Delocalization of the Laplace eigenmodes on Anosov surfaces

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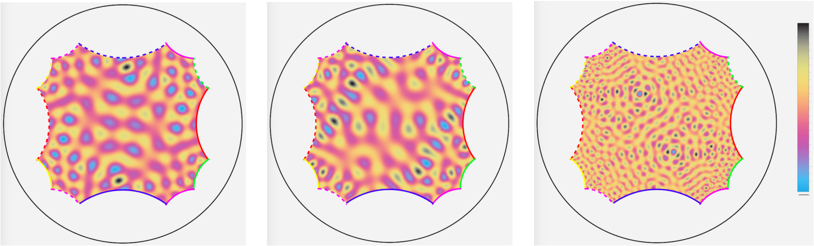
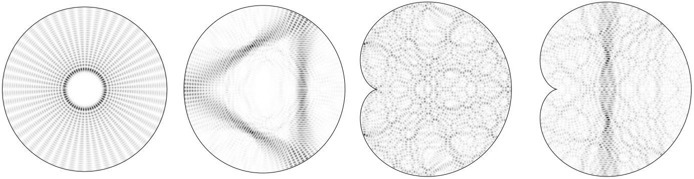


Eigenmodes of a compact hyperbolic surface © Alex Strohmaier

Physical motivation: waves on planar domains

Drum: acoustic waves on a compact domain $\Omega \subset \mathbb{R}^2$. Dirichlet Laplacian $\Delta = \Delta_\Omega \rightsquigarrow$ discrete set of stationary modes $(u_j(x))_{j\in\mathbb{N}}$ with wavelengths h_j ($\equiv$ inverse frequencies) :

$$-h_j^2 \Delta u_j = u_j, \quad u_j \restriction_{\partial\Omega} = 0, \quad \|u_j\|_{L^2(\Omega)} = 1 \quad \text{(Helmholtz eq.)}$$



Plots ©Arnd Bäcker: deformations of the disk $\rightsquigarrow$ cardioid

Main question: How are the u_j distributed over Ω?

Focus on the high frequency regime $\Longleftrightarrow h_j \ll 1$.

- distribution at the macroscopic scale $\ell \sim 1$
- fluctuations at the microscopic scale ($h_j \lesssim \ell \ll 1$)

High frequency / semiclassical regime

General strategy: to get informations about the eigenmodes,

$$-h_j^2 \Delta u_j = u_j,$$

study the long time evolution:

$$\text{half-wave equation} \quad i\partial_t u(t) = \sqrt{-\Delta}\, u(t), \qquad u(0) \in L^2(\Omega)\,,$$

solved by the half-wave propagator $u(t) = U^t u(0)$, $U^t = e^{-it\sqrt{-\Delta}}$.

In the semiclassical regime $h_j \to 0$, this wave dynamics is connected with the ray dynamics

$$\Phi^t : (x, \xi) \in S^*\Omega \mapsto \Phi^t(x, \xi) \in S^*\Omega \quad \text{(unit momentum shell)}$$

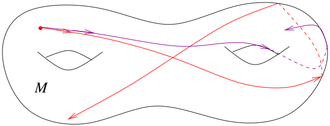
Quantum chaos: *focus on situations where the ray dynamics Φ^t is chaotic.*

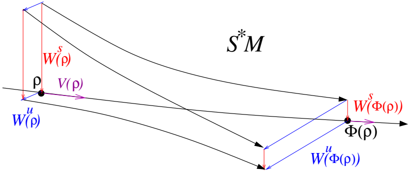
Geometric Quantum chaos: manifolds of negative curvature

Geometric drum: smooth compact Riemannian *surface* (M, g) $(\partial M = \emptyset)$

– Wave dynamics : Laplace-Beltrami operator Δ_g on $L^2(M, g)$, propagator $U^t = \exp(-it\sqrt{-\Delta_g})$.

– Ray dynamics = geodesic flow $\Phi^t : S^*M \to S^*M$.





