

# Quantum-classical motion of charged particles in interaction with scalar fields

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## Purposes

### 1 Bohr's correspondence principle :

<i>Quantum dynamic</i>	$\xrightarrow{\hbar \rightarrow 0}$	<i>Classical dynamic</i>
Nelson model		Particle-field equation

We study the transition by **Wigner measure** approach.



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### 2 To exhibit the global well-posedness for the particle-field equation.





# The classical system

Consider  $n$  **fixed** particles in  $\mathbb{R}^d$  with  $d \in \mathbb{N}^*$ , interacting with scalar meson field. The particle-field system reads for all  $j \in \{1, \dots, n\}$

$$\begin{aligned} \partial_t p_j &= -\nabla_{q_j} V(q) - \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk ; \\ \partial_t q_j &= \nabla f_j(p_j) ; \\ i\partial_t \alpha &= \omega(k) \alpha(k) + \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j} \end{aligned} \tag{PFE}$$

- ▶  $q_j, p_j$  are positions-momenta and  $M_j$  are the masses.
- ▶  $\alpha : \mathbb{R}^d \rightarrow \mathbb{C}$  describes the field,  $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$  is the form factor.
- ▶  $V : \mathbb{R}^{dn} \rightarrow \mathbb{R}$  is the potential.
- ▶  $f_j(p_j) = \begin{cases} \sqrt{p_j^2 + M_j^2} & \text{(relativistic)} \\ p_j^2/2M_j & \text{(non-relativistic)} \end{cases}$
- ▶  $\omega(k) = \sqrt{k^2 + m_f^2} \geq m_f > 0$  is the dispersion relation;



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## References

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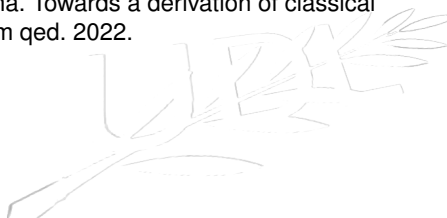
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The associated Hamiltonian is defined as follows

$$\begin{aligned}
 H(p, q, \alpha) &= \sum_{j=1}^n f_j(p_j) + V(q_1, \dots, q_n) + \int_{\mathbb{R}^d} \overline{\alpha(k)} \omega(k) \alpha(k) dk \\
 &+ \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk.
 \end{aligned}$$

The solution  $u = (p, q, \alpha)$  belongs to the classical phase-space :

$$X^\sigma := \mathbb{R}_p^{dn} \times \mathbb{R}_q^{dn} \times \mathcal{G}^\sigma,$$

where  $\mathcal{G}^\sigma$  is the weighted  $L^2$  Lebesgue space endowed with the norm :

$$\|\alpha\|_{\mathcal{G}^\sigma}^2 = \langle \alpha, \omega(\cdot)^{2\sigma} \alpha \rangle_{L^2} = \|\omega^\sigma \alpha\|_{L^2}^2.$$

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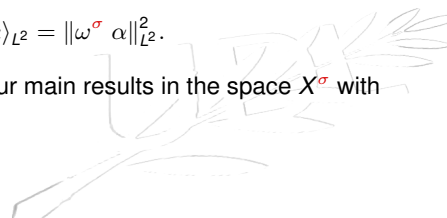
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The energy space is  $X^{1/2}$ . We will give our main results in the space  $X^\sigma$  with  $\sigma \in [\frac{1}{2}, 1]$ .



# The Quantum system



## The Quantum system : Nelson Hamiltonian

The Nelson Hamiltonian is defined as follows

$$\hat{H} = \sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + d\Gamma(\omega) + \sum_{j=1}^n (a_{\hbar} + a_{\hbar}^*) \left( \frac{\chi}{\sqrt{\omega}} e^{-2\pi k \cdot \hat{q}_j} \right).$$

The Hilbert space of the quantized particle-field system is

$$\mathcal{H} := L^2(\mathbb{R}_x^{dn}, \mathbb{C}) \otimes \Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C})),$$

where  $\Gamma_s$  is the symmetric Fock space

$$\Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C})) := \bigoplus_{m=0}^{+\infty} L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_s m} \simeq \bigoplus_{m=0}^{+\infty} L_s^2(\mathbb{R}^{dm}, \mathbb{C}).$$

We denote by  $\mathcal{F}^m := L_s^2(\mathbb{R}^{dm}, \mathbb{C})$  the symmetric  $L^2$  space over  $\mathbb{R}^{dm}$ .

