Gibbs measures for Hamiltonian PDEs: KMS property and completeness

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Introduction

Gibbs measures is a concept, originated from statistical physics, taking different forms in various topics:

- Probability
  - (Gaussian) Random fields
  - Stochastic PDEs
  - Stochastic quantization
- Dynamical System
  - Anosov diffeomorphisms
  - SRB (Sinai-Ruelle-Bowen) measures
  - Ergodic propreties
- Mathematical Physics
  - Quantum field theory
  - Lattice (gases) systems
  - Open systems

Outline

Hamiltonian systems: finite dimension

- $E \equiv \mathbb{R}^{2n}$ a finite dimensional phase-space.
- $h : E \to \mathbb{R}$ a $\mathscr{C}^1$-Hamiltonian function.
- J: skew-symmetric matrix with $J^2 = -Id_{2n}$.

A Hamiltonian dynamical system is defined through the vector field $X : E \to E$,

$$X(u) = J\nabla h(u) \equiv -i\partial_{\bar{z}} h(u)\,, \qquad \forall u \in E, \tag{1}$$

and the field equation:

$$\dot{u}(t) = X(u(t))\,. \tag{2}$$

- Complemented with an initial condition $u(t_0) = u_0 \in E$ at a fixed initial time t_0.
- Peano Theorem provides existence of local solutions for the equation (2).
- Completeness of the vector field X is in general not guaranteed.

Gibbs measures: finite dimension

Assume that

$$z_\beta := \int_E e^{-\beta h(u)}\, dL < +\infty\,, \tag{3}$$

for some $\beta > 0$ and with dL the Lebesgue measure on E.

Definition (Gibbs measure)

The Gibbs measure of the Hamiltonian system (1)-(2), at inverse temperature $\beta > 0$, is the Borel probability measure given by

$$\mu_\beta = \frac{e^{-\beta h(\cdot)}\, dL}{\int_E e^{-\beta h(u)}\, dL} \equiv \frac{1}{z_\beta}\, e^{-\beta h(\cdot)}\, dL\,. \tag{4}$$

Gibbs measures: finite dimension

Theorem (Invariance)

If the Hamiltonian vector field X is complete then by the Liouville theorem the Gibbs measure μ_β is invariant with respect to the Hamiltonian flow.
i.e.: if ϕ_t denotes the global Hamiltonian flow then for all $t \in \mathbb{R}$,

$$(\phi_t)_\sharp \mu_\beta = \mu_\beta \,.$$

Equivalently, for all B Borel sets of E,

$$\mu_\beta\left((\phi_t)^{-1}(B)\right) = \mu_\beta\,(B)\,.$$

- However, Gibbs measures are more than invariant measures.
- They satisfy more properties reflecting their statistical stability:
  - Entropy condition.
  - Classical Kubo-Martin-Schwinger (KMS) condition.

Gibbs measures: finite dimension

Theorem (Invariance)

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Gaussian measures: Linear Hamiltonian system

Consider a positive operator $A : D(A) \subseteq H \to H$ such that,

$$\exists c > 0, \quad A \geq c\mathbb{1} \,. \tag{5}$$

A Hamiltonian dynamical system is given by the quadratic Hamiltonian function,

$$h_0 : D(A^{1/2}) \to \mathbb{R}, \qquad h_0(u) = \frac{1}{2}\langle u, Au\rangle \,. \tag{6}$$

In this case, the vector field is a linear operator $X_0 : D(A) \to H$,

$$X_0(u) = -iAu,$$

and the linear field equation governing the dynamics of the system is:

$$\dot{u}(t) = X_0(u(t)) = -iAu(t) \,. \tag{7}$$

Gaussian measures: Compactness condition

We suppose that the operator A admits a compact resolvent.