- Negative curvature $\Longrightarrow$ the geodesic flow is hyperbolic (Anosov).

$$\forall \rho = (x, \xi) \in S^*M, \quad T_\rho(S^*M) = \mathbb{R}V(\rho) \oplus E^u(\rho) \oplus E^s(\rho)$$

$$\forall t \geq 0, \quad \|d\Phi^t_{\restriction E^s(\rho)}\| \leq Ce^{-\lambda t}, \quad \|d\Phi^{-t}_{\restriction E^u(\rho)}\| \leq Ce^{-\lambda t}$$

$E^s(\rho)$ tangent to the stable manifold $W^s(\rho)$.

[Hopf, Anosov]: Φ^t is ergodic and mixing w.r.t. the Liouville measure μ_L on S^*M: strong chaos.

Quantum ergodicity

Theorem (Quantum Ergodicity)

[SCHNIRELMAN'73, ZELDITCH'87, COLIN DE VERDIÈRE'85...] *Assume Φ^t is ergodic on S^*M. Then there exists a subsequence $\mathcal{S} \subset \mathbb{N}$ of density 1 such that, for any open set $V \subset M$,*

$$\lim_{\mathcal{S} \ni j \to \infty} \int \mathbb{1}_V(x) \, |u_j(x)|^2 \, dv_g(x) = \frac{\mathrm{Vol}_g(V)}{\mathrm{Vol}_g(M)} .$$

Observables $a(x, \xi) \in C_c^\infty(T^*M) \mapsto \mathrm{Op}_h(a) = a(x, hD_x)$ allow to test the localization both in position and momentum (h-Fourier).

$\rightsquigarrow$ Quantum Ergodicity can be lifted to phase space: $\forall a \in C_c^\infty(T^*M)$,

$$\lim_{\mathcal{S} \ni j \to \infty} \langle u_j, \mathrm{Op}_{h_j}(a) u_j \rangle_{L^2} = \int_{S^*M} a(x, \xi) \, d\mu_L(x, \xi),$$

μ_L the normalized Liouville measure on S^*M.

Quantum Unique Ergodicity conjecture [RUDNICK-SARNAK'94]: all eigenmodes u_j becomes equidistributed when $j \to \infty$ ($\mathcal{S} = \mathbb{N}$).

Constraining the localization of eigenmodes

If (exceptional) non-QUE eigenstates exist, how localized can they be?

One can always extract $\mathcal{E} \subset \mathbb{N}$, such that $(u_j)_{j\in\mathcal{E}}$ admits asymptotic localization properties:

$$\forall a \in C_c^\infty(T^*M), \qquad \lim_{\mathcal{E}\ni j\to\infty} \langle u_j, \mathrm{Op}_{h_j}(a) u_j\rangle_{L^2} = \int_{S^*M} a(x,\xi)\, d\mu_{sc}(x,\xi)\,.$$

- μ_{sc} the semiclassical measure associated with $(u_j)_{j\in\mathcal{E}}$. Probability measure on S^*M, invariant through Φ^t.

$\exists$ **many** invariant measures (ex: δ_γ on closed geodesics, measures supported on fractal invariant sets). Which ones can appear as μ_{sc}?

- [Anantharaman'06,Anantharaman-Koch-N., Rivière'10]: Lower bounds on the support of μ_{sc} and the Kolmogorov-Sinai entropy $H_{KS}(\mu_{sc})$.

$$\text{Ex: } (M,g) \text{ of curvature } -1: \qquad \dim(\mathrm{supp}\,\mu_{sc}) \geq 2, \qquad H_{KS}(\mu_{sc}) \geq \frac{1}{2} H_{\max}\,.$$

These bounds forbid $\mu_{sc} = \delta_\gamma$, but allow μ_{sc} to be supported on (not too thin) "fractal" flow-invariant subsets of S^*M.

