Quantum-classical motion of charged particles in interaction with scalar fields

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Purposes

1. **Bohr's correspondence principle :**

$$\boxed{\underset{\text{Nelson model}}{\textit{Quantum dynamic}} \underset{\hbar\to 0}{\longrightarrow} \underset{\text{Particle-field equation}}{\textit{Classical dynamic}}}$$

We study the transition by Wigner measure approach.

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1. **Bohr's correspondence principle** :

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   We study the transition by Wigner measure approach.

2. To exhibit the global well-posedness for the particle-field equation.

The classical system

The classical system

Consider n fixed particles in $\mathbb{R}^d$ with $d \in \mathbb{N}^*$, interacting with scalar meson field. The particle-field system reads for all $j \in \{1, \cdots, n\}$

$$
\begin{cases}
\partial_t p_j = -\nabla_{q_j} V(q) - \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}\right] dk \, ; \\
\partial_t q_j = \nabla f_j(p_j) \, ; \\
i\partial_t \alpha = \omega(k)\, \alpha(k) + \sum_{j=1}^{n} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j}
\end{cases}
\tag{PFE}
$$

- q_j, p_j are positions-momenta and M_j are the masses.
- $\alpha : \mathbb{R}^d \to \mathbb{C}$ describes the field, $\chi : \mathbb{R}^d \to \mathbb{R}$ is the form factor.
- $V : \mathbb{R}^{dn} \to \mathbb{R}$ is the potential.
- $f_j(p_j) = \begin{cases} \sqrt{p_j^2 + M_j^2} & \text{(relativistic)} \\ p_j^2/2M_j & \text{(non-relativistic)} \end{cases}$.
- $\omega(k) = \sqrt{k^2 + m_f{}^2} \geq m_f > 0$ is the dispersion relation ;

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The associated Hamiltonian is defined as follows

$$H(p,q,\alpha) = \sum_{j=1}^{n} f_j(p_j) + V(q_1,\cdots,q_n) + \int_{\mathbb{R}^d} \overline{\alpha(k)}\,\omega(k)\,\alpha(k)\,dk$$
$$+ \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k\cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k\cdot q_j}\right] dk.$$

The solution $u = (p,q,\alpha)$ belongs to the classical phase-space :

$$X^{\sigma} := \mathbb{R}_p^{dn} \times \mathbb{R}_q^{dn} \times \mathcal{G}^{\sigma},$$

where $\mathcal{G}^{\sigma}$ is the weighted L^2 Lebesgue space endowed with the norm :

$$\|\alpha\|_{\mathcal{G}^{\sigma}}^2 = \langle \alpha, \omega(\cdot)^{2\sigma}\,\alpha\rangle_{L^2} = \|\omega^{\sigma}\,\alpha\|_{L^2}^2.$$

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$$+ \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k\cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k\cdot q_j}\right] dk.$$

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The energy space is $X^{1/2}$. We will give our main results in the space X^{σ} with $\sigma \in [\frac{1}{2}, 1]$.

The Quantum system

The Quantum system : Nelson Hamiltonian

The Nelson Hamiltonian is defined as follows

$$\hat{H} = \sum_{j=1}^{n} f_j(\hat{p}_j) + V(\hat{q}) + d\Gamma(\omega) + \sum_{j=1}^{n} (a_\hbar + a_\hbar^*)(\frac{\chi}{\sqrt{\omega}}\, e^{-2\pi k\cdot \hat{q}_j}).$$

The Hilbert space of the quantized particle-field system is

$$\mathcal{H} := L^2(\mathbb{R}_x^{dn}, \mathbb{C}) \otimes \Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C}))\,,$$

where Γ_s is the symmetric Fock space

$$\Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C})) := \bigoplus_{m=0}^{+\infty} L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_s m} \simeq \bigoplus_{m=0}^{+\infty} L_s^2(\mathbb{R}^{dm}, \mathbb{C}).$$

We denote by $\mathcal{F}^m := L_s^2(\mathbb{R}^{dm}, \mathbb{C})$ the symmteric L^2 space over $\mathbb{R}^{dm}$.

Remark : $\hat{H}$ is a self adjoint operator.

