Phase transition in the Integrated Density of States of the Anderson model arising from a supersymmetric sigma model.

joint work with V. Rapenne, C. Rojas-Molina and X. Zeng

Margherita DISERTORI

University of Bonn & Hausdorff Center for Mathematics

GDR DynQua "Dynamique Quantique',
Cergy 2024

- some facts about Anderson model and integrated density of states
- Anderson model arising from the $H^{2|2}$ supersymmetric sigma model/vertex reinforced jump process
- Integrated density of states in this special case

Anderson model (discrete random Schrödinger)

electronic transport/wave propagation in disordered systems

the model on $\mathbb{Z}^d$: $H = H_0 + \lambda V \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}_{sym}$ random op.

- $H_0 = -\Delta =$ lattice Laplacian
  $(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, $-\Delta f(j) = \sum_{|k-j|=1} (f(j) - f(k))$
- $(Vf)(j) = V_j f(j)$, $\{V_j\}_{j \in \mathbb{Z}^d}$ random variables with prob. distr. $d\mathbb{P}(V)$
  $\to \mathrm{Var}(V_j) = 1$, $\mathbb{P}$ translation inv. and V (almost) indep
- $\lambda > 0$ parameter

heuristics:

- $\lambda=0$ $H=H_0=-\Delta$, $\sigma(-\Delta)=[0,4d]$, a.c. spectrum, delocalized eigenvectors
- $\lambda \gg 1$ $H \simeq \lambda V$ $\sigma(\lambda V)=\lambda \operatorname{supp} V$, p.p. spectrum, localized eigenvectors

Question: intermediate λ?

position of the spectrum

$\mathbb{P}$ translation invariant $\Rightarrow$ $\sigma(H)=$ deterministic interval a.s.

spectral type

- $d = 1$ localization $\forall \lambda > 0$
- $d \geq 2$ localization
  - $\forall \lambda > 0$ at the *edge* of the spectrum
  - for $\lambda \gg 1$ in the *bulk* of the spectrum
- *phase transition* on the *Bethe lattice:*
  - for $\lambda \gg 1$ localization
  - for $\lambda \ll 1$ delocalization in the bulk

Major open conjecture: on $\mathbb{Z}^d$ $d \geq 3$

for $\lambda \ll 1$: delocalization in the bulk of the spectrum

Integrated Density of States: $N(E,H):=\lim_{L\to\infty} N(E,H_{\Lambda_L})=\lim_{L\to\infty}\mathbb{E}[N(E,H_{\Lambda_L})]$

$\Lambda_L=[-L,L]^d\cap\mathbb{Z}^d,\quad H_{\Lambda_L}=H_{|\Lambda_L}\qquad N(E,H_{\Lambda_L})=\frac{\#\text{eigenvalues}_{\leq E}}{|\Lambda_L|}=\frac{\mathrm{Tr}\,(\mathbf{1}_{(-\infty,E)}(H_{\Lambda_L}))}{|\Lambda_L|}$

IDS for V i.i.d. Assume $d\mathbb{P}(V)=\prod_{j\in\mathbb{Z}^d}\mu(V_j)dV_j$ $\operatorname{supp}\mu=[0,\infty)$ plus some conditions on μ

- $\sigma(H)=[0,\infty)$ a.s.
- almost no spectrum at the lower edge: *Lifschitz tails*

for $0<E\ll 1$ $\qquad N(E,H)=c_1 e^{-c_2(E)^{-\frac{d}{2}}}\ll N(E,-\Delta)\propto(E)^{\frac{d}{2}}$

mechanism: $H_0+\lambda V\geq H_0\Rightarrow H$ minimal when $V_j=0\ \forall j$

many V_j small $\to$ very unlikely

- information on the spectral type?
  - edge: Lifschitz tails $\Rightarrow$ localization near 0, $\forall\lambda>0$
  - bulk: $N(E,H)$ cannot see the phase transition a.c/p.p :

$E\mapsto\mu(E)$ 'regular' $\Rightarrow E\mapsto N(E,H)$ 'regular' $\forall\lambda>0$

Anderson model arising from $H^{2|2}$

the model on $\mathbb{Z}^d$: $H_\beta = H_0 + \frac{1}{W} V(\beta) \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}_{sym}$ random op.