Main result: in 2D, all eigenstates fully delocalize

Theorem ([DYATLOV-JIN'17])

(M, g) *compact hyperbolic surface (curvature= −1).*
Then, for any open set $\emptyset \neq \mathcal{V} \subset S^*M$*, there exists* $c_\mathcal{V} > 0$ *s.t. all semiclassical measures satisfy* $\mu_{sc}(\mathcal{V}) \geq c_\mathcal{V}$.

Corollary

For any open set $\emptyset \neq V \subset M$, $\exists c_V > 0$ *such that* $\forall j \in \mathbb{N}$, $\|u_j\|_{L^2(V)} \geq c_V$.

Theorem ([DYATLOV-JIN-N.'22])

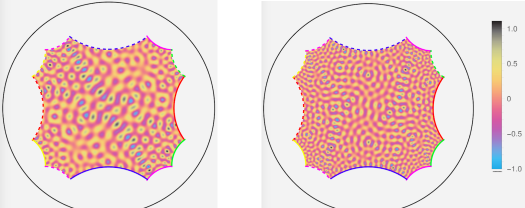
Generalize to (M, g) *compact Riemannian surface carrying an Anosov geodesic flow, e.g. surface of variable negative curvature.*

Remark: on any compact (M, g), Carleman estimates $\Rightarrow \|u_j\|_{L^2(V)} \geq e^{-C/h_j}$.
Sharp on the round sphere ("equatorial" spherical harmonics).

Here, the lower bounds $c_\mathcal{V}, c_V$ are uniform w.r.t. h_j!

Eigenmodes of a hyperbolic surface

(© Alex Strohmaier)



Quantum partition of unity

Notations: set $h = h_j$, $u_h = u_j$, satisfies $(-h^2\Delta - 1)u_h = 0$.

Our goal: for $\mathcal{V} \subset S^*M$ take $a_1 \in C_c^\infty(T^*M)$, $a_1 \approx \mathbb{1}_\mathcal{V}$, and show that

$$\exists C_{a_1} > 0, \quad \text{for all eigenstates } u_h, \qquad \boxed{\|u_h\| \leq C_{a_1} \| \mathrm{Op}_h(a_1) u_h \|}$$

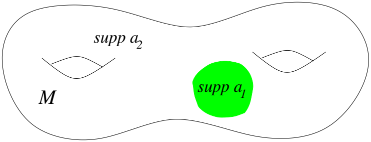
Strategy (inspired by [ANANTHARAMAN'06]):

- complete a_1 to form a smooth partition of unity near S^*M:

$$a_1 + a_2 = 1 \quad \text{near } S^*M, \qquad a_2 \approx \mathbb{1}_{S^*M \setminus \mathcal{V}}.$$

- $\rightsquigarrow$ Quantum partition of unity: $A_1 + A_2 \equiv Id$ near S^*M:

$$A_1 u_h + A_2 u_h = u_h + \mathcal{O}(h^\infty)_{L^2}, \quad \text{where } A_i \stackrel{\text{def}}{=} \mathrm{Op}_h(a_i).$$



- *Evolve* this quantum partition through the wave propagator:

$$A_1(t) + A_2(t) \equiv Id \quad \text{near } S^*M, \quad \text{where } A_i(t) \stackrel{\text{def}}{=} U^{-t} A_i U^t.$$

Evolving the quantum partition

- Quantum-classical correspondence (Egorov theorem):

$$A_i(t) = U^{-t} A_i U^t = \mathrm{Op}_h(a_i \circ \Phi^t) + \mathcal{O}_{a_i,t}(h),$$

holds as long as $a_i \circ \Phi^t$ oscillates on scales $\gg h^{1/2}$ (admissible symbol).