- $\hat{p}_j = -i\hbar\nabla_{x_j}$ is the momentum operator ;
- $\hat{q}_j = x_j$ is the position operator ;
- $d\Gamma(\omega) : \mathcal{H} \to \mathcal{H}$ is the free field Hamiltonian

$$d\Gamma(\omega)\mid_{\mathcal{F}^m} = \hbar \sum_{j=1}^{m} \omega(k_j);$$

- $a_\hbar$ and $a_\hbar^*$ are the generalized $\hbar$ scaled annihilation-creation operators are defined as follows :

for every $\psi = \{\psi^m\}_{m\geq 0} \in \mathcal{H}$ and $F(k) := \sum_{j=1}^{n} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k\cdot\hat{q}_j}$

$$[a_\hbar(F)\psi(x)]^m(K_m) = \sqrt{\hbar(m+1)} \int_{\mathbb{R}^d} \overline{F(k)}\, \psi^{m+1}(x; K_m, k)\, dk;$$

$$[a_\hbar^*(F)\psi(x)]^m(K_m) = \frac{\sqrt{\hbar}}{\sqrt{m}} \sum_{j=1}^{m} F(k_j)\, \psi^{m-1}(x; k_1, \cdot, \hat{k}_j, \cdot, k_m)$$

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Assumptions and main results

Assumptions

• On V and χ :

$$V \in C_b^2(\mathbb{R}^{dn};\mathbb{R}) \qquad (C_0)$$

$$\omega(\cdot)^{\frac{3}{2}-\sigma}\chi(\cdot) \in L^2(\mathbb{R}^d;\mathbb{R}), \qquad \sigma \in [\frac{1}{2}, 1]. \qquad (C_1)$$

• Let $(\varrho_\hbar)_{\hbar\in(0,1)}$ be a family of density matrices on $\mathcal{H}$ of the particle-field quantum system. We assume that :

$$\exists C_0 > 0,\ \forall \hbar \in (0,1), \quad \mathrm{Tr}[\varrho_\hbar\, d\Gamma(\omega^{2\sigma})] \leq C_0, \qquad (Q_0)$$

$$\exists C_1 > 0,\ \forall \hbar \in (0,1), \quad \mathrm{Tr}[\varrho_\hbar\, (\hat{q}^2 + \hat{p}^2)] \leq C_1. \qquad (Q_1)$$

Question : Propagation of estimates (Q_0) and (Q_1) uniformly in times ?

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Global well-posedness of (PFE)

Theorem [S. Farhat, 2023]

Let $\sigma \in [\frac{1}{2}, 1]$ and assume (C_0) and (C_1) hold true.
Then, for any initial data $u_0 \in X^\sigma$, there exists a unique global strong solution $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap \mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$ of the particle-field equation (PFE). Moreover, the generalized global flow

$$\begin{array}{rcl} \Phi_t : X^\sigma & \longrightarrow & X^\sigma \\ u_0 & \longmapsto & u(t). \end{array}$$

is Borel measurable.

- The proof of the above result still require some classical properties : for example uniqueness of solutions to the particle-field equation and so on.
- Assumptions (C_0) and (C_1) amount to these properties.

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Wigner measures

Definition [Weyl operator]

The Weyl operator over the entire interacting Hilbert space $\mathcal{H}$

$$\xi = (p, q, \alpha) \in X^0 \longmapsto \mathcal{W}(\xi) := e^{i(p\cdot\hat{q} - q\cdot\hat{p})} \otimes e^{\frac{i}{\sqrt{2}}(a_\hbar(\alpha) + a_\hbar^*(\alpha))}$$

Definition [Wigner measure]

A Borel probability measure μ over X^0 is a Wigner measure of a family of density matrices $(\varrho_\hbar)_{\hbar\in(0,1)}$ on the Hilbert space $\mathcal{H}$ if and only if there exists a subset $\mathcal{E} \subset (0,1)$ with $0 \in \overline{\mathcal{E}}$ such that for any $\xi = (p_0, q_0, \alpha_0), \tilde{\xi} = (2\pi q_0, -2\pi p_0, \sqrt{2}\pi\alpha_0) \in X^0$:

$$\lim_{\hbar\to 0, \hbar\in\mathcal{E}} \mathrm{Tr}\left[\mathcal{W}(\tilde{\xi})\, \varrho_\hbar\right] = \int_{X^0} e^{2\pi i \Re e\langle \xi, u\rangle_{X^0}}\, d\mu(u) = \mathcal{F}^{-1}[\mu](\xi).$$

- Denote by $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E})$ the set of all Wigner measures of $(\varrho_\hbar)_{\hbar\in\mathcal{E}}$.
- $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E}) \neq \phi$ if some assumptions on $(\varrho_\hbar)_\hbar$ are imposed.
- In our approach, we need to prove $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E}) = \{\text{Singleton}\}$ up to extraction of subsequence ?