- $H_0 = -P = -\Delta - 2d \;\; P_{ij} = \mathbf{1}_{|i-j|=1} \;\; W > 0$ parameter
- $V_j(\beta) = 2\beta_j, \; \{\beta_j\}_{j \in \mathbb{Z}^d}$ random variables with prob. distr. $d\mathbb{P}_W(\beta)$

finite volume marginal of $d\mathbb{P}_W(\beta)$: $\Lambda \subset \mathbb{Z}^d$ finite [Sabot-Tarrès-Zeng 2015]

$$d\nu_{W,\Lambda}(\beta) = \left(\frac{2}{\pi\sqrt{W}}\right)^{|\Lambda|} \mathbf{1}_{H_{\beta,\Lambda}>0} \frac{1}{(\det H_{\beta,\Lambda})^{1/2}} e^{-\frac{W}{2}\left(\langle 1, H_{\beta,\Lambda} 1\rangle + \langle \eta, H^{-1}_{\beta,\Lambda} \eta\rangle - 2\langle \eta, 1\rangle\right)} \prod_j d\beta_j$$

- $H_{\beta,\Lambda} = (H_\beta)_{|\Lambda}, \; \langle f, g\rangle = \sum_{j\in\Lambda} f_j g_j$
- $\eta_j = \#\{k \notin \Lambda \mid |k-j| = 1\}$ "wired boundary conditions"

finite volume Laplace transform:

$$\mathbb{E}_\Lambda\left[e^{\langle f, \beta\rangle}\right] = \prod_{j\in\Lambda} \frac{e^{-\eta_j(\sqrt{1+f_j}-1)}}{\sqrt{1+f_j}} \prod_{|i-j|=1} e^{-W_{ij}(\sqrt{1+f_i}\sqrt{1+f_j}-1)}, \quad f_j \geq -1$$

some properties

- the r.v. β are independent at distance 2
- $\exists ! \mathbb{P}_W(\beta)$ on $\mathbb{R}^{\mathbb{Z}^d}$ with marginals $d\nu_{W,\Lambda}(\beta)$ (Kolmogorov ext. thm.)
- H_β is ergodic and $\sigma(H_\beta)=[0,\infty)$ a.s. [Sabot,Zeng 2019][Rapenne 2023]
- $\sigma(H_0)=[-2d,2d]$ but $\sigma(H_\beta)=[0,\infty)$ (due to the constraint $H_\beta>0$)

  edge in the middle of $\sigma(H_0)=[-2d,2d]$

  $\beta=0 \to H=H_0 \not> 0$: some V_j must be always large, main mechanism creating Lifschitz tails broken
- heuristics: here W enters in $\mathbb{P}(\beta)$

$$\mathbb{E}\left[\frac{2\beta_j}{W}\right]=2d+\frac{1}{W}\simeq\begin{cases}2d & W\gg 1\\ \frac{1}{W}\gg 1 & W\ll 1\end{cases} \qquad \mathrm{Var}\left[\frac{2\beta_j}{W}\right]=\frac{2d}{W}+\frac{2}{W^2}\begin{cases}\ll 1 & W\gg 1\\ \gg 1 & W\ll 1\end{cases}$$

strong disorder $W\ll 1$: $H_\beta\simeq\frac{2\beta}{W}$ weak disorder $W\gg 1$: $H_\beta\simeq 2d-P=-\Delta$

$$\to H_\beta\equiv-\Delta+\lambda V,\ \lambda=\frac{1}{W}$$

Origin: supersymmetric nonlinear sigma model $H^{2|2}$ [Zirnbauer 1996]

originally introduced as a toymodel for quantum diffusion

Model in finite volume $\Lambda \subset \mathbb{Z}^d$

- spin configuration $S_\Lambda : \Lambda \to \mathbb{R}^{3|2}$
- spin at one point S is a (super)vector $S = (x, y, z, \xi, \eta)$,

  x, y, z even, ξ, η odd elements in a real Grassmann algebra
- $\langle S, S' \rangle = xx' + yy' - zz' + \xi\eta' - \eta\xi'$
- nonlinear constraint $\langle S, S \rangle = -1 \Rightarrow z = \sqrt{1 + x^2 + y^2 + 2\xi\eta} \Rightarrow S \in H^{2|2}$
- $\eta_j \geq 0$ magnetic field in direction $e = (0, 0, 1, 0, 0)$
- "Gibbs" measure:

  $d\mu_\Lambda(S) = \prod_{|i-j|=1} e^{W(\langle S_i, S_j \rangle + 1)} \prod_j e^{W\eta_j(\langle e, S_j \rangle + 1)} \delta(\langle S_j, S_j \rangle + 1) \; dS_\Lambda$