**Remark :**  $\hat{H}$  is a self adjoint operator.

- ▶  $\hat{p}_j = -i\hbar\nabla_{x_j}$  is the momentum operator ;
- ▶  $\hat{q}_j = x_j$  is the position operator ;
- ▶  $d\Gamma(\omega) : \mathcal{H} \rightarrow \mathcal{H}$  is the free field Hamiltonian

$$d\Gamma(\omega) |_{\mathcal{F}^m} = \hbar \sum_{j=1}^m \omega(k_j);$$

- ▶  $a_{\hbar}$  and  $a_{\hbar}^*$  are the generalized  $\hbar$  scaled annihilation-creation operators are defined as follows :

for every  $\psi = \{\psi^m\}_{m \geq 0} \in \mathcal{H}$  and  $F(k) := \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j}$

$$[a_{\hbar}(F)\psi(x)]^m(K_m) = \sqrt{\hbar(m+1)} \int_{\mathbb{R}^d} \overline{F(k)} \psi^{m+1}(x; K_m, k) dk;$$

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# Assumptions and main results



# Assumptions

- On  $V$  and  $\chi$  :

$$V \in \mathcal{C}_b^2(\mathbb{R}^{dn}; \mathbb{R}) \tag{C_0}$$

$$\omega(\cdot)^{\frac{3}{2}-\sigma} \chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R}), \quad \sigma \in \left[\frac{1}{2}, 1\right]. \tag{C_1}$$

- Let  $(\varrho_{\hbar})_{\hbar \in (0,1)}$  be a family of density matrices on  $\mathcal{H}$  of the particle-field quantum system. We assume that :

$$\exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar} d\Gamma(\omega^{2\sigma})] \leq C_0, \tag{Q_0}$$

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# Global well-posedness of (PFE)

## Theorem [S. Farhat, 2023]

Let  $\sigma \in [\frac{1}{2}, 1]$  and assume  $(C_0)$  and  $(C_1)$  hold true. Then, for any initial data  $u_0 \in X^\sigma$ , there exists a unique global strong solution  $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap \mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$  of the particle-field equation (PFE). Moreover, the generalized global flow

$$\begin{aligned} \Phi_t : X^\sigma &\longrightarrow X^\sigma \\ u_0 &\longmapsto u(t). \end{aligned}$$

is Borel measurable.

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# Wigner measures

## Definition [Weyl operator]

The Weyl operator over the **entire interacting** Hilbert space  $\mathcal{H}$

$$\xi = (p, q, \alpha) \in X^0 \mapsto \mathcal{W}(\xi) := e^{i(p \cdot \hat{q} - q \cdot \hat{p})} \otimes e^{\frac{i}{\sqrt{2}}(a_h(\alpha) + a_h^*(\alpha))}$$

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A Borel probability measure  $\mu$  over  $X^0$  is a Wigner measure of a family of density matrices  $(\varrho_h)_{h \in (0,1)}$  on the Hilbert space  $\mathcal{H}$  if and only if there exists a subset  $\mathcal{E} \subset (0, 1)$  with  $0 \in \overline{\mathcal{E}}$  such that for any  $\xi = (p_0, q_0, \alpha_0), \tilde{\xi} = (2\pi q_0, -2\pi p_0, \sqrt{2}\pi \alpha_0) \in X^0$  :

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- ▶ Denote by  $\mathcal{M}(\varrho_h, h \in \mathcal{E})$  the set of all Wigner measures of  $(\varrho_h)_{h \in \mathcal{E}}$ .
- ▶  $\mathcal{M}(\varrho_h, h \in \mathcal{E}) \neq \emptyset$  if some assumptions on  $(\varrho_h)_h$  are imposed.
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# Wigner measures