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A Borel probability measure μ over X^0 is a Wigner measure of a family of density matrices $(\varrho_\hbar)_{\hbar\in(0,1)}$ on the Hilbert space $\mathcal{H}$ if and only if there exists a subset $\mathcal{E} \subset (0,1)$ with $0 \in \overline{\mathcal{E}}$ such that for any $\xi = (p_0, q_0, \alpha_0), \tilde{\xi} = (2\pi q_0, -2\pi p_0, \sqrt{2}\pi\alpha_0) \in X^0$:

$$\lim_{\hbar\to 0, \hbar\in\mathcal{E}} \mathrm{Tr}\left[\mathcal{W}(\tilde{\xi})\ \varrho_h\right] = \int_{X^0} e^{2\pi i \Re e\langle \xi, u\rangle_{X^0}}\, d\mu(u) = \mathcal{F}^{-1}[\mu](\xi).$$

- Denote by $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E})$ the set of all Wigner measures of $(\varrho_\hbar)_{\hbar\in\mathcal{E}}$.
- $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E}) \neq \phi$ if some assumptions on $(\varrho_\hbar)_\hbar$ are imposed.
- In our approach, we need to prove $\mathcal{M}(\varrho_\hbar,\ \hbar \in \mathcal{E}) = \{\text{Singleton}\}$ up to extraction of subsequence ?

Classical limit : Bohr Correspondence principle

Theorem [S. Farhat, 2023]

Let $\sigma \in [\frac{1}{2}, 1]$ and assume (C_0) and (C_1) hold true. Let $(\varrho_\hbar)_{\hbar \in (0,1)}$ be a family of density matrices on $\mathcal{H}$ satisfying (Q_0) and (Q_1). Assume that

$$\mathcal{M}(\varrho_{\hbar_\ell},\, \ell \in \mathbf{N}) = \{\mu_0\}.$$

Then for all times $t \in \mathbb{R}$, we have

$$\mathcal{M}(\underbrace{e^{-i\frac{t}{\hbar_\ell}\hat{H}}\varrho_{\hbar_\ell}\, e^{i\frac{t}{\hbar_\ell}\hat{H}}}_{:=\varrho_{\hbar_\ell}(t)},\, \ell \in \mathbf{N}) = \{\mu_t\},$$

where $\mu_t \in \mathcal{P}(X^0)$ satisfies

(i) $\mu_t(X^\sigma) = 1$.

(ii) $\mu_t = (\Phi_t)_\sharp \mu_0$, where Φ_t is the global flow of the particle-field equation.

The convergence is rephrased according to the following commutative diagram :

$$\begin{array}{ccc}
\varrho_\hbar & \xrightarrow{\; e^{-i\frac{t}{\hbar}\hat{H}_\hbar}\,(\cdot)\, e^{i\frac{t}{\hbar}\hat{H}_\hbar}\;} & \varrho_\hbar(t) \\
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The quantum dynamical system

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- We work with the interaction representation with

$$\tilde{\varrho}_\hbar(t) := e^{i\frac{t}{\hbar}d\Gamma(\omega)}\varrho_\hbar(t)\, e^{-i\frac{t}{\hbar}d\Gamma(\omega)}$$

- The Wigner measures of $\tilde{\varrho}_\hbar(t)$ are the limits of the map

$$\lim_{\hbar\to 0} \underbrace{\mathrm{Tr}\left[\mathcal{W}(\tilde{\xi})\,\tilde{\varrho}_\hbar(t)\right]}_{(1)} = \int_{X^0} e^{2\pi i \Re e\langle \xi, u\rangle}\, d\tilde{\mu}_t(u). \qquad (1)$$

- For all $\xi = (p_0, q_0, \alpha_0) \in X^{1/2}$, for all $\hbar \in (0,1)$ and for all $t, t_0 \in \mathbb{R}$, the quantum dynamical system of (1) is

$$\mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_\hbar(t)\right] = \mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_\hbar(t_0)\right] - i\int_{t_0}^{t} \mathrm{Tr}\left(\frac{1}{\hbar}\left[\mathcal{W}(\xi), \hat{H}_k(s)\right]\tilde{\varrho}_\hbar(s)\right) ds$$

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The commutator in the Duhamel formula can be expanded as follows :

$$\frac{1}{\hbar}\left[\mathcal{W}(\xi), \hat{H}_k(s)\right] = \Big[\underbrace{B_0(s,\hbar,\xi)}_{\text{Main term}} + \hbar \underbrace{B_1(s,\hbar,\xi)}_{\text{Remainder term}}\Big]\, \mathcal{W}(\xi).$$

We have with $\hat{H}_0 = d\Gamma(\omega) + \sum_{j=1}^{n} f_j(\hat{p}_j)$ and $S = (\hat{H}_0 + 1)^{1/2}$

$$\|S^{-1} B_0(s,\hbar,\xi) S^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \|\xi\|_{X^0}\, \|\chi\|_{L^2};$$

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A single Wigner measure for all times