- There exists an orthonormal basis in H of eigenvectors $\{e_j\}_{j\in\mathbb{N}}$ of A with their eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$,

$$A e_j = \lambda_j \, e_j \,. \tag{8}$$

- Furthermore, assume the following assumption: $\exists s \geq 0$,

$$\boxed{\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{1+s}} < +\infty} \tag{9}$$

We note that $\{e_j, ie_j\}_{j\in\mathbb{N}}$ is an O.N.B of $H_{\mathbb{R}}$.

Gaussian measures: Weighted Sobolev spaces

Weighted Sobolev spaces w.r.t the operator A are constructed as follows: For any $r \in \mathbb{R}$, define the inner product:

$$\forall x, y \in \mathcal{D}(A^{\frac{r}{2}}) , \qquad \langle x, y \rangle_{H^r} := \langle A^{r/2} x, A^{r/2} y \rangle .$$

- H^s denotes the Hilbert space $(\mathcal{D}(A^{s/2}), \langle \cdot, \cdot \rangle_{H^s})$ with $s \geq 0$.
- H^{-s} denotes the completion of the pre-Hilbert space $(\mathcal{D}(A^{-s/2}), \langle \cdot, \cdot \rangle_{H^{-s}})$.
- Hilbert rigging: One has the canonical continuous and dense embedding,

$$H^s \subseteq H \subseteq H^{-s} .$$

Note that H^{-s} identifies with the dual space of H^s relatively to the inner product of H.

Gaussian measures: infinite dimension

The free Gibbs measure written formally as

$$\mu_{\beta,0} \equiv \frac{e^{-\beta h_0(\cdot)}\, du}{\int e^{-\beta h_0(u)}\, du},$$

is rigorously defined as a Gaussian measure on the Hilbert space H^{-s}.

Definition

▷ The mean-vector of $\mu \in \mathcal{P}(H^{-s})$ is the vector $m \in H^{-s}$ such that:

$$\langle f, m \rangle_{H^{-s}_{\mathbb{R}}} = \int_{H^{-s}} \langle f, u \rangle_{H^{-s}_{\mathbb{R}}}\, d\mu, \qquad \forall f \in H^{-s}.$$

If $m = 0$, one says that μ is a centered measure.

▷ The covariance operator of $\mu \in \mathcal{P}(H^{-s})$ is a linear operator $Q : H^{-s}_{\mathbb{R}} \to H^{-s}_{\mathbb{R}}$ such that:

$$\langle f, Q\, g \rangle_{H^{-s}_{\mathbb{R}}} = \int_{H^{-s}} \langle f, u - m \rangle_{H^{-s}_{\mathbb{R}}} \langle u - m, g \rangle_{H^{-s}_{\mathbb{R}}}\, d\mu, \quad \forall f, g \in H^{-s}.$$

▷ $\mu \in \mathcal{P}(H^{-s})$ is Gaussian if $B \mapsto \mu(\{y \in H^{-s} : \langle x, y \rangle_{H^{-s}_{\mathbb{R}}} \in B\})$ are Gaussian measures on $\mathbb{R}$.

Gaussian measures: infinite dimension

The following result is well-known.

Theorem

There exists a unique centred Gaussian measure on H^{-s}, denoted $\mu_{\beta,0}$, such that its covariance operator is $\beta^{-1}A^{-(1+s)}$, i.e.: for all $f, g \in H^{-s}$

$$\frac{1}{\beta}\langle f, A^{-(1+s)} g\rangle_{H_{\mathbb{R}}^{-s}} = \int_{H^{-s}} \langle f, u\rangle_{H_{\mathbb{R}}^{-s}} \, \langle u, g\rangle_{H_{\mathbb{R}}^{-s}} \, d\mu_{\beta,0} \,, \qquad (10)$$

Moreover, the characteristic function of $\mu_{\beta,0}$ is given for any $v \in H^{-s}$ by

$$\int_{H^{-s}} e^{i\langle v,u\rangle_{H_{\mathbb{R}}^{-s}}} \, d\mu_{\beta,0}(u) = e^{-\frac{1}{2\beta}\langle v, A^{-(1+s)} v\rangle_{H_{\mathbb{R}}^{-s}}} \,. \qquad (11)$$

i.e: Centred Gaussian measures are Gibbs measures over infinite dimensional spaces.

