Boundary states of the Robin magnetic Laplacian

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Collaboration with L. Le Treust, N. Raymond and S. Vũ Ngọc

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We assume that the magnetic field is constant equal to 1 (and associated with a smooth vector potential $\mathbf{A}$):

$$B := \partial_1 A_2 - \partial_2 A_1 = 1\,.$$

Classical facts

The operator $\mathscr{L}_h$ is defined by the Friedrichs extension theorem, from the closed semi-bounded quadratic form, defined on $H^1(\Omega)$ by

$$u \mapsto \mathscr{Q}_{h,\mathbf{A}}(\psi) = \int_\Omega |(-ih\nabla - \mathbf{A})\psi|^2 \mathrm{d}x + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 \mathrm{d}s\,.$$

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- Describe the eigenvalues $(\lambda_j(h,\gamma))_{j\geqslant 1}$ in the semiclassical limit $h \to 0$.
- Presenting techniques to pass from energies defined in Ω to effective energies defined in simpler sets.

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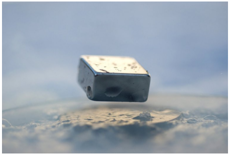

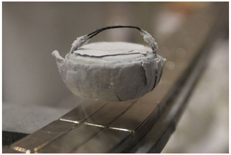

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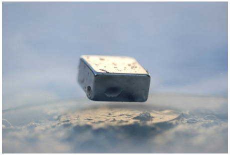

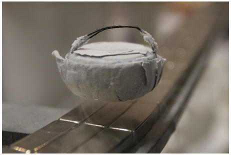

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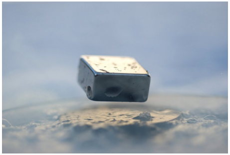

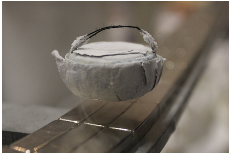

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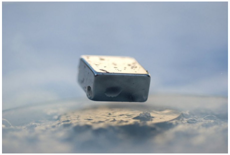

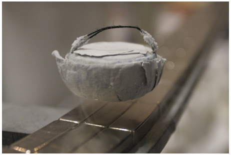

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- Extend our knowledge of the spectrum of magnetic Schrödinger operators in the semiclassical limit: finding normal forms for magnetic operators, describing the tunneling effect ...

Robin-de Gennes operator

Our results are expressed in terms of the eigenvalues of de Gennes operator with Robin boundary condition acting as

$$H[\gamma,\sigma] = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t-\sigma)^2\,,$$

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Proposition (Dauge-Helffer'93, De Bièvre-Pulé'99, Kachmar'06, Fahs'23)

Let us fix $n \geqslant 1$. When $\gamma \in \mathbb{R}$, the function $\mu_n(\gamma,\cdot)$ is analytic and

$$\lim_{\sigma\to-\infty} \mu_n(\gamma,\sigma) = +\infty\,, \qquad \lim_{\sigma\to+\infty} \mu_n(\gamma,\sigma) = 2n-1.$$

Moreover, $\mu_n(\gamma,\cdot)$ has a unique minimum attained at $\sigma = \xi_{n-1}(\gamma)$, but not attained at infinity. This minimum is non-degenerate. The function $\mu_n(\gamma,\cdot)$ is decreasing on $(-\infty,\xi_{n-1}(\gamma))$ and increasing on $(\xi_{n-1}(\gamma),+\infty)$. In addition, we have, for all $n\geqslant 2$,

$$2n-3 < \Theta^{[n-1]}(\gamma) := \inf_{\sigma\in\mathbb{R}} \mu_n(\gamma,\sigma) < 2n-1.$$

When $\gamma = +\infty$, that is when the Robin condition is replaced by the Dirichlet condition, $\mu_n(+\infty,\cdot)$ is still smooth, but now decreasing from $+\infty$ to $2n-1$.