The next step is to prove that we can extract a subsequence $(\hbar_\ell)_{\ell\in\mathbb{N}}$ and a family of Borel probability measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ such that for all $t \in \mathbb{R}$

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Ideas of proof :

- **Propagation of quantum estimates :** (Q_0) **and** (Q_1)
  - $\mathrm{Tr}\left[\tilde{\varrho}_\hbar(t)\, d\Gamma(\omega^{2\sigma})\right] \leq C_1\ \mathrm{Tr}\left[\varrho_\hbar\, (d\Gamma(\omega^{2\sigma}) + 1)\right] e^{C_2|t|} \leq C_3.$
  - $\mathrm{Tr}\left[\tilde{\varrho}_\hbar(t)\, (\hat{p}^2 + \hat{q}^2)\right] \leq C_1\ \mathrm{Tr}\left[\varrho_\hbar\, (\hat{H}_0 + \hat{p}^2 + \hat{q}^2 + 1)\right] e^{C_2|t|} \leq C_3,$
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$\mathrm{Tr}[B_0(s,\hbar_\ell,\xi)\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)] =$ **Non-interacting terms** $+$ **Interacting terms**.

- **Non-interacting terms**

$$= -\sum_{j=1}^{n}\Big[\mathrm{Tr}[\nabla_{q_j} V(\hat{q})\cdot q_{0j}\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)] + \mathrm{Tr}[\nabla f_j(\hat{p}_j)\cdot p_{0j}\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)]\Big].$$

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$$= +\sum_{j=1}^{n} \mathrm{Tr}[(a_{\hbar_\ell} + a^*_{\hbar_\ell})(e^{-2\pi i k\cdot\hat{q}_j}\ \tilde{F}_j(\hbar_\ell,k))\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)]$$

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The characteristic equations

- The limit of the Duhamel is formulated as follows for all $t, t_0 \in \mathbb{R}$ and $y \in X^\sigma$

$$\int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} \, d\tilde{\mu}_t(u) = \int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} \, d\tilde{\mu}_{t_0}(u) + 2\pi i \int_{t_0}^{t} \int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} \, \Re e \langle v(s,u), y \rangle_{X^\sigma} \, d\tilde{\mu}_s(u) \, ds, \tag{C}$$

(IPFE) : the interaction representation of (PFE)

$$\begin{cases} \dfrac{du}{dt} = v(t, u(t)) = \Phi^f_{-t} \circ \underbrace{\mathcal{N}}_{\text{Non-linearity of (PFE)}} \circ \Phi^f_t(u(t)), \\ u(0) = u_0 \in X^\sigma. \end{cases} \tag{IPFE}$$

- $\Phi^f_t : X^\sigma \longrightarrow X^\sigma$ is the free flow defined as follows

$$\Phi^f_t(p, q, \alpha) = (p, q, e^{-it\omega(k)}\alpha).$$

- The non-autonomous vector field v is as follows $v : \mathbb{R} \times X^\sigma \to X^\sigma$ satisfying

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Equivalence between characteristic and Liouville equation

Definition : Liouville equation

A family of Borel probability measures $\{\tilde{\mu}_t\}_{t\in I}$ on X^σ is a measure-valued solution of the Liouville equation associated to the vector field $v : \mathbb{R} \times X^\sigma \to X^\sigma$ if and only if for all $\phi \in \mathcal{C}^\infty_{0,cyl}(I \times X^\sigma)$:

$$\int_I \int_{X^\sigma} \{\partial_t \phi(t,u) + \Re e\langle v(t,u), \nabla\phi(t,u)\rangle_{X^\sigma}\} d\tilde{\mu}_t(u)\, dt = 0, \qquad \text{(L)}$$

$\mathcal{C}^\infty_{0,cyl}(I \times X^\sigma)$ is the cylindrical functional space.

Then, thanks to the regular properties of $\tilde{\mu}_t$ and of the vector field $v(t,u)$, we have the following are equivalent :

Equivalence between Liouville equation and Characteristic equation

$\{\tilde{\mu}_t\}_{t\in I}$ solves the Liouville equation (L) $\Leftrightarrow$ $\{\tilde{\mu}_t\}_{t\in I}$ solves the characteristic equation (C).

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Definition : Liouville equation

A family of Borel probability measures $\{\tilde{\mu}_t\}_{t\in I}$ on X^σ is a measure-valued solution of the Liouville equation associated to the vector field $v : \mathbb{R} \times X^\sigma \to X^\sigma$ if and only if for all $\phi \in \mathcal{C}^\infty_{0,cyl}(I \times X^\sigma)$:

$$\int_I \int_{X^\sigma} \{\partial_t \phi(t,u) + \Re e \langle v(t,u), \nabla \phi(t,u) \rangle_{X^\sigma}\} d\tilde{\mu}_t(u)\, dt = 0, \qquad \text{(L)}$$

$\mathcal{C}^\infty_{0,cyl}(I \times X^\sigma)$ is the cylindrical functional space.