  $W =$ inverse temperature

From $H^{2|2}$ to H_β

- *horospherical coordinates:* $(x,y,\xi,\eta)\to(u,s,\bar\psi,\psi)$

$d\mu_\Lambda(S)\to d\mu_\Lambda(u,s,\bar\psi,\psi)=$

$$e^{-W\sum_{|i-j|=1}(\cosh(u_i-u_j)-1)-W\sum_j\eta_j(\cosh u_j-1)]}e^{-\frac{W}{2}(s,Ms)-W(\bar\psi,M\psi)}\;(due^{-u}dsd\bar\psi d\psi)^\Lambda$$

$$(s,Ms)=\sum_{|i-j|=1}e^{u_i+u_j}(s_i-s_j)^2+\sum_{j\in\Lambda}\eta_j e^{u_j}s_j^2$$

- *u−marginal:*

$$d\rho_\Lambda(u)=e^{-W\sum_{|i-j|=1}(\cosh(u_i-u_j)-1)-W\sum_j\eta_j(\cosh u_j-1)]}\sqrt{\det(e^{-u}Me^{-u})}\left(\frac{W}{2\pi}\right)^{\frac{|\Lambda|}{2}}du_\Lambda$$

$$e^{-u}Me^{-u}=H_{\beta(u)}=-P+\frac{1}{W}2\beta(u),\qquad \frac{1}{W}2\beta(u)_j=\sum_{|k-j|=1}e^{u_k-u_j}+\eta_j e^{-u_j}$$

- $H^{2|2}\leftrightarrow H_\beta$: $\mathbb{E}^u_\Lambda[f(\beta(u))]=\mathbb{E}^\beta_\Lambda[f(\beta)]$

From $H^{2|2}$ to vertex reinforced jump process (VRJP)

VRJP (Werner 2000): continuous time process $(X_t)_{t\geq 0}$ on $\mathbb{Z}^d$.

$$\mathbb{P}\Big(X_{t+dt}=j|\ X_t=i,(X_s)_{s\leq t}\Big)=dt\ \mathbf{1}_{|i-j|=1}W(1+L_j(t))+o(dt)$$

$$L_j(t)=\int_0^t \mathbf{1}_{X_s=j}ds= \text{ total time spent at } j \text{ up to time } t$$

$(1+L_j(t))\to$ history dependence: prefers to come back

- $H^{2|2}\leftrightarrow$ *VRJP:*

VRJP in finite volume Λ=mixture of Markov jump processes

$$\mathbb{P}_\Lambda^{VRJP}[\cdot]'='\int \mathbb{P}_\Lambda^{W(u)}[\cdot]d\rho_\Lambda(u), \qquad W(u)_{ij}=We^{u_i+u_j}\mathbf{1}_{|i-j|=1}$$

$\to u-$marginal=mixing measure for VRJP [Sabot-Tarrès-Zeng]

three related models: $H^{2|2}$, VRJP, H_β

Phase transitions in $d \geq 3$

- $H^{2|2}$: disordered for $W \ll 1$ 'ordered' for $W \gg 1$

  [D.-Spencer, D.-Spencer-Zirnbauer 2010]

- VRJP: recurrent for $W \ll 1 \leftrightarrow$ transient for $W \gg 1$

  [Sabot-Tarrès 2013]

Conjecture: phase transition for H_β in $d \geq 3$

- localization $W \ll 1 \leftrightarrow$ extended for $W \gg 1$

  only localization proved [Collevecchio-Zeng 2021]

IDS for H_β

Theorem [D.-Rapenne-Rojas Molina-Zeng 2023]

- phase transition in $d \geq 3$ near the edge $0 < E \ll 1$:
  $N(E) \simeq \sqrt{E}$ for $W \ll 1$ $\leftrightarrow$ $N(E) \leq E$ for $W \gg 1$
- no Lifschitz tails in

  $d=1$ (any disorder $W>0$)

  $d \geq 2$ (strong disorder $W \ll 1$)

Precisely $\forall d \geq 1$ we have:

$$N(E) \leq 2\sqrt{\tfrac{W}{\pi}}\sqrt{E} \qquad \forall W>0, E>0$$
$$N(E) \geq c_W (|\log E|)^{-d}\sqrt{E} \qquad \forall 0<W<W_c(d),\ 0<E<E_0(W,d)$$

Note: $W_c(1)=\infty,\ W_c(d)=\frac{C}{d}, d\geq 2,\quad C=\frac{\sqrt{\pi}}{\Gamma(1/4)2^{3/4}}$

