nonlinear terms}$$

Inverting the Laplacian on $\mathbb{R}^3$

- **Behaviour of Δ in weighted Sobolev spaces on $\mathbb{R}^3$:**

$$\left(\check{u},\check{X}\right)=\sum_{j=1}^{q}\sum_{\ell=-(j-1)}^{j-1}\left\langle\Phi\left(e+\check{g},\check{\pi}\right),W_{j\ell}\right\rangle V_{j\ell}+O\left(r^{-q-\delta}\right),$$

where $\{W_{j\ell}\}=\{\text{harmonic polynomials}\}=\ker\left(\Delta^*\right)$.

- **Cancelling the scalar products?**

$$\left\langle\Phi\left(e+\check{g},\check{\pi}\right),W_{j\ell}\right\rangle=\left\langle(\check{g},\check{\pi}),D\Phi[e,0]^*(W_{j\ell})\right\rangle+\cdots$$

$\Longrightarrow$ **linear obstructions** come from $\ker\left(D\Phi[e,0]^*\right)\cap\ker\left(\Delta^*\right)$.

Theorem (Moncrief 1975)
If (Σ,g,π) comes from $(\mathcal{M},\mathbf{g})$ Ricci-flat, then

$\ker\left(D\Phi[g,\pi]^*\right)=\{$*projection on Σ of Killing vector fields of* $(\mathcal{M},\mathbf{g})\}$.

Symmetries in general relativity

- $\{\text{linear obstructions}\} = \ker(D\Phi[e,0]^*) \cap \ker(\Delta^*)$,
- $\ker(D\Phi[g,\pi]^*) = \{\textit{projection of Killing vector fields}\}$.

$(\mathcal{M}, \mathbf{g})$	Killing vector fields	$\dim\ker(D\Phi[g,\pi]^*)$
Minkowski	$\partial_t,\ \partial_i,\ x_i\partial_j - x_j\partial_i,\ t\partial_i + x_i\partial_t$	10
Schwarzschild	$\partial_t,\ x_i\partial_j - x_j\partial_i$	4
Kerr	$\partial_t,\ \partial_\phi$	2

Bad news: $\ker(D\Phi[e,0]^*) \subset \ker(\Delta^*) \Longrightarrow$ modify the ansatz!

Modification of the ansatz

$$\begin{cases} g = (1+\breve{u})^4(e+\breve{g}), \\ \pi = \breve{\pi} + L_e\breve{X}, \end{cases} \longrightarrow \begin{cases} g = (1+\breve{u})^4\left(e + \chi_{\vec{p}}(g_{\vec{p}} - e) + \breve{g} + \breve{g}\right), \\ \pi = \chi_{\vec{p}}\pi_{\vec{p}} + \breve{\pi} + \breve{\pi} + L_e\breve{X}. \end{cases}$$

Cancelling the scalar products:

- $W_{j\ell} \notin \ker\left(D\Phi[e,0]^*\right)$: $(\breve{g}, \breve{\pi})$ compactly supported and solve the linear system

$$\Big\langle (\breve{g}, \breve{\pi}), D\Phi[e,0]^*(W_{j\ell}) \Big\rangle = -\Big\langle (\breve{g}, \breve{\pi}), D\Phi[e,0]^*(W_{j\ell}) \Big\rangle + \cdots$$

- $W_{j\ell} \in \ker\left(D\Phi[e,0]^*\right)$: truncated black hole $(\chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}})$ and solve for the parameter $\vec{p}$ the system

$$\Big\langle D\Phi[e,0]\left(\chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right), W_{j\ell} \Big\rangle = -\left\langle \frac{1}{2} D^2\Phi[e,0](\breve{g}, \breve{\pi}), W_{j\ell} \right\rangle + \cdots$$

System for the black hole parameter

- **Truncated black hole:** $\vec{p} = (m, y, a, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \times B_{\mathbb{R}^3}(0,1)$ and $(g_{\vec{p}}, \pi_{\vec{p}})$ = data of a translated, rotated and boosted Kerr + $\chi_{\vec{p}}$ smooth cutoff to avoid singularities

- **System for $\vec{p}$:**

$$\left\langle D\Phi[e,0]\left(\chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right), W_{j\ell}\right\rangle = -\left\langle \frac{1}{2}D^2\Phi[e,0](\breve{g}, \breve{\pi}), W_{j\ell}\right\rangle$$

$$\text{with } W_{j\ell} = \begin{pmatrix} \partial_t \\ \partial_i \\ t\partial_i + x_i\partial_t \\ x_i\partial_j - x_j\partial_i \end{pmatrix} \Longrightarrow \begin{cases} \gamma m = \frac{1}{4}\|\nabla\breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2} \\ \qquad - \frac{1}{2}\|\mathrm{div}\breve{g}\|^2_{L^2} + \frac{1}{4}\|\nabla\mathrm{tr}\breve{g}\|^2_{L^2} - \frac{1}{2}\|\mathrm{tr}\breve{\pi}\|^2_{L^2} \\ \gamma m v = \langle\breve{\pi}, \nabla\breve{g}\rangle \\ m y = \left\langle(\nabla\breve{g})^2, r\right\rangle + \left\langle(\breve{\pi})^2, r\right\rangle \\ m a = \left\langle(\nabla\breve{g})^2, r\right\rangle + \left\langle(\breve{\pi})^2, r\right\rangle \end{cases}$$

$$\left(\gamma = \text{Lorentz factor} = \frac{1}{\sqrt{1-|v|^2}}\right)$$

Positive mass theorem $\longrightarrow (\breve{g}, \breve{\pi})$ TT!