Then, thanks to the regular properties of $\tilde{\mu}_t$ and of the vector field $v(t,u)$, we have the following are equivalent :

Equivalence between Liouville equation and Characteristic equation

$\{\tilde{\mu}_t\}_{t\in I}$ solves the Liouville equation (L) $\Leftrightarrow$ $\{\tilde{\mu}_t\}_{t\in I}$ solves the characteristic equation (C).

Proof of the main results

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- Using measure theoretical techniques, we have :

  ⇒ Almost sure existence of unique **global** solutions to (PFE) with a generalized global flow

$$\boxed{\Phi_t = \Phi_t^f \circ \tilde{\Phi}_t.}$$

- To get rid of almost sureness, we select a special choice of family of density matrices which is coherent states centered at initial data.
- It remains to check

$$\mu_t = (\Phi_t)_\sharp \mu_0, \qquad \Phi_t = \Phi_t^f \circ \tilde{\Phi}_t.$$

  - The important tool to do is the following link :

$$\boxed{\underbrace{\mathcal{M}(\varrho_\hbar(t), \hbar \in (0,1))}_{=\{\mu_t\}} = \{(\Phi_t^f)_\sharp \tilde{\mu}_t,\ \tilde{\mu}_t \in \underbrace{\mathcal{M}(\tilde{\varrho}_\hbar(t),\ \hbar \in (0,1))}_{=\{\tilde{\mu}_t\}}\}}$$

  - We have, by probabilistic representation, that

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Thank you for your attention !

Slides for more details about the presentation

Almost sure existence result

Theorem 2 [Z. Ammari, M. Falconi and F. Hiroshima, 2022]

In a separable Hilbert space $\mathcal{H}$, consider the initial value problem (IPFE) with a vector field $v : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ continuous and bounded on bounded sets. Let $I \ni 0$ be a bounded open interval and assume

(i) $\exists t \in \mathbb{R} \to \mu_t \in \mathcal{P}(\mathcal{H})$ a weakly narrowly continuous solution to (L) satisfying

$$\int_I \int_{\mathcal{H}} \|v(t,u)\|_{\mathcal{H}} \, d\mu_t(u) \, dt < +\infty. \tag{Int}$$

(ii) Uniqueness of the solutions to (IPFE) over any I.

Then for μ_0-almost all initial conditions x in $\mathcal{H}$, there exists a (unique) global strong solution (i.e. $u(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathcal{H})$) to (IPFE). Moreover, the set

$$\mathfrak{G} := \{x \in \mathcal{H} : \exists u(\cdot) \text{ a global strong solution of (IPFE) with } u(0) = x\},$$

is **Borel** subset of $\mathcal{H}$ with $\mu_0(\mathfrak{G}) = 1$ and for any time $t \in \mathbb{R}$ the map

$$\begin{aligned} \Phi_t : \mathfrak{G} &\longrightarrow \mathfrak{G} \\ x &\longmapsto \Phi_t(x) = u(t). \end{aligned}$$

is **Borel measurable**.

Existence of unique global solutions

- Apply Theorem 2 with the following choices :
  - $\mu_t \equiv \tilde{\mu}_t$;
  - $\mathcal{H} \equiv X^\sigma$.

  $\Rightarrow$ Almost sure existence of unique **global** solutions for (IPFE) with a generalized global flow $\tilde{\Phi}_t$.

- We have this equivalence : **(PFE)** $\underset{\Phi^f_t}{\overset{\Phi^f_{-t}}{\rightleftharpoons}}$ **(IPFE)**

  $\Rightarrow$ Almost sure existence of unique **global** solutions to (PFE) with a generalized global flow

$$\boxed{\Phi_t = \Phi^f_t \circ \tilde{\Phi}_t.}$$

- To get rid of almost sureness, we select a special choice of family of density matrices which is coherent states centered at initial data.

Probabilistic representation

The crucial tool that was used for constructing the above generalized global flow for (IPFE) is the following :

Probabilistic representation

There exists $\eta \in \mathscr{P}(X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma))$ satisfying :

(i) $\eta(\mathcal{F}_I) = 1$ where

$$\mathcal{F}_I := \left\{(u_0, u(\cdot)) \in X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) : \; u(\cdot) \text{ satisfies (IPFE) on } I \text{ with } u_0\right\}$$

(ii) $\tilde{\mu}_t = (e_t)_\sharp \eta, \quad \forall t \in I$, where the map

$$\begin{array}{rcl} e_t \; : X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) & \longrightarrow & X^\sigma \\ (u_0, u(\cdot)) & \longmapsto & e_t(u_0, u(\cdot)) = u(t). \end{array}$$

is the evaluation map.