On the contrary $\forall d \geq 3$ we have:

$$N(E, H_\beta) \leq C' E \qquad \forall E>0, W>W_0(d)$$

lower bound: strategy of the proof

- $N(E,H_\beta)\geq\frac{1}{|\Lambda_L|}\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{E}\Big)\quad\forall L>0$

$$H^D_{\beta,\Lambda_L}=H_{\beta,\Lambda_L}+M_{2d-n},\qquad M_{2d-n}f(j)=(2d-n_j)f(j),\ n_i=\sum_{|k-i|=1,k\in\Lambda}1$$

no direct info on $(H^D_{\beta,\Lambda_L})^{-1}(0,0)\to$ use info on $H^{-1}_{\beta,\Lambda_L}(0,0)$

- $0<(H^D_{\beta,\Lambda_L})^{-1}(0,j)\leq H^{-1}_{\beta,\Lambda_L}(0,j)\forall j$ (rand. walk repr.) hence

$$(H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq H^{-1}_{\beta,\Lambda_L}(0,0)-(2d-1)\sum_{j\in\partial\Lambda_L}H^{-1}_{\beta,\Lambda_L}(0,j)^2$$

- $W\ll 1\Rightarrow\mathbb{P}(\Omega_{loc}):=\mathbb{P}\Big(H^{-1}_{\beta,\Lambda_L}(0,j)\leq e^{-c|j|}\forall j\in\partial\Lambda_L\Big)\simeq 1$

- on $\Omega_{loc},H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{1}{E}\Rightarrow(H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{E}-L^{d-1}e^{-c2L}\geq\frac{1}{2E}$

$$\Rightarrow\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{E}\Big)\geq\mathbb{P}\Big(H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{2}{E}\cap\Omega_{loc}\Big)$$

lower bound: strategy of the proof

- conditioned on β_{0^c} $(H_{\beta,\Lambda_L})^{-1}(0,0)=\frac{1}{y}$ $d\rho_a(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(a-y)^2}{2y}}\frac{1}{\sqrt{y}}dy$

$$a=a(\beta_{0^c})=\sum_{j\in\partial\Lambda_L}\frac{H^{-1}_{\beta,\Lambda_L}(0,j)}{H^{-1}_{\beta,\Lambda_L}(0,0)}\eta_j$$

$$\Rightarrow\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\geq\mathbb{E}\Big[\mathbf{1}_{\Omega_{loc}}\int_0^{WE}d\rho_a(y)\Big]$$

add to Ω_{loc} : $(H_{\beta,\Lambda_L})^{-1}(0,0)\leq e^{cL/2}$

$$\Rightarrow 0<a\leq Ke^{-cL/2},\quad \mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\geq\mathbb{E}\Big[\mathbf{1}_{\Omega_{loc}}\int_{We^{-cL}}^{WE}d\rho_a(y)\Big]$$

using the bound on a

$$\int_{We^{-cL}}^{WE}d\rho_a(y)=\frac{1}{\sqrt{2\pi}}\int_{We^{-cL}}^{WE}e^{-\frac{(a-y)^2}{2y}}\frac{1}{\sqrt{y}}dy\simeq\int_{We^{-cL}}^{WE}\frac{1}{\sqrt{y}}dy\simeq\sqrt{E}$$

upper bound: strategy of the proof

- in finite volume $N(E,H_{\beta,\Lambda_L}) \leq \frac{1}{|\Lambda_L|} \sum_{j\in\Lambda_L} \mathbb{E}_{W,\Lambda_L}\left[\mathcal{L}_{\rho_{a_j}}(2E)\right]$

$\mathcal{L}_{\rho_{a_j}} =$ Lévy concentration of the conditional measure $\rho_{a_j(\beta_{j^c})}$

$$\mathcal{L}_\mu(\varepsilon) := \sup_x \mu([x, x+\varepsilon])$$

- $d \geq 1$: use $\mathcal{L}_{\rho_{a_j}}(\varepsilon) \leq c\sqrt{\varepsilon}$
- $d \geq 3$: use $\rho_{a_j}(y) \leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a} + \frac{1}{\sqrt{a}}\right)$ together with (use relation with $H^{2|2}$ and $u-$marginal)

$$\mathbb{E}[1/a] = \mathbb{E}_u[e^{-u_0} H^{-1}_{\beta(u),\Lambda_L}(0,0)] \leq C/W$$

Some open problems

- $d \geq 3$: complete the phase diagram for IDS
- spectral phase transition for H_β from the phase transition in the IDS?
- effect of the constraint $H_\beta > 0$?

  take $H = -P + \lambda V$ V centered i.i.d. bounded

  Question: add the constraint $H > 0$. Do we lose the Lipshitz tails?

THANK-YOU!