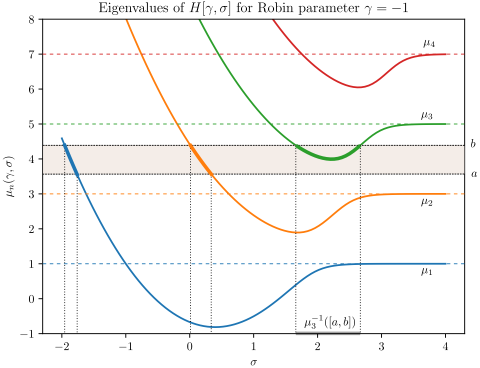
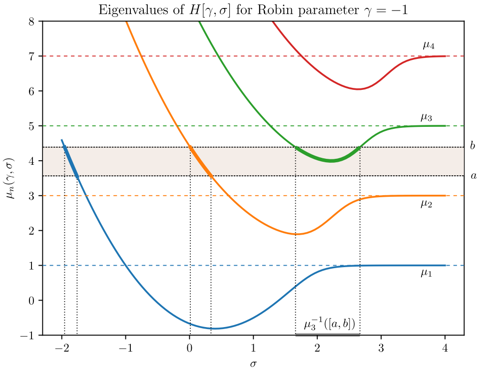


Figure: $\sigma \mapsto \mu_k(-1,\sigma)$, for $k = 1,\dots,4$.



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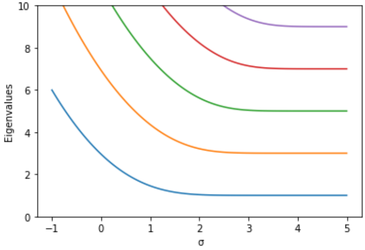

Figure: $\sigma \mapsto \mu_k(+\infty,\sigma)$, for $k = 1,\ldots,5$.

An overview of the known results

- The case $\gamma = 0$ has been studied a lot in the last 25 years, see especially the seminal work Helffer-Morame'01 where a two-term expansion of the groundstate energy is obtained by variational means (test functions, partition of the unity), and a tunneling result has even been proved recently.

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- When $\gamma \neq 0$, only the smallest eigenvalue has been estimated.

$$\lambda_1(\gamma, h) \underset{h \to 0}{=} \Theta^{[0]}(\gamma) h - \kappa_{\max} C(\gamma) h^{\frac{3}{2}} + o(h^{\frac{3}{2}}),$$

where $\kappa_{\max}$ is the maximum curvature of the boundary, and $C(\gamma) > 0$.

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✍ Goal: If we consider the spectrum in a given spectral window (not only the lowest eigenvalues), we want to study the spectral properties of $\mathscr{L}_h$ in the semiclassical regime $h \to 0$ and for $\gamma \in \mathbb{R} \cup \{+\infty\}$.

Magnetic Laplacian on the Euclidean plane

- If we forget the boundary condition, the operator acts as the magnetic Laplacian with constant magnetic field, on the Euclidean plane $\mathbb{R}^2$

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$$\{(2n-1)h\,, n \geqslant 1\},$$

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- This suggests considering the potential eigenvalues of $\mathscr{L}_h$ in a window of the form $[ha, hb]$ with:
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⚠ In fact, this decay does not really follow from the usual Agmon estimates, since we want to consider eigenvalues between two consecutive Landau levels.

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- If $p \in S_{\mathbb{T}_{2L}\times\mathbb{R}}(1)$, then $\mathrm{Op}^{\mathrm{W}}_{\hbar} p$ defines a bounded operator from $e^{i\theta\cdot}L^2(\mathbb{T}_{2L})$ to $e^{i\theta\cdot}L^2(\mathbb{T}_{2L})$.

