

Boundary states of the Robin magnetic Laplacian

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Collaboration with L. Le Treust, N. Raymond and S. Vũ Ngọc

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- 1 Introduction and motivations
- 2 An overview of the known results
- 3 Main result and applications

Robin magnetic Laplacian

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We assume that the magnetic field is constant equal to 1 (and associated with a smooth vector potential \mathbf{A}):

$$B := \partial_1 A_2 - \partial_2 A_1 = 1:$$

The operator L_h is defined by the Friedrichs extension theorem, from the closed semi-bounded quadratic form, defined on $H^1(W)$ by

$$u \nabla Q_{h,\mathbf{A}}(y) = \int_W (i\hbar \tilde{\nabla} - \mathbf{A})y \cdot \overline{y} dx + gh^{\frac{3}{2}} \int_{\partial W} |y|^2 ds:$$

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- [**A gauge invariance property:** The quadratic form $Q_{h;\mathbf{A}}$ is gauge invariant, in the sense that it does not change under the transformation $(u; \mathbf{A}) \nabla (e^{ibf} u; \mathbf{A} + \tilde{N}f)$:

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- Presenting techniques to pass from energies defined in W to effective energies defined in simpler sets.

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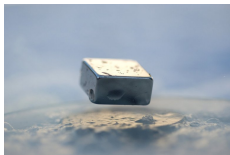
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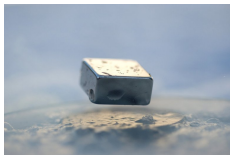
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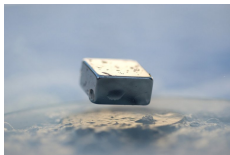
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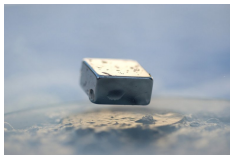
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- Extend our knowledge of the spectrum of magnetic Schrödinger operators in the semiclassical limit: finding normal forms for magnetic operators, describing the tunneling effect ...

Robin-de Gennes operator

Our results are expressed in terms of the eigenvalues of de Gennes operator with Robin boundary condition acting as

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Proposition (Dauge-Helffer'93, De Bièvre-Pulé'99, Kachmar'06, Fahs'23)

Let us $x_n > 1$. When $g \in \mathbb{R}$, the function $m_n(g; \cdot)$ is analytic and

$$\lim_{s \rightarrow +\infty} m_n(g; s) = +\infty; \quad \lim_{s \rightarrow -\infty} m_n(g; s) = 2n - 1;$$

Moreover, $m_n(g; \cdot)$ has a **unique minimum** attained at $s = x_{n-1}(g)$, but not attained at infinity. This minimum is **non-degenerate**. The function $m_n(g; \cdot)$ is decreasing on $(-\infty; x_{n-1}(g))$ and increasing on $(x_{n-1}(g); +\infty)$. In addition, we have, for all $n > 2$,

$$2n - 3 < Q^{[n-1]}(g) := \inf_{s \in \mathbb{R}} m_n(g; s) < 2n - 1;$$

When $g = +\infty$, that is when the Robin condition is replaced by the Dirichlet condition, $m_n(+\infty; \cdot)$ is still smooth, but now decreasing from $+\infty$ to $2n - 1$.

Figure: $s \nabla m_k(\cdot; s)$; for $k = 1, \dots, 4$:

Figure: $s \nabla m_k(-1; s)$; for $k = 1, \dots, 4$:

Figure: $s \nabla m_k(+\infty; s)$; for $k = 1, \dots, 5$:

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- â The case $g = 0$ has been studied a lot in the last 25 years, see especially the seminal work Helffer-Morame'01 where a **two-term expansion of the groundstate energy** is obtained by variational means (test functions, partition of the unity), and a tunneling result has even been proved recently.



B. Helffer and A. Morame. *Magnetic bottles in connection with superconductivity*. *J. Funct. Anal.*, 185(2):604-680, 2001.



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$$I_1(g; h)_{h \downarrow 0} = Q^{[0]}(g)h - k_{\max} C(g)h^{\frac{3}{2}} + o(h^{\frac{3}{2}});$$

where k_{\max} is the maximum curvature of the boundary, and $C(g) > 0$:



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- **Goal:** If we consider the **spectrum in a given spectral window (not only the lowest eigenvalues)**, we want to study the spectral properties of L_h in the semiclassical regime $h \rightarrow 0$ and for $g \in \mathbb{R} \setminus \{0\}$.