- *Compose* these evolved partitions for times $t = 0, 1, 2, \cdots, n$. Write the resulting sum in terms of words $\boldsymbol{w} = w_0 w_1 \cdots w_{n-1}$, $w_j \in \{1, 2\}$

$$\sum_{|\boldsymbol{w}|=n} A_{\boldsymbol{w}} \equiv Id \quad \text{near } S^*M, \quad \text{with} \quad A_{\boldsymbol{w}} = A_{w_{n-1}}(n-1) \cdots A_{w_1}(1) A_{w_0}.$$

- Egorov Thm + semiclassical calculus ($\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) \approx \mathrm{Op}_h(a \times b)$):

$$A_{\boldsymbol{w}} = A_{w_{n-1}}(n-1) \cdots A_{w_1}(1) A_{w_0} \approx \mathrm{Op}_h(a_{\boldsymbol{w}}), \quad \text{for}$$
$$a_{\boldsymbol{w}} = a_{w_{n-1}} \circ \Phi^{n-1} \times \cdots \times a_{w_1} \circ \Phi^1 \times a_{w_0},$$

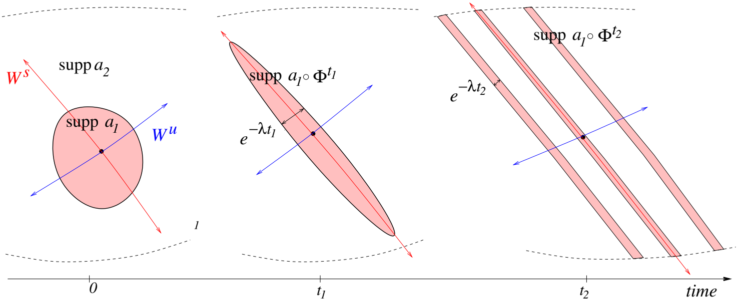
provided all $a_{w_j}(j)$ are admissible.

$\{a_{\boldsymbol{w}} : |\boldsymbol{w}| = n\}$ (smooth) partition *refined* by the flow Φ^t
$\{A_{\boldsymbol{w}} : |\boldsymbol{w}| = n\}$ *refined* quantum partition (near S^*M)

Pushing the dynamics to large times (curv.$= -1$)

Until what times j do $a_{w_j} \circ \Phi^j$ remain admissible (oscillate on scales $\gg h^{1/2}$)?

- Hyperbolicity of $\Phi^t \Longrightarrow a_j \circ \Phi^t$ elongated along W^s, contracted along W^u, at the rate $\sim \exp(\pm\lambda t)$ (λ= Lyapunov exponent).



- Ehrenfest time $T_E \stackrel{\text{def}}{=} \frac{|\log h|}{\lambda}$, time for the dynamics to connect scales 1 to h.

$\rightsquigarrow a_{w_j} \circ \Phi^j$, and hence $a_{\mathbf{w}}$, are admissible provided $n < \frac{1}{2} T_E$.

"Good" words

$$\text{Our goal}\quad \|u_h\| = \|\sum_{|\boldsymbol{w}|=n} A_{\boldsymbol{w}} u_h\| \le C_{a_1} \|A_1 u_h\| \quad \text{for some } n > 0.$$

- One identifies a large family of good words:

Given $\alpha \in (0,1)$, we say that the n-word $\boldsymbol{w}$ is α-good if $\#\{w_j = 1\} \ge \alpha n$.

Dynamical interpretation: if $(x,\xi) \in \operatorname{supp} a_{\boldsymbol{w}}$, then the discrete orbit $(\Phi^j(\rho))_{0\le j<n}$ visits $\operatorname{supp} a_1$ at least αn times.

- Claim: for $n \le T_E/4$, α-good words are controlled by $A_1 u_h$:

$$\Big\| \sum_{\substack{|\boldsymbol{w}|=n \\ \boldsymbol{w}\ \alpha-\text{good}}} A_{\boldsymbol{w}} u_h \Big\| \le \frac{C_c}{\alpha} \|A_1 u_h\| + \mathcal{O}(h^\gamma).$$

Proof: Egorov thm + semiclassical calculus + easy tricks.

- This construction can be extended to times $n > T_E/2$ (symbols $a_{\boldsymbol{w}}$ are then nonadmissible). We will use it for $n \approx 2T_E$.

The remaining words are called "α-bad".