$$\gamma m = \frac{1}{4}\|\nabla \breve{g}\|_{L^2}^2 + \|\breve{\pi}\|_{L^2}^2 - \frac{1}{2}\|\mathrm{div}\breve{g}\|_{L^2}^2 + \frac{1}{4}\|\nabla \mathrm{tr}\breve{g}\|_{L^2}^2 - \frac{1}{2}\|\mathrm{tr}\breve{\pi}\|_{L^2}^2$$

Theorem (Schoen–Yau 1979, Witten 1982)
Let $m(g) = \int_{\mathbb{R}^3} DR[e,0](g-e)$. *If* $R(g) \geq 0$, *then* $m(g) \geq 0$ *and* $m(g) = 0 \Longrightarrow g \simeq e$.

$\Longrightarrow$ **need a coercive estimate on the mass!**

Proposition
Up to a choice of diffeo, every perturbation of Minkowski is of the form $(g, \pi) = \big((1+\breve{u})^4(e+\breve{g}), \breve{\pi} + L_e X\big)$ *with* $(\breve{g}, \breve{\pi})$ *TT.*

$$(\breve{g}, \breve{\pi}) \text{ TT implies } \gamma m = \frac{1}{4}\|\nabla \breve{g}\|_{L^2}^2 + \|\breve{\pi}\|_{L^2}^2$$

(In the spirit of Choquet-Bruhat–Marsden 1976)

Coupling with the boost $\longrightarrow (\breve{g}, \breve{\pi})$ in a cone!

$$\begin{cases} \gamma m = \frac{1}{4} \|\nabla \breve{g}\|_{L^2}^2 + \|\breve{\pi}\|_{L^2}^2 \\ \gamma m v = \langle \breve{\pi}, \nabla \breve{g} \rangle \end{cases} \Longrightarrow m = \sqrt{1 - J(\breve{g}, \breve{\pi})^2} \left(\frac{1}{4} \|\nabla \breve{g}\|_{L^2}^2 + \|\breve{\pi}\|_{L^2}^2 \right)$$

with the functional

$$J(\breve{g}, \breve{\pi}) := \frac{\sqrt{\sum_{k=1,2,3} \left(\int_{\mathbb{R}^3} \breve{\pi} \cdot \partial_k \breve{g} \right)^2}}{\frac{1}{4} \|\nabla \breve{g}\|_{L^2}^2 + \|\breve{\pi}\|_{L^2}^2}.$$

Proposition

We have $\sup_{(\breve{g}, \breve{\pi}) \in \cap \dot{H}^1 \times L^2 \cap TT^2} J(\breve{g}, \breve{\pi}) = 1$ *and the supremum is not reached.*

$\Longrightarrow$ we consider $(\breve{g}, \breve{\pi})$ in the cone $J(\breve{g}, \breve{\pi}) \leq 1 - \alpha$

Large y and a

- **Two norms for $(\breve{g}, \breve{\pi})$:**

$$\varepsilon := \|\breve{g}\|_{H^4_{-q-\delta}} + \|\breve{\pi}\|_{H^3_{-q-\delta-1}} \qquad \eta := \sqrt{\frac{1}{4}\|\nabla\breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2}} \lesssim \varepsilon$$

- **Dividing by the mass:**

$$\begin{cases} my = \left\langle (\nabla\breve{g})^2, r\right\rangle + \left\langle (\breve{\pi})^2, r\right\rangle \\ ma = \left\langle (\nabla\breve{g})^2, r\right\rangle + \left\langle (\breve{\pi})^2, r\right\rangle \end{cases} \Longrightarrow (y, a) = \frac{\left\langle (\nabla\breve{g})^2, r\right\rangle + \left\langle (\breve{\pi})^2, r\right\rangle}{\frac{1}{4}\|\nabla\breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2}} \lesssim \left(\frac{\varepsilon}{\eta}\right)^2$$

- **Cutoff $\chi_{\vec{p}}$ in far regions:**

$$\begin{aligned} \left\|\Phi\left(e + \chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right)\right\|_{L^2_{-q-\delta-2}} &\lesssim m|y|^{q+\delta-1} \\ &\lesssim \eta^{2(-q-\delta+2)}\varepsilon^{2(q+\delta-1)} \end{aligned}$$

Doesn't allow us to solve $\Delta\left(\breve{u}, \breve{X}\right) = \Phi\left(e + \chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right)$ ☹

Large y and a

- **Two norms for $(\breve{g}, \breve{\pi})$:**

$$\varepsilon := \|\breve{g}\|_{H^4_{-q-\delta}} + \|\breve{\pi}\|_{H^3_{-q-\delta-1}} \qquad \eta := \sqrt{\frac{1}{4}\|\nabla\breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2}} \lesssim \varepsilon$$