- The usual class $S_{\mathbb{R}^2}(1)$ given by

$$S_{\mathbb{R}^2}(1) = \{p \in \mathscr{C}^\infty(\mathbb{R}^2_{s,\sigma}) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha p| \leqslant C_\alpha\}.$$

- Let us recall the formula for the Weyl quantization:

$$(\mathrm{Op}^{\mathrm{W}}_{\hbar} p)\psi(x) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p\left(\frac{x+y}{2}, \eta\right)\psi(y)\mathrm{d}y\mathrm{d}\eta\,, \quad \forall \psi \in \mathscr{S}(\mathbb{R}),$$

  which defines, in virtue of the Calderón-Vaillancourt theorem, a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

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- To shorten the notation, we will sometimes write p^{W} instead of $\mathrm{Op}^{\mathrm{W}}_{\hbar} p$.

Main result

Theorem

The spectrum of $\mathscr{L}_h$ in $[ha,hb]$ coincides with that of $h\mathfrak{M}_h$ modulo $\mathscr{O}(h^2)$, where

$$\mathfrak{M}_h := \begin{bmatrix} m_1^{\mathsf{W}} & 0 & \cdots & 0 \\ 0 & m_2^{\mathsf{W}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m_N{}^{\mathsf{W}} \end{bmatrix}$$

is a bounded operator acting diagonally on $e^{i\theta(h)\cdot}L^2(\mathbb{T}_{2L})^N$. Here

$$\theta(h) = \frac{|\Omega|}{|\partial\Omega| h},$$

and each m_k^{W} is an $h^{\frac{1}{2}}$-pseudodifferential operator with symbol in $S_{\mathbb{T}_{2L}\times\mathbb{R}}(1)$.

Main result

Theorem

Let us denote by (s,σ) *the (canonical) variables in* $\mathbb{T}_{2L}\times\mathbb{R}$. *Then, we have:*

- *the principal symbol of* m_k^{W} *is* $\mu_k\circ\Xi_0(\sigma)$;
- *its subprincipal symbol is* $-\kappa(s)C_k\circ\Xi_0(\sigma)$ *with*

$$C_k(\sigma)=\left\langle\left((\tau-\sigma)\tau^2-\partial_\tau-2\tau(\sigma-\tau)^2\right)u_k^{[\gamma,\sigma]}(\tau),\,u_k^{[\gamma,\sigma]}(\tau)\right\rangle_{L^2(\mathbb{R}_+)},$$

where

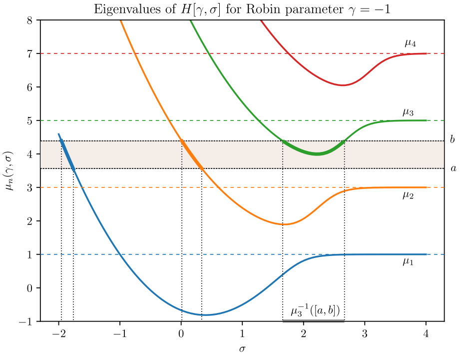
- $\Xi_0:\mathbb{R}\to\mathbb{R}$ *is a smooth, bounded with all its derivatives,*
- *for all* $k\in\{1,\ldots,N\}$, $\mu_k(\gamma,\Xi_0(\sigma))=\mu_k(\gamma,\sigma)$ *in a neighborhood of* $\mu_k^{-1}([a,b])$,
- $\kappa(s)$ *is the curvature of the boundary at the point of curvilinear abscissa* s.

Remark

For all $k\geqslant 1$, $C_k(\xi_{k-1}(\gamma))$ has the same sign as $\gamma_0^{[k-1]}-\gamma$, for some $\gamma_0^{[k-1]}$. This fact has important consequences on the asymptotics of the low-lying spectrum, which have not been observed before. In Kachmar'06, for $k=1$, it was stated that $C_1(\xi_0(\gamma))>0$ for all γ...

What are Ξ_0 and N ?

- Let $\Xi_0 : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded with all its derivatives, and increasing function such that for all $k \in \{1, \ldots, N\}$, $\mu_k(\gamma, \Xi_0(\sigma)) = \mu_k(\gamma, \sigma)$ in a neighborhood of $\mu_k^{-1}([a,b])$ and $\mu_k \circ \Xi_0$ takes its values in $(-\infty, a) \cup (b, +\infty)$ away from it.