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4 In fact, this decay does not really follow from the usual Agmon estimates, since we want to consider eigenvalues between two consecutive Landau levels.

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$$S_{\mathbb{R}^2}(1) = \{p \in C^\infty(\mathbb{R}_{s;S}^2) : \exists a \in \mathbb{N}^2; \exists C_a > 0 : |p| \leq C_a \langle s \rangle^{-a}\}$$

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â Let us recall the formula for the Weyl quantization:

$$(\text{Op}_h^W p)y(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{i(x-y)h} p\left(\frac{x+y}{2}; h\right) y(y) dy dh; \quad \forall y \in S(\mathbb{R});$$

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â Let $T_{2L} = \mathbb{R}/2L\mathbb{Z}$, and $L^2(T_{2L})$ be the subset of $L^2_{\text{loc}}(\mathbb{R})$ of $2L$ -periodic functions, equipped with the usual L^2 norm on $[0; 2L]$.

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Main result: preliminaries

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â To shorten the notation, we will sometimes write p^W instead of $\text{Op}_h^W p$.

Theorem

The spectrum of L_h in $[ha; hb]$ coincides with that of $h\mathfrak{M}_h$ modulo $O(h^2)$, where

$$\mathfrak{M}_h := \begin{matrix} & \begin{matrix} 2 & & 3 \end{matrix} \\ \begin{matrix} 6 \\ 6 \\ 6 \\ 6 \\ 4 \end{matrix} & \begin{matrix} m_1^W & 0 & 0 \\ 0 & m_2^W & \vdots \\ \vdots & & \ddots \\ 0 & 0 & m_N^W \end{matrix} \end{matrix} \begin{matrix} \\ \\ \\ \\ 5 \end{matrix}$$

is a bounded operator acting diagonally on $e^{iq(h)} L^2(\mathbb{T}_{2L})^N$. Here

$$q(h) = \frac{JWj}{j\|Wjh};$$

and each m_k^W is an $h^{\frac{1}{2}}$ -pseudodifferential operator with symbol in $S_{\mathbb{T}_{2L}} \mathcal{R}(1)$.

Main result

Theorem

Let us denote by $(s; s)$ the (canonical) variables in $\mathbb{T}_{2L} \subset \mathbb{R}$. Then, we have:

the principal symbol of m_k^W is $m_k(X_0(s))$;

its subprincipal symbol is $k(s)C_k(X_0(s))$ with

$$C_k(s) = \int_{\mathbb{R}} (t - s)t^2 \mathfrak{f}_t - 2t(s - t)^2 u_k^{[g;s]}(t); u_k^{[g;s]}(t) \in L^2(\mathbb{R}_+);$$

where

$X_0 : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth, bounded with all its derivatives,

for all $k \geq 1; \dots; Ng$, $m_k(g; X_0(s)) = m_k(g; s)$ in a neighborhood of $m_k^{-1}([a; b])$,

$k(s)$ is the curvature of the boundary at the point of curvilinear abscissa s .

Remark

For all $k > 1$, $C_k(x_{k-1}(g))$ has the same sign as $g_0^{[k-1]}(g)$, for some $g_0^{[k-1]}$. This fact has important consequences on the asymptotics of the low-lying spectrum, which have not been observed before. In Kachmar'06, for $k = 1$, it was stated that $C_1(x_0(g)) > 0$ for all $g \dots$

What are X_0 and N ?

- + Let $X_0 : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth, bounded with all its derivatives, and increasing function such that for all $k \geq 1, \dots, N$, $m_k(g; X_0(s)) = m_k(g; s)$ in a neighborhood of $m_k^{-1}([a; b])$ and $m_k \circ X_0$ takes its values in $(-\infty; a) \cup (b; +\infty)$ away from it.

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- + We let

$$N := \#\{k > 1 : m_k(g; \cdot)^{-1}([a; b]) \neq \emptyset\}$$

Application I: low-lying eigenvalues

Corollary (Spectrum at critical value)

Consider $g \in g_0^{[0]}$, and let $e = \text{sign}(g_0^{[0]}(g)) = \text{sign}(C_1(x_0(g)))$. Assume that $e k$ admits a unique maximum at s_{\max} , which is non-degenerate. Then, for all $j > 1$, uniformly when $j h^{\frac{1}{4}} = o(1)$,

$$I_j(g; h) = Q^{[0]}(g) h \left(k(s_{\max}) C_1(x_0(g)) h^{\frac{3}{2}} + \frac{h^{\frac{7}{4}} (2j - 1)^{j-1}}{2} \frac{1}{k_2 C_1(x_0(g)) m_1^{(j)}(g; x_0(g))} \right) + o(h^{\frac{7}{4}});$$

with $k_2 = k''(s_{\max})$.