Gibbs measures: infinite dimension

Nonlinear Hamiltonian system:

- Linear operator A satisfying the compactness condition in (5) and (8).
- Nonlinear functional $h^I : H^{-s} \to \mathbb{R}$ satisfying for some $\beta > 0$:

$$\forall p \in [1,\infty) \qquad e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \underbrace{\mathbb{D}^{1,p}(\mu_{\beta,0})}_{\text{Gross-Sobolev spaces}} \tag{12}$$

- The Hamiltonian function:

$$h(u) = \frac{1}{2}\langle u, Au\rangle + h^I(u) = h_0(u) + h^I(u)\,. \tag{13}$$

The vector field of the system is

$$X(u) = -iAu - i\nabla h^I(u) = X_0(u) + X^I(u)\,, \tag{14}$$

defines a (non-autonomous) field equation in the interaction representation:

$$\dot{u}(t) = e^{itA} X^I(e^{-itA} u(t))\,.$$

Gibbs measures: Cylindrical smooth functions

Definition

Let $\{f_j\}_{j\in\mathbb{N}}$ O.N.B of $H_{\mathbb{R}}$. Consider for $n\in\mathbb{N}$ the mapping $\pi_n : H^{-s}\to\mathbb{R}^{2n}$,

$$\pi_n(x) = \left(\langle x, f_1\rangle_{H_{\mathbb{R}}}, \dots, \langle x, f_{2n}\rangle_{H_{\mathbb{R}}}\right). \tag{15}$$

Define $\mathscr{C}^{\infty}_{c,cyl}(H^{-s})$ as the set of all functions $F : H^{-s}\to\mathbb{R}$ such that

$$F = \varphi\circ\pi_n \tag{16}$$

for some $n\in\mathbb{N}$ and $\varphi\in\mathscr{C}^{\infty}_{c}(\mathbb{R}^{2n})$.

The gradient of F at the point $u\in H^{-s}$ is

$$\nabla F(u) = \sum_{j=1}^{2n}\partial_j\varphi(\pi_n(u))\, f_j\,, \tag{17}$$

where $\partial_j\varphi$ are the partial derivatives with respect the $2n$ coordinates of φ.

Gibbs measures: Gross-Sobolev spaces

Lemma (Malliavin derivative)

The following linear operator is closable:

$$\begin{aligned} \nabla : \mathscr{C}^\infty_{c,cyl}(H^{-s}) \subset L^p(\mu_{\beta,0}) \quad &\longrightarrow \quad L^p(\mu_{\beta,0}; H^{-s}) \,, \\ F = \varphi \circ \pi_n \quad &\longmapsto \quad \nabla F = \sum_{j=1}^{2n} \partial_j \varphi(\pi_n(\cdot))\, f_j \,. \end{aligned}$$

The Malliavin derivative is the closure of such linear operator (still denoted by ∇).

Definition (Gross-Sobolev spaces)

The Gross-Sobolev space $\mathbb{D}^{1,p}(\mu_{\beta,0})$ is the closure domain of the Malliavin derivative ∇ with respect to the norm:

$$\|F\|^p_{\mathbb{D}^{1,p}(\mu_{\beta,0})} := \|F\|^p_{L^p(\mu_{\beta,0})} + \|\nabla F\|^p_{L^p(\mu_{\beta,0};H^{-s})} \,. \tag{18}$$

Gibbs measures: infinite dimension

Nonlinear Hamiltonian system:

- Linear operator A satisfying the compactness condition.
- Nonlinear energy functional $h^I : H^{-s} \to \mathbb{R}$ satisfying for some $\beta > 0$:

$$\forall p \in [1, \infty) \qquad e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \mathbb{D}^{1,p}(\mu_{\beta,0})\,. \tag{19}$$