How to control "bad" words?

There remains to estimate the contribution of $\sum_{\mathbf{w}\ \alpha-\mathrm{bad}} A_{\mathbf{w}} u_h$.

- combinatorics: if $\alpha > 0$ chosen small, there will be few α-bad words

$$\#\{|\mathbf{w}| = n \approx 2T_E,\ \mathbf{w}\ \alpha - \text{bad}\} \leq h^{-\varepsilon} \quad \text{out of } 2^n \sim h^{-2\log 2/\lambda}\ n\text{-words}$$

- all individual operators $A_{\mathbf{w}}$ of length $n \approx 2T_E$ are small:

Proposition

There exists $\beta > 0$ and $h_0 > 0$ such that, $\forall h < h_0$,

$$\boxed{\textit{for any word of length } |\mathbf{w}| \approx 2T_E, \quad \|A_{\mathbf{w}}\| \leq h^{\beta}.}$$

From there, easy to conclude the proof of the Theorem:

$$1 = \|u_h\| = \Big\| \sum_{|\mathbf{w}|=\lfloor 2T_E \rfloor} A_{\mathbf{w}} u_h \Big\| \leq \Big\| \sum_{\mathbf{w}\ \alpha-\mathrm{good}} A_{\mathbf{w}} u_h \Big\| + \Big\| \sum_{\mathbf{w}\ \alpha-\mathrm{bad}} A_{\mathbf{w}} u_h \Big\|$$

$$\leq \Big(\frac{C}{\alpha} \|A_1 u_h\| + \mathcal{O}(h^{\gamma}) \Big) + h^{-\varepsilon} h^{\beta}$$

choose α s.t. $\varepsilon < \beta \implies \|u_h\| \leq \dfrac{2C}{\alpha} \|A_1 u_h\|$ if $h < h_0$. □

Why do we need to take the time $n > T_E$?

Our goal: for all n-words $\mathbf{w}$, $\|A_{\mathbf{w}}\|_{L^2\to L^2} \leq h^{\beta}$.

$\ominus$ if $n \leq T_E$: conjugating by $U^{n/2}$, we find

$$U^{n/2} A_{\mathbf{w}} U^{-n/2} = \underbrace{A_{w_{n-1}}(n/2-1)\cdots A_{w_{n/2}}(0)}_{A^+_{\mathbf{w}^+}}\underbrace{A_{w_{n/2-1}}(-1)\cdots A_{w_0}(-n/2)}_{A^-_{\mathbf{w}^-}}$$

$A^+_{\bullet}$, resp. $A^-_{\bullet}$ describe refined partitions in the future, resp. in the past, of lengths $|\mathbf{w}^-| = |\mathbf{w}^+| = n/2 \leq T_E/2$, hence admissible

$$\Longrightarrow A^+_{\mathbf{w}^+} A^-_{\mathbf{w}^-} \approx \mathrm{Op}_h(a^+_{\mathbf{w}^+})\,\mathrm{Op}_h(a^-_{\mathbf{w}^-}) \approx \mathrm{Op}_h(a^+_{\mathbf{w}^+} a^-_{\mathbf{w}^-})$$

$$\rightsquigarrow \|A^+_{\mathbf{w}^+} A^-_{\mathbf{w}^-}\| \approx \|a^+_{\mathbf{w}^+} a^-_{\mathbf{w}^-}\|_{L^\infty} \lesssim 1.$$

$\Longrightarrow$ to obtain the bound h^{β}, the symbols $a^+_{\mathbf{w}^+}$, $a^-_{\mathbf{w}^-}$ cannot be admissible, hence we need $n = |\mathbf{w}| > T_E$.

Fractal Uncertainty Principle

To show $\|A^+_{w^+} A^-_{w^-}\|_{L^2 \to L^2} \leq h^\beta$ we use a "black box" result from 1D harmonic analysis.