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+ If $g = 0$, then $m_1^{(0)}(g; x_0(g)) = 6 C_1(x_0(g)) Q^{[0]}(g)$: Thus,

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4 It extends to any **any value of the Robin parameter** the result obtained by Fournais and Helffer (but without the uniformity in j) when $g = 0$.



S. Fournais and B. Helffer. *Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian*. Ann. Inst. Fourier (Grenoble), 56(1), 2006.

- ④ We can prove that the corresponding eigenfunctions are localized near the points of maximal curvature when $g < g_0^{[0]}$, but near the points of minimal curvature when $g > g_0^{[0]}$. This last fact was not known before.

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where

$$A_h = \frac{\eta_S^2 m(g; x_0(g))}{2} (D_s + a(h) - h^{\frac{1}{2}} x_0(g))^2 + C_g k^2(s);$$

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- ③ When the curvature k is constant, in the case $g \in \mathbb{R}$, we are in a degenerate situation rather similar to the case when $g = g_0^{[0]}$. We can prove an expansion in the form

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When $g = 0$ and $j = 1$, a similar estimate is described in Fournais-Helffer'10.



S. Fournais and B. Helffer. *Spectral methods in surface superconductivity*, volume 77 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, 2010.

Application II: excited eigenvalues

Theorem (Spectrum at Regular Values)

Let $[a; b]$ be an interval as before and without critical values.

Let $p(k)$ be the number of connected components of $m_k^{-1}([a; b])$.

For each $k = 1; \dots; N$, for each $q = 1; \dots; p(k)$, let $S_{k; q} \subset \mathbb{R}$ be an interval such that $m_k(g; \cdot)$ is a diffeomorphism from $S_{k; q}$ to a neighborhood of $[a; b]$.

Application II: excited eigenvalues

Theorem (Spectrum at Regular Values)

Then there exists a smooth map $S_{k;q} \ni s \mapsto f_{k;q}(s; h^{\frac{1}{2}}) \in \mathbb{R}$ with an asymptotic expansion

$$f_{k;q}(s; h^{\frac{1}{2}}) = f_{k;q,0}(s) + h^{\frac{1}{2}} f_{k;q,1}(s) +$$

s.t the spectrum of L_h in $[ha; hb]$ coincides, modulo $O(h^2)$, with the disjoint union

$$\bigcup_{k=1}^n \bigcup_{q=1}^n \{ f_{k;q}(s; h^{\frac{1}{2}}); s \in h^{\frac{1}{2}} (\frac{p}{L}Z + q(h)) \setminus S_{k;q} \} \cup [ha; hb]; \quad q(h) = \frac{W_j}{h_j \Gamma_j W_j}$$

Moreover, we have, when $s \in S_{k;q}$,

$$f_{k;q,0}(s) = m_k(g; s); \quad f_{k;q,1}(s) = hki C_k(s);$$

where hki is the average curvature:

$$hki = \frac{1}{2L} \int_0^{2L} k(s) ds = \frac{p}{L} \text{ (Gauss-Bonnet theorem):}$$

Application III: Precise Weyl formula

Theorem

Consider a and b as before. Then the number of eigenvalues of L_h in $[ha; hb]$ is

$$N(L_h; [ha; hb]) = \frac{L}{ph^{1=2}} \mathring{a} d_{k;q}^{[0]} + \frac{Lhk i}{p} \mathring{a} d_{k;q}^{[1]} + O(h^{1=2}) ;$$

where

$$d_{k;q}^{[0]} := a_{k;q} \quad b_{k;q} ; \quad d_{k;q}^{[1]} := \frac{C_k(b_{k;q})}{m_k^0(b_{k;q})} \quad \frac{C_k(a_{k;q})}{m_k^0(a_{k;q})} ;$$

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- When $g = 0$, the one-term asymptotics is a consequence of Frank'07.



R. L. Frank. *On the asymptotic number of edge states for magnetic Schrödinger operators*, Proc. Lond. Math. Soc. (3), 95(1):1-19, 2007.

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- 2 When $g = +\infty$, the one-term asymptotics is a consequence of the analysis by Cornean, Fournais, Frank, Helffer'13.



H. D. Cornean, S. Fournais, R. L. Frank, and B. Helffer. *Sharp trace asymptotics for a class of 2D-magnetic operators*, Ann. Inst. Fourier, 63(6):2457-2513, 2013.

Thank you !