The vector field of the system is

$$X(u) = -iAu - i\nabla h^I(u) = X_0(u) + X^I(u)\,. \tag{20}$$

Definition (Gibbs measure)

The Gibbs measure of the dynamical system (20), at inverse temperature $\beta > 0$, is:

$$\mu_\beta = \frac{e^{-\beta h^I(\cdot)}\, d\mu_{\beta,0}}{\int_{H^{-s}} e^{-\beta h^I(u)}\, d\mu_{\beta,0}} \equiv \frac{1}{z_\beta}\, e^{-\beta h^I(\cdot)}\, d\mu_{\beta,0}\,. \tag{21}$$

Question: Invariance of the Gibbs measures by the Hamiltonian flow?

Outline

Gibbs measures
- Finite dimension
- Gaussian measures: infinite dimension
- Statistical propreties

Initial value problems
- Statistical Liouville equation
- Completeness
- Applications

Statistical propreties: Entropy condition

The Gibbs measure μ_β is characterized through the following entropy functional.

Theorem (Gibbs variational principle)

Define the relative entropy for all $\mu, \nu \in \mathcal{P}(H^{-s})$ such that $\nu \ll \mu$ as:

$$\mathcal{E}(\nu|\mu) := \int_{H^{-s}} \frac{d\nu}{d\mu} \log\left(\frac{d\nu}{d\mu}\right) d\mu\,.$$

Then the Gibbs measure μ_β is the unique minimizer of the free energy functional:

$$E_{\mu_{\beta,0}}(\nu) = \mathcal{E}(\nu|\mu_{\beta,0}) + \beta \int_{H^{-s}} h^I \, d\nu = \mathcal{E}(\nu|\mu_\beta) - \log(z_\beta)\,, \tag{22}$$

where z_β is given by (21).

One knows that $\mathcal{E}(\nu|\mu_\beta)$ is non-negative with $\mathcal{E}(\nu|\mu_\beta) = 0$ if and only if $\nu = \mu_\beta$.

Statistical propreties: Poisson structure

Consider:

- The algebra of smooth bounded cylindrical functions $\mathscr{C}^\infty_{b,cyl}(H^{-s})$.
- $F, G \in \mathscr{C}^\infty_{b,cyl}(H^{-s})$ such that: $\forall u \in H^{-s}$,

$$F(u) = \varphi \circ \pi_n(u)\,, \qquad G(u) = \psi \circ \pi_m(u)\,, \tag{23}$$

with $\varphi \in \mathscr{C}^\infty_b(\mathbb{R}^{2n})$ and $\psi \in \mathscr{C}^\infty_b(\mathbb{R}^{2m})$ for some $n, m \in \mathbb{N}$.

Definition

Then, for all such $F, G \in \mathscr{C}^\infty_{b,cyl}(H^{-s})$, the Poisson bracket is:

$$\{F, G\}(u) := \sum_{j=1}^{\min(n,m)} \partial_j^{(1)}\varphi(\pi_n(u))\,\partial_j^{(2)}\psi(\pi_m(u)) - \partial_j^{(1)}\psi(\pi_m(u))\,\partial_j^{(2)}\varphi(\pi_n(u))\,. \tag{24}$$

Statistical propreties: KMS condition

The classical Kubo-Martin-Schwinger (KMS) condition, introduced by Gallavotti and Verboven, characterizes the Gibbs measures.

Definition (Classical KMS states)

A measure $\mu \in \mathcal{P}(H^{-s})$ is a classical KMS state, at inverse temperature β, for the Hamiltonian system (26)-(27) if and only if for all $F, G \in \mathscr{C}^{\infty}_{c,cyl}(H^{-s})$,

$$\int_{H^{-s}} \{F, G\}(u)\, d\mu = \beta \int_{H^{-s}} \langle \nabla F(u), X(u) \rangle \, G(u)\, d\mu\,, \tag{25}$$

with the Poisson bracket $\{\cdot, \cdot\}$ defined in (24).

