Small eigenvalues and metastability for semiclassical Boltzmann operators

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16th conference of the GDR DynQua "Quantum Dynamics"

February 2nd, 2024

Introduction

We are interested in Boltzmann equations of the form

$$\begin{cases} \partial_t u + X_0(u) + Q(u) = 0 \\ u_{|t=0} = u_0 \in L^1(\mathbb{R}^{2d}) \end{cases} \tag{1}$$

We aim at studying the return to equilibrium of the solutions of (1).

Here, Q is linear $\rightsquigarrow$ spectral analysis of the operator associated to (1) with

$$X_0 = v \cdot \partial_x - \partial_x W \cdot \partial_v$$

- Non-linear case without potential : Shizuta-Asano, Ukai, Desvillettes-Villani (1974-2005)
- Spatially homogeneous case : Barranger, Mouhot (2005-2006)
- Hypocoercivity : Hérau, Villani ($\sim$2006); Dolbeault-Mouhot-Schmeiser (2010); or recently Carrapatoso, Mischler, Robbe, Bernou, Tristani...

Denoting $h > 0$ a parameter proportional to the temperature of the system, (1) becomes

$$\begin{cases} h\partial_t f + v \cdot h\partial_x f - \partial_x W \cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \in L^2(\mathbb{R}^{2d}) \end{cases} \tag{2}$$

$\rightsquigarrow$ **semiclassical** study, i.e in the limit $h \to 0$ ("*low temperature*" regime) of the spectrum of the operator

$$\begin{aligned} P_h &= v \cdot h\partial_x - \partial_x W \cdot h\partial_v + Q_h \\ &= X_0^h + Q_h \end{aligned}$$

associated to equation (2).

Notations and assumptions

- $W \in \mathcal{C}^\infty(\mathbb{R}^d_x, \mathbb{R})$ is a **Morse** coercive function, at most quadratic at infinity with $n_0 \in \mathbb{N}_{\geq 2}$ local minima.
- Π_h is the **orthogonal projector** on $\mathrm{e}^{-\frac{v^2}{2h}} L^2(\mathbb{R}^d_x)$,

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We consider the "**linear relaxation**" collision operator (BGK model)

$$Q_h = h(\mathrm{Id} - \Pi_h)$$

i.e

$$Q_h f(x,v) = h\left(f(x,v) - \int_{w\in\mathbb{R}^d} f(x,w)\mathrm{e}^{-\frac{w^2}{2h}}\mathrm{d}w\, \mathrm{e}^{-\frac{v^2}{2h}}\right)$$

Small eigenvalues of semiclassical operators

Quantum side :

- Simon, Robert, Martinez, Helffer-Sjöstrand (1980's); Dimassi (1990's) $\rightsquigarrow$ *Schrödinger*
- Helffer-Sjöstrand (1980's), Helffer-Morame; Fournais-Helffer (2000's), Bonnaillie-Noël-Hérau-Raymond (2016-2022) $\rightsquigarrow$ *Magnetic Laplacian*

Probabilistic side :

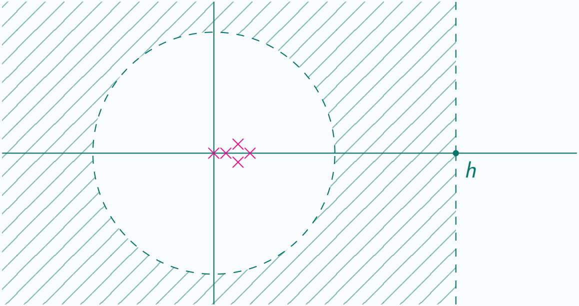
- Helffer-Sjöstrand (1980's); Helffer-Klein-Nier, Bovier-Gayrard-Klein (2000's) $\rightsquigarrow$ *Witten Laplacian*
- Hérau-Hitrik-Sjöstrand (2008) $\rightsquigarrow$ *Kramers-Fokker-Planck*
- Robbe (2016) $\rightsquigarrow$ *Boltzmann equation*
- Le Peutrec-Michel, Bony-Le Peutrec-Michel (2020's) $\rightsquigarrow$ *KFP without supersymmetry*

$\rightarrow$ We adapt the methods from the latter to some **non local** frameworks.