## Definition [Weyl operator]

The Weyl operator over the **entire interacting** Hilbert space  $\mathcal{H}$

$$\xi = (p, q, \alpha) \in X^0 \mapsto \mathcal{W}(\xi) := e^{i(p \cdot \hat{q} - q \cdot \hat{p})} \otimes e^{\frac{i}{\sqrt{2}}(a_{\hbar}(\alpha) + a_{\hbar}^*(\alpha))}$$

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A Borel probability measure  $\mu$  over  $X^0$  is a Wigner measure of a family of density matrices  $(\varrho_{\hbar})_{\hbar \in (0,1)}$  on the Hilbert space  $\mathcal{H}$  if and only if there exists a subset  $\mathcal{E} \subset (0, 1)$  with  $0 \in \bar{\mathcal{E}}$  such that for any  $\xi = (p_0, q_0, \alpha_0), \tilde{\xi} = (2\pi q_0, -2\pi p_0, \sqrt{2}\pi \alpha_0) \in X^0$  :

$$\lim_{\hbar \rightarrow 0, \hbar \in \mathcal{E}} \text{Tr} \left[ \mathcal{W}(\tilde{\xi}) \varrho_{\hbar} \right] = \int_{X^0} e^{2\pi i \Re \theta(\xi, u)} d\mu(u) = \mathcal{F}^{-1}[\mu](\xi).$$

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# Classical limit : Bohr Correspondence principle

## Theorem [S. Farhat, 2023]

Let  $\sigma \in [\frac{1}{2}, 1]$  and assume  $(C_0)$  and  $(C_1)$  hold true. Let  $(\varrho_{\hbar})_{\hbar \in (0,1)}$  be a family of density matrices on  $\mathcal{H}$  satisfying  $(Q_0)$  and  $(Q_1)$ . Assume that

$$\mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbf{N}) = \{\mu_0\}.$$

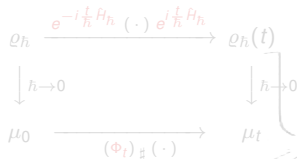
Then for all times  $t \in \mathbb{R}$ , we have

$$\underbrace{\mathcal{M}(e^{-i\frac{t}{\hbar_\ell} H} \varrho_{\hbar_\ell} e^{i\frac{t}{\hbar_\ell} H}, \ell \in \mathbf{N})}_{:= \varrho_{\hbar_\ell}(t)} = \{\mu_t\},$$

where  $\mu_t \in \mathcal{P}(X^0)$  satisfies

- (i)  $\mu_t(X^\sigma) = 1$ .
- (ii)  $\mu_t = (\Phi_t)_\# \mu_0$ , where  $\Phi_t$  is the global flow of the particle-field equation.

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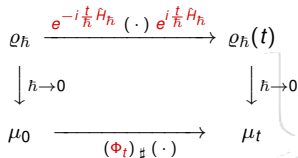
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- ▶ We work with the interaction representation with

$$\tilde{\varrho}_h(t) := e^{i\frac{t}{\hbar}d\Gamma(\omega)} \varrho_h(t) e^{-i\frac{t}{\hbar}d\Gamma(\omega)}$$

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The commutator in the Duhamel formula can be expanded as follows :

$$\frac{1}{\hbar} [\mathcal{W}(\xi), \hat{H}_k(s)] = \left[ \underset{\substack{\uparrow \\ \text{Main} \\ \text{term}}}{B_0(s, \hbar, \xi)} + \hbar \underset{\substack{\uparrow \\ \text{Remainder} \\ \text{term}}}{B_1(s, \hbar, \xi)} \right] \mathcal{W}(\xi).$$

We have with  $\hat{H}_0 = d\Gamma(\omega) + \sum_{j=1}^n f_j(\hat{p}_j)$  and  $S = (\hat{H}_0 + 1)^{1/2}$

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The next step is to prove that we can extract a subsequence  $(\hbar_\ell)_{\ell \in \mathbb{N}}$  and a family of Borel probability measures  $(\tilde{\mu}_t)_{t \in \mathbb{R}}$  such that for all  $t \in \mathbb{R}$