Here the Hamiltonian function is

$$h(u) = \frac{1}{2}\langle u, Au\rangle + h^{I}(u) = h_0(u) + h^{I}(u)\,, \tag{26}$$

and the vector field of the system is

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Statistical propreties: KMS condition

Assume for all $p \in [1, \infty)$:

$$e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \mathbb{D}^{1,p}(\mu_{\beta,0}) \,. \tag{28}$$

Theorem (KMS principle (V. Sohinger-ZA))

Let $\mu \in \mathcal{P}(H^{-s})$ such that $\mu \ll \mu_{\beta,0}$ and suppose that

$$\frac{d\mu}{d\mu_{\beta,0}} \in \mathbb{D}^{1,2}(\mu_{\beta,0}) \,.$$

Then μ is a classical KMS state of the Hamiltonian system (26)-(27) if and only if μ is equal to the Gibbs measure, i.e.:

$$\mu_\beta = \frac{e^{-\beta h^I} \, \mu_{\beta,0}}{\int_{H^{-s}} e^{-\beta h^I(u)} d\mu_{\beta,0}} = \mu \,.$$

Outline

Initial value problem

Let $(B, ||\cdot||)$ be a real separable dual Banach space: There exists $(E, \|\cdot\|_E)$ a real Banach space such that B is the topological dual of E and $(B, ||\cdot||)$ is separable.

Goal

Our main purpose is to prove the almost sure existence of global solutions to the initial value problem

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)), \\ \gamma(0) = x \in B, \end{cases} \tag{29}$$

when $v : \mathbb{R} \times B \to B$ is a Borel vector field.

Ingredients:

- A stationary probability measure satisfying a statistical Liouville equation.
- Integrability assumption on the vector field v.

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Biorthogonal system

Definition (Fundamental strongly total biorthogonal system)

We say that the families $\{e_k\}_{k\in\mathbb{N}}$ and $\{e_k^*\}_{k\in\mathbb{N}}$ in E, E^* respectively form a *fundamental strongly total biorthogonal system* if the following properties hold.

- $\mathrm{Span}\{e_k, k \in \mathbb{N}\}$ is dense in E (*fundamental*),
- $\mathrm{Span}\{e_k^*, k \in \mathbb{N}\}$ is dense in $B = E^*$ (*strongly total*),
- $\langle e_{k'}^*, e_k\rangle = \delta_{k',k}, \ \forall k, k' \in \mathbb{N}$ (*biorthogonal*).

□

Note that such an object exists in our framework.

Cylindrical test functions

This allows us to define a convenient class of cylindrical test functions.

Definition (Cylindrical test functions)

A function $F : B = E^* \to \mathbb{R}$ belongs to $\mathscr{C}^\infty_{c,cyl}(B)$ (resp. $\mathscr{S}_{cyl}(B)$ or $\mathscr{C}^\infty_{b,cyl}(B)$) if there exists $n \in \mathbb{N}$ and $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ (resp. $\mathscr{S}(\mathbb{R}^n)$ or $\mathscr{C}^\infty_b(\mathbb{R}^n)$) such that

$$F(u) = \varphi(\langle u, e_1\rangle, \dots \langle u, e_n\rangle), \quad \forall u \in B = E^*. \tag{30}$$

Here, $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space and we have the inclusions

$$\mathscr{C}^\infty_{c,cyl}(B) \subset \mathscr{S}_{cyl}(B) \subset \mathscr{C}^\infty_{b,cyl}(B)\,.$$

□

In particular, we have

$$\nabla F(u) = \sum_{k=1}^{n} \partial_k \varphi(\langle u, e_1\rangle, \dots \langle u, e_n\rangle)\, e_k \in E\,. \tag{31}$$

Statistical Liouville equation

Let $\mathscr{P}(B)$ denote the space of Borel probability measures on $(B, ||\cdot||)$.