Generalization : Z. Ammari, S. Farhat and V. Sohinger "Almost sure existence of global solutions for general initial value problems."

Global well-posedness of the particle-field equation

- Let $u_0 = (z_0, \alpha_0) \in X^\sigma$ and consider the coherent vectors repectively in the particle and Fock spaces

$$W_1(\frac{\sqrt{2}}{i\hbar} z_0)\psi, \qquad W_2(\frac{\sqrt{2}}{i\hbar}\alpha_0)\Omega$$

  - $\psi(x) = (\pi\hbar)^{-dn/4}\, e^{-x^2/2\hbar} \in L^2(\mathbb{R}^{dn}, dx)$ is the **normalized gaussian** function on the particles.
  - Ω is the **vacuum vector** on the Fock space.

  Then, the following projection

$$\boxed{\mathcal{C}_\hbar(u_0) = \Big| W_1(\tfrac{\sqrt{2}}{i\hbar} z_0)\psi \otimes W_2(\tfrac{\sqrt{2}}{i\hbar}\alpha_0)\Omega \Big\rangle \Big\langle W_1(\tfrac{\sqrt{2}}{i\hbar} z_0)\psi \otimes W_2(\tfrac{\sqrt{2}}{i\hbar}\alpha_0)\Omega \Big|}$$

  gives rise to a family of coherent states.

- We have

$$\boxed{\mathcal{M}(\mathcal{C}_\hbar(u_0), \hbar \in (0,1)) = \{\delta_{u_0}\} : \text{Dirac measure centered on } u_0}$$

- Since $u_0 \in X^\sigma$, this implies

$$\boxed{(\mathcal{C}_\hbar(u_0))_\hbar \text{ satisfies } (Q_0) \text{ and } (Q_1).}$$

Let $u_0 \in X^\sigma$ and let $\varrho_\hbar = \mathcal{C}_\hbar(u_0)$.

- Apply Theorem A with the measure $\tilde{\mu}_t$ to get the

  $$\boxed{\text{GWP of (IPFE) } \tilde{\mu}_0\text{-almost all initial data in } X^\sigma}$$

  with a generalized global flow $\tilde{\Phi}_t$.

- We have also

  $$\boxed{\tilde{\mu}_0(\mathfrak{G}) = \delta_{u_0}(\mathfrak{G}) = 1.}$$

  This implies $u_0 \in \mathfrak{G}$.

- GWP of (PFE) with a generalized global flow

  $$\boxed{\Phi_t(u_0) = \Phi_t^f \circ \tilde{\Phi}_t(u_0),}$$

The classical limit : Validity of Bohr's correspondance

Goal

To prove the second property : $\mu_t = (\Phi_t)_\sharp \mu_0,\ \Phi_t = \Phi_t^f \circ \tilde{\Phi}_t$.

We have, by probabilistic representation, that

$$\boxed{\tilde{\mu}_t = (\tilde{\Phi}_t)_\sharp \tilde{\mu}_0.}$$

The important tool to do that is the following link :

$$\boxed{\underbrace{\mathcal{M}(\varrho_\hbar(t), \hbar \in (0,1))}_{=\{\mu_t\}} = \left\{ (\Phi_t^f)_\sharp \tilde{\mu}_t,\ \tilde{\mu}_t \in \underbrace{\mathcal{M}(\tilde{\varrho}_\hbar(t),\ \hbar \in (0,1))}_{=\{\tilde{\mu}_t\}} \right\}}$$

This implies using the two boxes :

$$\boxed{\begin{aligned} \mu_t &= (\Phi_t^f)_\sharp \tilde{\mu}_t = (\Phi_t^f \circ \tilde{\Phi}_t)_\sharp \tilde{\mu}_0 \\ &= (\Phi_t)_\sharp \tilde{\mu}_0 = (\Phi_t)_\sharp \mu_0 \end{aligned}}$$

and where we have used $\tilde{\mu}_0 = \mu_0$ as a consequence of

$$\tilde{\varrho}_\hbar(0) = \varrho_\hbar(0) = \varrho_\hbar.$$

Globally defined quantum dynamical system

The last part of the Duhamel formula is well-defined :