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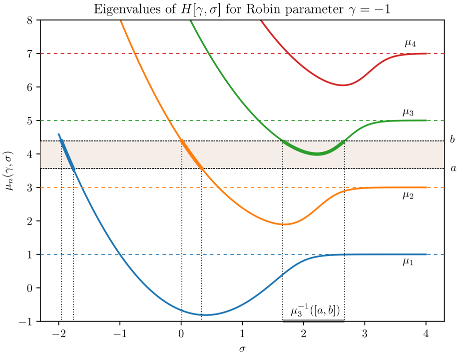

- We let

$$N := \#\{k \geqslant 1 : \mu_k(\gamma, \cdot)^{-1}([a,b]) \neq \emptyset\}.$$

Application I: low-lying eigenvalues

Corollary (Spectrum at critical value)

Consider $\gamma \neq \gamma_0^{[0]}$, and let $\varepsilon = \mathrm{sign}(\gamma_0^{[0]} - \gamma) = \mathrm{sign}(C_1(\xi_0(\gamma)))$. Assume that $\varepsilon\kappa$ admits a unique maximum at $s_{\max}$, which is non-degenerate. Then, for all $j \geqslant 1$, uniformly when $jh^{\frac{1}{4}} = o(1)$,

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa(s_{\max})C_1(\xi_0(\gamma))h^{\frac{3}{2}} + \frac{h^{\frac{7}{4}}(2j-1)}{2}\sqrt{k_2 C_1(\xi_0(\gamma))\mu_1''(\gamma, \xi_0(\gamma))} + o(h^{\frac{7}{4}}),$$

with $k_2 = -\kappa''(s_{\max})$.

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☞ If $\gamma = 0$, then $\mu_1''(\gamma,\xi_0(\gamma)) = 6C_1(\xi_0(\gamma))\sqrt{\Theta^{[0]}(\gamma)}$. Thus,

$$\lambda_j(0,h) = \Theta^{[0]}(\gamma)h - \kappa_{\max}C_1(\xi_0(\gamma))h^{\frac{3}{2}} + \frac{h^{\frac{7}{4}}(2j-1)}{2}C_1(\xi_0(\gamma))\Theta^{[0]}(\gamma)^{1/4}\sqrt{6k_2} + o(h^{\frac{7}{4}}).$$

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✔ It extends to any any value of the Robin parameter the result obtained by Fournais and Helffer (but without the uniformity in j) when $\gamma = 0$.

S. Fournais and B. Helffer. *Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian.* Ann. Inst. Fourier (Grenoble), 56(1), 2006.

Comments

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$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h + h^2\lambda_j(\mathscr{A}_h) + o(h^2),$$

where

$$\mathscr{A}_h = \frac{\partial_\sigma^2 \mu(\gamma, \xi_0(\gamma))}{2}(D_s + \alpha(h) - h^{-\frac{1}{2}}\xi_0(\gamma))^2 + C_\gamma \kappa^2(s),$$

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$$\lambda_j(\gamma,h) = \Theta^{[0]}(\gamma)h - \kappa C_1(\xi_0(\gamma))h^{\frac{3}{2}} + h^2\lambda_j(\mathscr{A}_h) + o(h^2).$$

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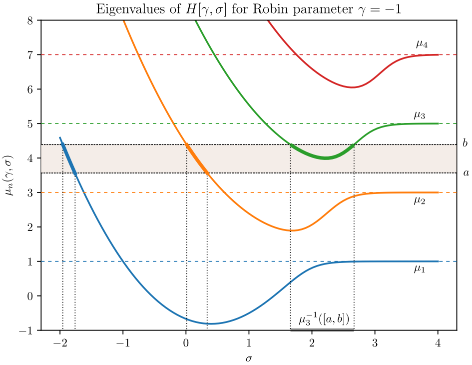
When $\gamma = 0$ and $j = 1$, a similar estimate is described in Fournais-Helffer'10.