Definition (Narrow continuity)

We say a curve $(\mu_t)_{t\in\mathbb{R}}$ in $\mathscr{P}(B)$ is narrowly continuous if for any bounded continuous real-valued function $F \in \mathscr{C}_b(B,\mathbb{R})$, the map $t \in \mathbb{R} \mapsto \int_B F(u)\,\mu_t(du)$ is continuous. □

Definition (Statistical Liouville equation)

A narrowly continuous curve $(\mu_t)_{t\in\mathbb{R}}$ in $\mathscr{P}(B)$ satisfies the statistical Liouville equation with respect to the Borel vector field $v : \mathbb{R}\times B \to B$ if:

$$\frac{d}{dt}\int_B F(u)\,\mu_t(du) = \int_B \langle v(t,u), \nabla F(u)\rangle\,\mu_t(du), \quad \forall F \in \mathscr{C}^\infty_{c,cyl}(B), \tag{32}$$

in the sense of distributions on $\mathbb{R}$.

Outline

General principle: completeness

Theorem (S. Farhat-V. Sohinger-ZA)

Let B be a separable dual Banach space and $v : \mathbb{R} \times B \to B$ a Borel vector field. Assume that there exists $(\mu_t)_{t\in\mathbb{R}}$ a narrowly continuous curve in $\mathscr{P}(B)$ such that:

- We have:
$$t \in \mathbb{R} \mapsto \int_B \|v(t,u)\| \, \mu_t(du) \in L^1_{loc}(\mathbb{R}, dt), \tag{33}$$
- and $(\mu_t)_{t\in\mathbb{R}}$ satisfies the statistical Liouville equation (32).

Then there exists a universally measurable subset $\mathscr{G}$ of B of total measure $\mu_0(\mathscr{G}) = 1$ such that for any $x \in \mathscr{G}$ there exists a global mild solution to the initial value problem (29).

Proof:

- Construction of a path measure concentrated on global solutions.
- Use of the measurable projection theorem.

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General principle: optimal condition?

We borrow an example from the work of A. Cruzeiro.

Example

Consider the time-independent $\mathscr{C}^\infty$-vector field $v : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$v(u) = \Big(q^2, (2q - q^3)e^{p^2/2} \int_p^{+\infty} e^{-s^2/2} ds\Big), \qquad u = (q, p) \in \mathbb{R}^2. \tag{34}$$

- The standard centered Gaussian measure on $\mathbb{R}^2$ satisfies the statistical Liouville equation (32) with the above vector field v.
- The initial value problem with this v leads to the ODE

$$\dot{q}(t) = q(t)^2 ,$$

which has for each initial condition $q(0) \neq 0$ a unique non global solution.
- The vector field v does not satisfy the integrability condition (33).
- Hence, the integrability condition (33) can be interpreted as an almost sure non-blow up criterion.

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Application: NLS equation

Consider the spatial domain $\mathbb{T}^d$ for $d = 1, 2$.

- Nonlinear Schrödinger equation (NLS).

$$\begin{cases} i\partial_t u_t(x) = \overbrace{\big(-\Delta + \mathbb{1}\big)}^{\equiv A} u_t(x) + |u_t(x)|^2\, u_t(x) \\ u_0(x) = \varphi(x) \in H^{-s} \equiv H^{-s}(\mathbb{T}^d)\,. \end{cases}$$

- Hamiltonian function:

$$h(u) = \frac{1}{2}\int_{\mathbb{T}^d} \bar{u}(x)(\mathbb{1} - \Delta)u(x)\, dx + \underbrace{\frac{1}{4}\int_{\mathbb{T}^d} |u(x)|^4\, dx}_{\text{OK if } d=1}\,.$$

- The vector field:

$$X(u) = -iAu - i\nabla h^I(u)$$

Application: NLS equation

- For $d = 1$ take $s = 0$.
- For $d = 2$ take $s > 0$ and change h^I by the Wick ordered functional:

$$h^I(u) \equiv \frac{1}{4}\int_{\mathbb{T}^2} :|u|^4:\ dx = \lim_{n\to\infty} \frac{1}{4}\int_{\mathbb{T}^2} |P_n(u)|^4 dx - \frac{1}{2}\left(\int_{\mathbb{T}^2} |P_n(u)|^2 dx\right)^2 .$$

Proposition

Then for all $p \in [1, \infty)$:

$$e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \mathbb{D}^{1,p}(\mu_{\beta,0})\,. \tag{35}$$

Theorem (Bourgain)

The NLS equation is GWP μ_β-almost surely and the Gibbs measure μ_β is flow invariant.