Theorem ([Dyatlov-Zahl'16, Bourgain-Dyatlov'18])

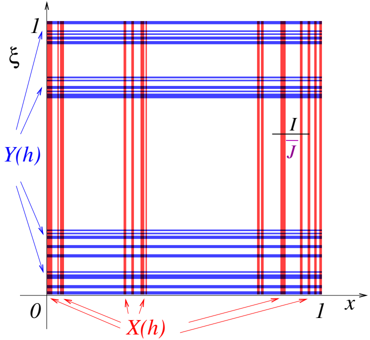
Let X, $Y \subset [0,1]$ be ***fractal*** *sets, and for $h \in (0,1)$ call $X(h)$, $Y(h)$ their h-neighbourhoods.*
Then $\exists \beta = \beta(X,Y) > 0$ s.t., for any $0 < h < h_0$,

$$\|\mathbb{1}_{X(h)}(x)\, \mathbb{1}_{Y(h)}(hD_x)\|_{L^2 \to L^2} \leq h^\beta .$$

A function $u \in L^2(\mathbb{R})$ cannot be essentially localized on $X(h)$, if its h-Fourier transform is essentially localized on $Y(h)$.

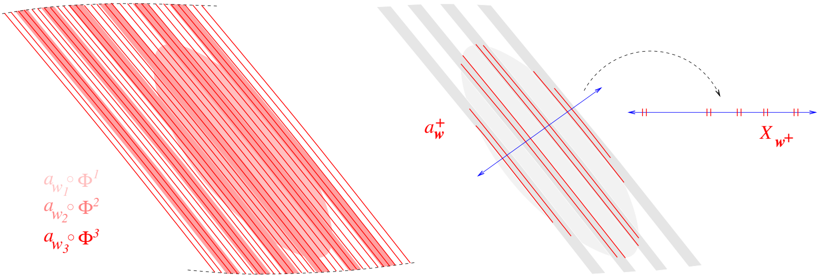
A fractal set X is ν-**porous** for some $\nu \in (0,1)$:
For any interval $I \subset [0,1]$, $\exists J \subset I$, s.t. $|J| \geq \nu |I|$ and $J \cap X = \emptyset$.

This is not a volume effect:
holds if $|X(h)| \times |Y(h)| \gg h$



The support of $a_{\mathbf{w}}^{+}$ is fractal

Remember the foliation of S^*M by stable, resp. unstable leaves.



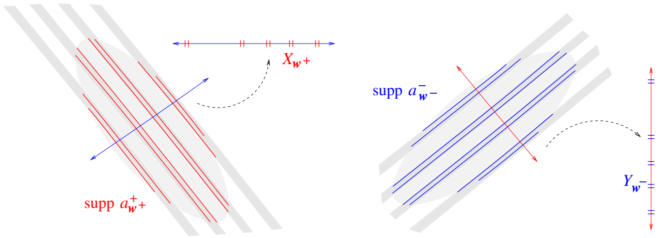
When $m \gg 1$, $a_1 \circ \Phi^m$ is supported on "stable stripes" of widths $\sim e^{-\lambda m}$, separated by stripes of widths $\sim e^{-\lambda m}$.

$\Longrightarrow$ the support of $a_{\mathbf{w}^+}^{+} = \prod_{m=0}^{n/2-1} a_{w_m^+} \circ \Phi^m$ is porous transversely to W^s, on the scales $1 \searrow e^{-\lambda n/2}$.

Ex: for $|\mathbf{w}^+| \approx T_E$, the set $X_{\mathbf{w}^+} \subset \mathbb{R}$ is porous on scales $1 \searrow h$.

- Similarly: porosity of $\operatorname{supp} a_{\mathbf{w}^-}^{-}$ transversely to W^u.

Future and past fractal symbols



Symbols $a^+_{\boldsymbol{w}^+}$ and $a^-_{\boldsymbol{w}^-}$, associated with words $\boldsymbol{w}^\pm$ of length $n/2 \sim T_E$.
supp $a^+_{\boldsymbol{w}^+}$ is fractal transversely to W^s, on scales $1 \searrow h$.
supp $a^-_{\boldsymbol{w}^-}$ is fractal transversely to W^u, on scales $1 \searrow h$.