S. Fournais and B. Helffer. *Spectral methods in surface superconductivity*, volume 77 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, 2010.

Application II: excited eigenvalues

Theorem (Spectrum at Regular Values)

- Let $[a,b]$ be an interval as before and without critical values.
- Let $p(k)$ be the number of connected components of $\mu_k^{-1}([a,b])$.
- For each $k=1,\ldots,N$, for each $q=1,\ldots,p(k)$, let $\Sigma_{k,q}\subset\mathbb{R}$ be an interval such that $\mu_k(\gamma,\cdot)$ is a diffeomorphism from $\Sigma_{k,q}$ to a neighborhood of $[a,b]$.



Application II: excited eigenvalues

Theorem (Spectrum at Regular Values)

Then there exists a smooth map $\Sigma_{k,q} \ni \sigma \mapsto f_{k,q}(\sigma, h^{\frac{1}{2}}) \in \mathbb{R}$ with an asymptotic expansion

$$f_{k,q}(\sigma, h^{\frac{1}{2}}) \sim f_{k,q,0}(\sigma) + h^{\frac{1}{2}} f_{k,q,1}(\sigma) + \cdots$$

s.t the spectrum of $\mathscr{L}_h$ in $[ha, hb]$ coincides, modulo $\mathscr{O}(h^2)$, with the disjoint union

$$\left(\bigsqcup_{k=1}^{N} \bigsqcup_{q=1}^{p(k)} \left\{ h f_{k,q}(\sigma, h^{\frac{1}{2}}),\, \sigma \in h^{\frac{1}{2}} \left(\tfrac{\pi}{L}\mathbb{Z} + \theta(h)\right) \cap \Sigma_{k,q} \right\} \right) \cap [ha, hb], \quad \theta(h) = \frac{|\Omega|}{h|\partial\Omega|}.$$

Moreover, we have, when $\sigma \in \Sigma_{k,q}$,

$$f_{k,q,0}(\sigma) = \mu_k(\gamma, \sigma), \qquad f_{k,q,1}(\sigma) = -\langle\kappa\rangle C_k(\sigma),$$

where $\langle\kappa\rangle$ is the average curvature:

$$\langle\kappa\rangle = \frac{1}{2L}\int_0^{2L} \kappa(s)\mathrm{d}s = \frac{\pi}{L} \text{ (Gauss-Bonnet theorem).}$$

Application III: Precise Weyl formula

Theorem

Consider a and b as before. Then the number of eigenvalues of $\mathscr{L}_h$ in $[ha, hb]$ is

$$N(\mathscr{L}_h, [ha, hb]) = \left\lfloor \frac{L}{\pi h^{1/2}} \sum_{k,q} \delta_{k,q}^{[0]} + \frac{L\langle\kappa\rangle}{\pi} \sum_{k,q} \delta_{k,q}^{[1]} + \mathscr{O}(h^{1/2}) \right\rfloor,$$

where

$$\delta_{k,q}^{[0]} := \left|\alpha_{k,q} - \beta_{k,q}\right|, \qquad \delta_{k,q}^{[1]} := \frac{C_k(\beta_{k,q})}{\left|\mu_k'(\beta_{k,q})\right|} - \frac{C_k(\alpha_{k,q})}{\left|\mu_k'(\alpha_{k,q})\right|},$$

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1. When $\gamma = 0$, the one-term asympotics is a consequence of Frank'07.

R. L. Frank. *On the asymptotic number of edge states for magnetic Schrödinger operators*, Proc. Lond.Math. Soc. (3), 95(1):1-19, 2007.

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2. When $\gamma = +\infty$, the one-term asympotics is a consequence of the analysis by Cornean, Fournais, Frank, Helffer'13.

H. D. Cornean, S. Fournais, R. L. Frank, and B. Helffer. *Sharp trace asymptotics for a class of 2D-magnetic operators*, Ann. Inst. Fourier, 63(6):2457-2513, 2013.

Thank you !