GWP: Existence and uniqueness of global solutions in H^{-s} for μ_β-almost all initial data.

Bourgain's method

Strategy:

- Projection over finite modes.
- Finite dimensional Hamiltonian system (locally) WP.
- Convergence of the truncated Gibbs measures towards the Gibbs measure μ_β.
- Uniform local well-posedness theory:

$$\|u_0\|_{H^{-s}} \leq K \implies \text{LWP on } [-T, T], T \sim K^{-\delta}$$

  combined to the invariance of the truncated Gibbs measures allow to extend truncated local solutions to global solutions over almost total sets.
- Convergence of global truncated solutions towards solutions of NLS over a total set.
- Invariance of the Gibbs measure μ_β.

Application of the general principle

Consider the same framework: The Hamiltonian function and vector field are

$$h(u) = \frac{1}{2}\langle u, Au\rangle + G^I(u)\,, \qquad X(u) = -iAu - i\nabla G^I(u)\,. \tag{36}$$

Assume:

$$e^{-\beta G^I(\cdot)} \in L^2(\mu_{\beta,0}) \qquad \text{and} \qquad G^I \in \mathbb{D}^{1,2}(\mu_{\beta,0})\,.$$

Proposition

Consider the time-dependent push-forward Gibbs measures

$$\nu_t = (e^{itA})_\sharp \mu_\beta. \tag{37}$$

Then for all $t \in \mathbb{R}$ and any $F \in \mathscr{C}^\infty_{b,cyl}(H^{-s})$,

$$\frac{d}{dt}\int_{H^{-s}} F(u)\,\nu_t(du) = \int_{H^{-s}} \langle v(t,u), \nabla F(u)\rangle\,\nu_t(du). \tag{38}$$

Application of the general principle

Corollary (S. Farhat-V. Sohinger-ZA)

The above initial value problem (36) admits global solutions $\mu_{\beta,0}$-almost surely.

Other applications:

- Euler equation,
- Modified surface quasi-geostrophic equation,
- Wave or Klein-Gordon equations,
- Hartree equations...

Application of the general principle

Corollary (S. Farhat-V. Sohinger-ZA)

The above initial value problem (36) admits global solutions $\mu_{\beta,0}$-almost surely.

Other applications:

- Euler equation,
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Some problems

Further questions:

- Derivation of classical KMS boundary condition from the quantum KMS boundary condition.
- Study of ergodic properties of nonlinear PDEs.

Some old and recent advances

Old results:

- **Rigorous construction of Gibbs measures:** QFT literature in the 1970's.
  → Nelson, Glimm-Jaffe, Simon, Høgh-Krohn, Guerra ...
- **Gibbs measures for (de)-focusing NLS:** Lebowitz-Rose-Speer (1988).
  → A weighted Gibbs measure is constructed but invariance is conjectured.
- **GWP and proof of invariance:** Bourgain and Zhidkov (1990's).
  → Almost sure GWP on low-regularity Sobolev spaces.

Recent results:

- **Application to supercritical wave equation:** Burq and Tzvetkov
  → Improvement of deterministic (dispersive) estimates.
- **Transport of Gaussian measures:** Genovese, Tzvetkov and Visciglia...
  → Absolute continuity of transported Gaussian measures by Hamiltonian flows.
- **Further results:** Bringmann, Cacciafesta-De Suzzoni, Kenig-Mendelson, Nahmod-Oh-Rey-Bellet-Staffilani, Oh-Thomann...