$$\begin{aligned}
&\mathrm{Tr}\Big(\frac{1}{\hbar}\big[\mathcal{W}(\xi), \hat{H}_k(s)\big]\ \tilde{\varrho}_\hbar(s)\Big) \\
&= \mathrm{Tr}\Big[\underbrace{S^{-1}\ B_0(s,\hbar,\xi)\ S^{-1}}_{\in\mathcal{L}(\mathcal{H})}\underbrace{S\,\mathcal{W}(\xi)\ S^{-1}}_{\in\mathcal{L}(\mathcal{H})}\underbrace{S\ \tilde{\rho}_\hbar(s)\ S}_{\in\mathcal{L}^1(\mathcal{H})}\Big] \\
&+ \hbar\mathrm{Tr}\Big[\underbrace{S^{-1}\ B_1(\hbar,s,\xi)\ S^{-1}}_{\in\mathcal{L}(\mathcal{H})}\underbrace{S\,\mathcal{W}(\xi)\ S^{-1}}_{\in\mathcal{L}(\mathcal{H})}\underbrace{S\ \tilde{\rho}_\hbar(s)\ S}_{\in\mathcal{L}^1(\mathcal{H})}\Big]
\end{aligned}$$

$\rightarrow$ The second term in the last two lines is a consequence of Weyl -Heisenberg operator estimates ;

$\rightarrow$ The last term is a consequence of Assumption (Q_0) and (Q_1) together with equivalence between $\hat{H}$ and $\hat{H}_0$.

The next step is to pass to the limit in the Duhamel formula as $\hbar$ tends to zero. So that, we prove that there exists a subsequence $(\hbar_\ell)_{\ell\in\mathbb{N}}$ such that

$$\boxed{\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t),\ \ell\in\mathbb{N}) = \{\text{Singelton}\}}$$

A single Wigner measure for all times

Proposition. (Wigner measure for all times)

Let $(\varrho_\hbar)_\hbar$ be a family of density matrices satisfying (Q_0) and (Q_1). Then, for any sequence $(\hbar_n)_{n\in\mathbb{N}}$ such that $\hbar_n \underset{n\to\infty}{\longrightarrow} 0$, we can extract a subsequence $(\hbar_\ell)_{\ell\in\mathbb{N}}$ such that $\hbar_\ell \underset{\ell\to\infty}{\longrightarrow} 0$ and a family of Borel probability measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ such that for all $t \in \mathbb{R}$,

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t);\ \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

Moreover, for any compact interval, there exists $C > 0$ such that for $t \in J$:

$$\int_{X^0} \|u\|^2_{X^\sigma}\, d\tilde{\mu}_t(u) \leq C.$$

- To prove the above proposition, we have to use the following result [Z. Ammari, F. Nier (2008)]

Let $(\varrho_\hbar)_{\hbar\in(0,1)}$ satisfies : $\exists C > 0,\ \forall \hbar \in (0,1),\ \mathrm{Tr}[\varrho_\hbar\,(\hat{p}^2 + \hat{q}^2 + \hat{N}_\hbar)] < C.$
Then : $\forall\ \hbar_n \underset{n\to\infty}{\longrightarrow} 0,\ \exists \hbar_\ell \underset{\ell\to\infty}{\longrightarrow} 0;\ \mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbb{N}) = \{\mu\}.$

Sketch of the proof :

Recall that

$$\exists C_0 > 0,\ \forall \hbar \in (0,1), \quad \mathrm{Tr}[\varrho_\hbar\, d\Gamma(\omega^{2\sigma})] \leq C_0, \qquad (Q_0)$$

$$\exists C_1 > 0,\ \forall \hbar \in (0,1), \quad \mathrm{Tr}[\varrho_\hbar\, (\hat{q}^2 + \hat{p}^2)] \leq C_1. \qquad (Q_1)$$

- Let $(\varrho_\hbar)_{\hbar\in(0,1)}$ satisfies (Q_0) and (Q_1). Then, the family of states

$$\boxed{(\tilde{\varrho}_h(t))_{\hbar\in(0,1)}}$$

satisfy (Q_0) and (Q_1) uniformly for any $t \in \mathbb{R}$ in every arbitrary compact time interval.

Indeed, we have the following inequalities with some $C_1,\ C_2,\ C_3 > 0$

- $\mathrm{Tr}\left[\tilde{\varrho}_\hbar(t)\, d\Gamma(\omega^{2\sigma})\right] \leq C_1\ \mathrm{Tr}\left[\varrho_\hbar\, (d\Gamma(\omega^{2\sigma}) + 1)\right] e^{C_2|t|} \leq C_3.$
- $\mathrm{Tr}\left[\tilde{\varrho}_\hbar(t)\, (\hat{p}^2 + \hat{q}^2)\right] \leq C_1\ \mathrm{Tr}\left[\varrho_\hbar\, (\hat{H}_0 + \hat{p}^2 + \hat{q}^2 + 1)\right] e^{C_2|t|} \leq C_3,$

- For each fixed $t_0 \in \mathbb{R}$:

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_0);\ \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_0}\}, \quad \int_{X^0} \underbrace{\|u\|^2_{X^\sigma}}_{=p^2+q^2+\|\alpha\|^2_{\mathcal{G}^\sigma}} \, d\tilde{\mu}_{t_0}(u) \leq C.$$

For all $\mu \in \mathcal{M}(\varrho_\hbar, \hbar \in (0,1))$, we have the implications below

- $\mathrm{Tr}[\varrho_\hbar(\hat{p}^2 + \hat{q}^2)] \leq C \Rightarrow \quad \int_{X^0} (p^2 + q^2)\, d\mu(u) \leq C$;
- $\mathrm{Tr}[\varrho_\hbar\ \hat{N}_\hbar] \leq C \Rightarrow \quad \int_{X^0} \|\alpha\|^2_{L^2} d\mu(u) \leq C$;
- $\mathrm{Tr}[\varrho_\hbar\ d\Gamma(\omega^{2\sigma})] \leq C \Rightarrow \quad \int_{X^0} \|\alpha\|^2_{\mathcal{G}^\sigma} d\mu(u) \leq C$.

- We use the above localization estimates, the diagonal extraction method and the prokhorov's theorem to prove for all times.

Convergence of the interacting terms

Let $\varphi(k) := 2\pi ik.q_{0j}\, F_j(k)$, we have for $u = (p, q, \alpha) \in X^0$

$$\left|\mathrm{Tr}[a^*_{\hbar_\ell}(e^{-2\pi ik\cdot\hat{q}_j}\ \tilde{F}_j(\hbar_\ell, k))\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)] - \int_{X^0} \langle \alpha, e^{-2\pi ik\cdot q_j}\ \varphi(.)\rangle\ e^{Q(\xi,u)} d\tilde{\mu}_s(u)\right|$$

$$+\left|\mathrm{Tr}[a^*_{\hbar_\ell}(e^{-2\pi ik\cdot\hat{q}_j}\ \varphi(.))\ \mathcal{W}(\xi)\ \tilde{\varrho}_{\hbar_\ell}(s)] - \int_{X^0} \langle \alpha, e^{-2\pi ik\cdot q_j}\ \varphi(.)\rangle\ e^{Q(\xi,u)} d\tilde{\mu}_s(u)\right| \cdots (2) \to 0$$

- (1) goes to zero as $\ell \to \infty$ by lebesgue dominated convergence theorem.
- (2) goes to zero as $\ell \to \infty$ by exploiting the following convergence for all $\varphi \in L^2(\mathbb{R}^d_k)$:

$$\lim_{\ell\to\infty} \mathrm{Tr}\left[a_{\hbar_\ell}(e^{-2\pi ik\cdot\hat{q}_j}\varphi)\,\mathcal{W}(\xi)\ \varrho_{\hbar_\ell}\right] = \int_{X^0} \langle e^{-2\pi i\,k\cdot q_j}\varphi, \alpha\rangle_{L^2(\mathbb{R}^d_k)}\ e^{Q(\xi,u)} d\mu(u)$$

$$\lim_{\ell\to\infty} \mathrm{Tr}\left[a^*_{\hbar_\ell}(e^{-2\pi ik\cdot\hat{q}_j}\varphi)\,\mathcal{W}(\xi)\ \varrho_{\hbar_\ell}\right] = \int_{X^0} \langle \alpha, e^{-2\pi i\,k\cdot q_j}\varphi\rangle_{L^2(\mathbb{R}^d_k)}\ e^{Q(\xi,u)} d\mu(u)$$

Equivalence between characteristic and Liouville equation

$$\boxed{\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t),\ \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.}$$

Lemma [Regular properties of the Wigner Measure $\tilde{\mu}_t$]

The Wigner measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ extracted in above arguments satisfy

(i) $\tilde{\mu}_t(X^\sigma) = 1$ i.e. $\tilde{\mu}_t$ concentrates on X^σ.

(ii) $\mathbb{R} \ni t \longmapsto \tilde{\mu}_t \in \mathcal{P}(X^\sigma)$ is **weakly narrowly continuous**.

Lemma [Continuity, integrability and boundedness]

Assume (C_0) and (C_1) are satisfied.

Then, the vector field $v : \mathbb{R} \times X^\sigma \longrightarrow X^\sigma$ is **continuous** and **bounded** on bounded subsets of $\mathbb{R} \times X^\sigma$. Moreover, for any bounded open interval I,

$$\int_I \int_{X^\sigma} \|v(t,u)\|_{X^\sigma}\, d\tilde{\mu}_t(u)\, dt < +\infty. \tag{Int}$$

Equivalence between Liouville equation and Characteristic equation

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