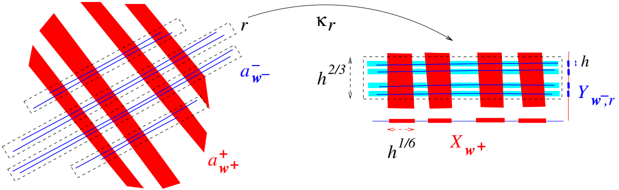
- $X_{\boldsymbol{w}^+}$, $Y_{\boldsymbol{w}^-}$ are porous sets on scales $1 \searrow h$, like the sets $X(h)$, $Y(h)$ in the FUP. We want to relate the product $A^+_{\boldsymbol{w}^+} A^-_{\boldsymbol{w}^-}$ with the product $\mathbb{1}_{X_{\boldsymbol{w}^+}}(x)\mathbb{1}_{Y_{\boldsymbol{w}^-}}(hD_x)$ appearing in the FUP
- Can we *straighten* the stable/unstable foliations through a symplectomorphism κ, so that they become vertical / horizontal?

Variable curvature: straightening $C^{2-\varepsilon}$ foliations

$\ominus$ in general the foliations W^s, W^u are only $C^{2-\epsilon}$ regular
$\Rightarrow$ impossible to straighten them by a smooth symplectomorphism.

Treat past and future differently: $|\mathbf{w}^+| \approx T_E/6$, $|\mathbf{w}^-| \approx T_E$

- $A^+_{\mathbf{w}^+} \approx \mathrm{Op}_h(a^+_{\mathbf{w}^+})$ (admissible), $\mathrm{supp}\, a^+_{\mathbf{w}^+}$ porous on scales $1 \searrow h^{1/6}$.
- $A^-_{\mathbf{w}^-}$ not admissible, but **is microlocalized** on $\mathrm{supp}\, a^-_{\mathbf{w}^-}$ down to scale h.



- split $\mathrm{supp}\, a^-_{\mathbf{w}^-}$ into "unstable clusters" of thickness $\sim h^{2/3}$.
- each cluster r is (almost) straightened by κ_r, quantized by an FIO $\mathcal{U}_{\kappa_r}$

$\Rightarrow \mathcal{U}_{\kappa_r} A^+_{\mathbf{w}^+} A^-_{\mathbf{w}^-,r} \mathcal{U}^*_{\kappa_r}$ controlled by $\mathbb{1}_{X_{\mathbf{w}^+}}(x) \mathbb{1}_{Y_{\mathbf{w}^-,r}}(hD_x) \overset{FUP}{\rightsquigarrow} \|A^+_{\mathbf{w}^+} A^-_{\mathbf{w}^-,r}\| \leq h^\beta$.

- clusters almost orthogonal to e.o. $\overset{Cotlar-Stein}{\rightsquigarrow} \|A^+_{\mathbf{w}^+} A^-_{\mathbf{w}^-}\| \leq h^\beta$ □

Conclusions

- Our results are restricted to *surfaces*. Some higher-dimensional versions of the FUP exist [HAN-SCHLAG'18,COHEN'22,'23, BACKUS-LENG-TAO'23], but there are counterexamples.
- Quantitative estimates on the constant c_γ can be obtained (require estimates of β [JIN-ZHANG'17]), very small.
- Can we apply the method to surfaces with boundary? To chaotic Euclidean domains?
- $\exists$ counterexamples to QUE ($\mu_{sc} \neq \mu_L$) for quantized Anosov maps, e.g. for the "quantum cat map" on $\mathbb{T}^2$ [FAURE-N-DEBIÈVRE'03].

  Full delocalization can also be proved in this setup [SCHWARTZ'21], and (under conditions) for higher dimensional "cat maps" [DYATLOV-JEZEQUEL'21].

Thank you for your attention