- **Dividing by the mass:**

$$\begin{cases} my = \left\langle (\nabla\breve{g})^2, r \right\rangle + \left\langle (\breve{\pi})^2, r \right\rangle \\ ma = \left\langle (\nabla\breve{g})^2, r \right\rangle + \left\langle (\breve{\pi})^2, r \right\rangle \end{cases} \Longrightarrow (y, a) = \frac{\left\langle (\nabla\breve{g})^2, r \right\rangle + \left\langle (\breve{\pi})^2, r \right\rangle}{\frac{1}{4}\|\nabla\breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2}} \lesssim \left(\frac{\varepsilon}{\eta}\right)^{\frac{2}{2q+2\delta-1}}$$

- **Cutoff $\chi_{\vec{p}}$ in far regions:**

$$\begin{aligned} \left\|\Phi\left(e + \chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right)\right\|_{L^2_{-q-\delta-2}} &\lesssim m|y|^{q+\delta-1} \\ &\lesssim \eta\varepsilon \end{aligned}$$

Allows us to solve $\Delta\left(\breve{u}, \breve{X}\right) = \Phi\left(e + \chi_{\vec{p}}(g_{\vec{p}} - e), \chi_{\vec{p}}\pi_{\vec{p}}\right)$ ☺

Final version of the theorem

Theorem (Fang–Szeftel–T 2023)

Let $q \in \mathbb{N}^*$, $0 < \delta < 1$. *Let* $(\breve{g}, \breve{\pi})$ *TT-tensors such that*

$$\|\breve{g}\|_{H^4_{-q-\delta}} + \|\breve{\pi}\|_{H^3_{-q-\delta-1}} \leq \varepsilon$$

$$J(\breve{g}, \breve{\pi}) \leq 1 - \alpha, \qquad \eta^2 := \frac{1}{4} \|\nabla \breve{g}\|^2_{L^2} + \|\breve{\pi}\|^2_{L^2} > 0$$

There exists a solution of (C) *of the form*

$$(g, \pi) = \left((1 + \breve{u})^4 \left(e + \chi_{\vec{p}} (g_{\vec{p}} - e) + \breve{g} + \breve{g} \right), \chi_{\vec{p}} \pi_{\vec{p}} + \breve{\pi} + \breve{\pi} + L_e \breve{X} \right),$$

$$m \sim \eta^2, \qquad |y| + |a| \lesssim \left(\frac{\varepsilon}{\eta} \right)^{\frac{2}{2q+2\delta-1}}, \qquad |v| \leq 1 - \frac{\alpha}{2},$$

$$\|\breve{u}\|_{H^2_{-q-\delta}} + \left\| \breve{X} \right\|_{H^2_{-q-\delta}} + \|\breve{g}\|_{W^{2,\infty}} + \|\breve{\pi}\|_{W^{1,\infty}} \lesssim \eta \varepsilon.$$

Uniqueness

Proposition

If
$$\begin{cases} (1+\check{u}_1)^4\left(e+\chi_{\vec{p}_1}(g_{\vec{p}_1}-e)+\check{g}_1+\breve{g}_1\right) \\ \qquad\qquad = (1+\check{u}_2)^4\left(e+\chi_{\vec{p}_2}(g_{\vec{p}_2}-e)+\check{g}_2+\breve{g}_2\right), \\ \chi_{\vec{p}_1}\pi_{\vec{p}_1}+\check{\pi}_1+\breve{\pi}_1+L_e\check{X}_1=\chi_{\vec{p}_2}\pi_{\vec{p}_2}+\check{\pi}_2+\breve{\pi}_2+L_e\check{X}_2, \end{cases}$$

then $\left(\vec{p}_1,\check{u}_1,\breve{g}_1,\check{g}_1,\breve{\pi}_1,\check{X}_1,\check{\pi}_1\right)=\left(\vec{p}_2,\check{u}_2,\breve{g}_2,\check{g}_2,\breve{\pi}_2,\check{X}_2,\check{\pi}_2\right)$.

1. Asymptotics $\Longrightarrow \vec{p}_1=\vec{p}_2$
2. $\check{g}$ TT and $\breve{g}$ traceless $\Longrightarrow \check{u}_1=\check{u}_2$
3. $\check{g}$ TT and $\mathrm{div}_{|\{\breve{g}\}}$ injective $\Longrightarrow \breve{g}_1=\breve{g}_2$ and $\check{g}_1=\check{g}_2$
4. $(\Delta\mathrm{tr}+\mathrm{divdiv})L_e=0$, $\check{\pi}$ TT and $(\Delta\mathrm{tr}+\mathrm{divdiv})_{|\{\breve{\pi}\}}$ injective $\Longrightarrow \breve{\pi}_1=\breve{\pi}_2$
5. $\check{\pi}$ TT and $\mathrm{div}L_{e|H^2_{-q-\delta}}$ injective $\Longrightarrow \check{X}_1=\check{X}_2$ and $\check{\pi}_1=\check{\pi}_2$

Thank you for your attention!