Preliminary result

Theorem 1 [Robbe 2016]

- P_h admits $\mathbf{0}$ as a **simple eigenvalue**
- $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq h\}$ consists of exactly $\mathbf{n_0}$ **eigenvalues** that are $O(\mathrm{e}^{-c/h})$
- The resolvent estimate $\boldsymbol{(P_h - z)^{-1} = O(1/h)}$ holds uniformly on $\{\mathrm{Re}\, z \leq h\} \backslash B(0, \frac{h}{2})$



Main result :

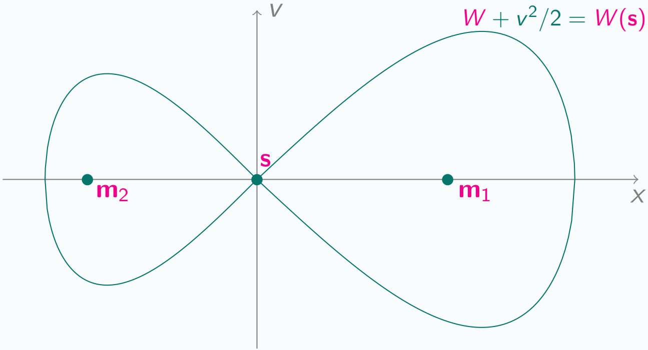
Theorem

Let $\mathbf{m}$ *a local minimum of* W.
There exist $S(\mathbf{m})$, $a(\mathbf{m}) > 0$ *that we can compute and such that*

$$\lambda(\mathbf{m}, h) \sim h\, a(\mathbf{m}) \mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}.$$

Strategy Heuristic

To simplify, let us suppose that we have the following picture in $\mathbb{R}^{2d}$:



In particular, $n_0 = 2$ and $W(\mathbf{m}_1) < W(\mathbf{m}_2)$.

We take

$$f_{ker}(x,v) = \exp\left(-\frac{W(x)+v^2/2}{h}\right) \in \mathrm{Ker}\, P_h.$$

We aim at choosing f the closest possible to the **generalized eigenspace** of P_h :

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$\rightarrow$ denoting

$$\Pi_0 = \frac{1}{2i\pi}\int_{|z|=\frac{h}{2}} (z-P_h)^{-1}\mathrm{d}z,$$

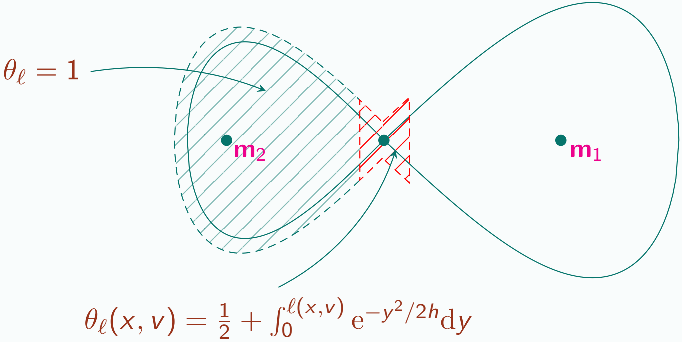
we have

$$\begin{aligned}(1-\Pi_0)f &= \frac{1}{2i\pi}\int_{|z|=\frac{h}{2}} \left(z^{-1}-(z-P_h)^{-1}\right) f\,\mathrm{d}z \\ &= \frac{-1}{2i\pi}\int_{|z|=\frac{h}{2}} z^{-1}(z-P_h)^{-1}P_h f\,\mathrm{d}z.\end{aligned}$$

$\rightsquigarrow$ We want $P_h f$ to be **the smallest possible**.

We define the **gaussian quasimode**

$$g_\ell(x,v) = \theta_\ell(x,v)\exp\left(-\frac{W(x)+v^2/2}{h}\right)$$



$$\theta_\ell(x,v) = \frac{1}{2} + \int_0^{\ell(x,v)} \mathrm{e}^{-y^2/2h}\mathrm{d}y \quad \Longrightarrow \quad \nabla\theta_\ell = \mathrm{e}^{-\ell^2/h}\nabla\ell$$

We easily compute

$$X_0^h g_\ell = \omega_\ell^{tran}(x,v)\exp\left(-\frac{W + \frac{v^2}{2} + \ell^2}{h}\right), \qquad \text{with } \omega_\ell^{tran} = O_{L^\infty}(1)$$

"X_0^h sends the ℓ-**gaussian quasimode** on the ℓ-**elliptized** phase."

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What we expected:

$$Q_h g_\ell \overset{?}{=} \omega_\ell^{col}(x,v)\exp\left(-\frac{W+\frac{v^2}{2}+\ell^2}{h}\right), \qquad \text{with } \omega_\ell^{col} = O_{L^\infty}(1)$$

$$\rightsquigarrow P_h g_\ell \overset{?}{=} \omega_\ell(x,v)\exp\left(-\frac{W+\frac{v^2}{2}+\ell^2}{h}\right)$$

What actually happens:

Proposition

There exists a vector L_y depending on $y \in [0,1]$ such that

$$Q_h g_\ell(x,v) = \int_0^1 \partial_y(L_y)\, \mathrm{e}^{-\frac{W(x)+v^2/2+L_y\cdot(x,v)^2}{h}}\, \mathrm{d}y \ \cdot (x,v)$$

Problem :

$$X_0^h g_\ell \approx \mathrm{e}^{-\frac{1}{h}\left(W+\frac{v^2}{2}+\ell^2\right)} \quad \not\approx \quad Q_h g_\ell \approx \int_0^1 \mathrm{e}^{-\frac{1}{h}\left(W+\frac{v^2}{2}+L_y^2\right)} \mathrm{d}y$$

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Idea : $\quad g_\ell \quad \rightsquigarrow \quad f_{\ell,k} := \int_0^1 k(\gamma) g_{L_\gamma} \mathrm{d}\gamma$

Proposition

$$\left(X_0^h + Q_h\right) f_{\ell,k} = \int_0^1 \omega_{\ell,k}(z,x,v)\, \mathrm{e}^{-\frac{1}{h}\left(W(x)+\frac{v^2}{2}+L_z\cdot(x,v)^2\right)}\mathrm{d}z$$

Last step of proof

Choosing the **optimal** ℓ and k and denoting $\tilde{\lambda}_h := \langle P_h f_{\ell,k}, f_{\ell,k} \rangle$, we then get

Lemma

- $\tilde{\lambda}_h \sim h\,\tilde{a}\,\mathrm{e}^{-\frac{2S(\mathbf{m}_2)}{h}}$ *with* $\tilde{a}$, $S(\mathbf{m}_2) > 0$ *explicit*
- $\|P_h f_{\ell,k}\|^2 = o(h\tilde{\lambda}_h)$.

Conclusion : We go from quasimodes to actual **eigenfunctions** :

$$(f_{ker}, f_{\ell,k}) \xrightarrow{\Pi_0} (f_{ker}, \Pi_0 f_{\ell,k}) \xrightarrow[\text{Schmidt}]{\text{Gram}} (f_{ker}, \boldsymbol{\varphi_2})$$

Interaction matrix : $\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\lambda_h} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \underbrace{\langle \boldsymbol{P_h \varphi_2, \varphi_2} \rangle}_{=\tilde{\lambda}_h(1+o(1))} \end{pmatrix}$

□

Return to equilibrium and metastability

Corollary [Helffer-Sjöstrand 2010, Bony et al. 2021, N. 2022]

There exist some projectors $\mathbb{P}_1, \ldots, \mathbb{P}_{n_0}$ *such that*

$$f_t = \mathbb{P}_1 f_0 + O\left(\mathrm{e}^{-t\lambda_2}\|f_0\|\right)$$

as well as some time intervals $[t_k^-, t_k^+]$ *during which*

$$f_t = \sum_{j=1}^{k} \mathbb{P}_j f_0 + O\left(h^\infty \|f_0\|\right) \qquad \forall t \in [t_k^-, t_k^+].$$

$$
\begin{aligned}
\mathrm{e}^{-tP_h} &= \mathrm{e}^{-tP_h}\Pi_0 + O\left(\mathrm{e}^{-cht}\right) \qquad \text{(Gearhart-Prüss)} \\
&= \sum_{j=1}^{n_0} \mathrm{e}^{-t\lambda_j}\mathbb{P}_j + O\left(\mathrm{e}^{-cht}\right) \\
&= \mathbb{P}_1 + O\left(\mathrm{e}^{-t\lambda_2}\right) \quad \rightsquigarrow \textbf{return to equilibrium}
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e^{-tP_h} + O\left(e^{-cht}\right) &= \sum_{j=1}^{n_0} e^{-t\lambda_j}\mathbb{P}_j \\
&= \sum_{j=1}^{k}\mathbb{P}_j + \sum_{j=2}^{k}\left(e^{-t\lambda_j} - 1\right)\mathbb{P}_j + \sum_{j=k+1}^{n_0} e^{-t\lambda_j}\mathbb{P}_j \\
&\overset{t\geqslant t_k^-}{=} \sum_{j=1}^{k}\mathbb{P}_j + \sum_{j=2}^{k}\left(e^{-t\lambda_j} - 1\right)\mathbb{P}_j + O\left(h^\infty\right) \\
&\overset{t\leqslant t_k^+}{=} \sum_{j=1}^{k}\mathbb{P}_j + O\left(h^\infty\right) \quad \rightsquigarrow \textbf{metastability} \qquad \square
\end{aligned}
$$

Perspectives

- ▷ Consider cases with multiple *collision invariants* [Post-doc with F. Hérau and D. Le Peutrec] :

$$\operatorname{Ker} Q_h = \operatorname{Vect}\left(e^{-v^2/2h}, v_1 e^{-v^2/2h}, \ldots, v_d e^{-v^2/2h}, v^2 e^{-v^2/2h}\right).$$

- ▷ Consider the linearized Boltzmann and Landau operators [Carrapatoso et al.]

- ▷ Study on domains bounded in space [Nier, Lelièvre et al., Bernou et al.] : $\mathbb{R}^{2d} \rightsquigarrow \Omega \times \mathbb{R}^d_v$.

Thank you !