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$$\lim_{\ell \rightarrow \infty} \text{Tr}[\mathcal{W}(\xi) \tilde{\varrho}_{\hbar_\ell}(t)] = \lim_{\ell \rightarrow \infty} \text{Tr}[\mathcal{W}(\xi) \tilde{\varrho}_{\hbar_\ell}(t_0)] - i \int_{t_0}^t \lim_{\ell \rightarrow \infty} \text{Tr}[B_0(s, \hbar_\ell, \xi) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_\ell}(s)] ds$$

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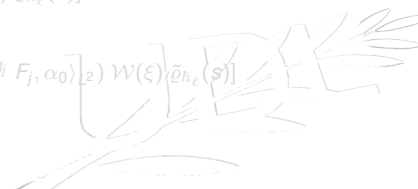
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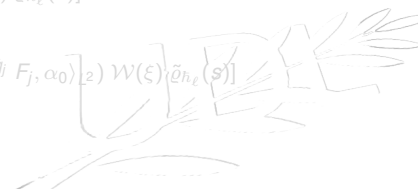
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# The characteristic equations

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(IPFE) : the interaction representation of (PFE)

$$\begin{cases} \frac{du}{dt} = v(t, u(t)) = \Phi_{-t}^f \circ \underbrace{\mathcal{N}}_{\text{Non-linearity of (PFE)}} \circ \Phi_t^f(u(t)), \\ u(0) = u_0 \in X^\sigma. \end{cases} \tag{IPFE}$$

- ▶  $\Phi_t^f : X^\sigma \rightarrow X^\sigma$  is the free flow defined as follows

$$\Phi_t^f(p, q, \alpha) = (p, q, e^{-it\omega(k)} \alpha).$$

- ▶ The non-autonomous vector field  $v$  is as follows  $v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$  satisfying

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$$\begin{cases} \frac{du}{dt} = v(t, u(t)) = \Phi_{-t}^f \circ \underbrace{\mathcal{N}}_{\text{Non-linearity of (PFE)}} \circ \Phi_t^f(u(t)), \\ u(0) = u_0 \in X^\sigma. \end{cases} \tag{IPFE}$$

- ▶  $\Phi_t^f : X^\sigma \rightarrow X^\sigma$  is the free flow defined as follows

$$\Phi_t^f(p, q, \alpha) = (p, q, e^{-it\omega(k)} \alpha).$$

- ▶ The non-autonomous vector field  $v$  is as follows  $v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$  satisfying

$$\int_I \int_{X^\sigma} \|v(t, u)\|_{X^\sigma} d\tilde{\mu}_t(u) dt < +\infty. \tag{Int}$$



# The characteristic equations

- ▶ The limit of the Duhamel is formulated as follows for all  $t, t_0 \in \mathbb{R}$  and  $y \in X^\sigma$

$$\int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} d\tilde{\mu}_t(u) = \int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} d\tilde{\mu}_{t_0}(u) + 2\pi i \int_{t_0}^t \int_{X^0} e^{2\pi i \Re e \langle y, u \rangle_{X^\sigma}} \Re e \langle v(s, u), y \rangle_{X^\sigma} d\tilde{\mu}_s(u) ds, \tag{C}$$

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## Equivalence between characteristic and Liouville equation

### Definition : Liouville equation

A family of Borel probability measures  $\{\tilde{\mu}_t\}_{t \in I}$  on  $X^\sigma$  is a measure-valued solution of the Liouville equation associated to the vector field

$v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$  if and only if for all  $\phi \in C_{0,cyl}^\infty(I \times X^\sigma)$  :

$$\int_I \int_{X^\sigma} \{ \partial_t \phi(t, u) + \Re e \langle v(t, u), \nabla \phi(t, u) \rangle_{X^\sigma} \} d\tilde{\mu}_t(u) dt = 0, \quad (L)$$

$C_{0,cyl}^\infty(I \times X^\sigma)$  is the cylindrical functional space.

Then, thanks to the regular properties of  $\tilde{\mu}_t$  and of the vector field  $v(t, u)$ , we have the following are equivalent :

### Equivalence between Liouville equation and Characteristic equation

$\{\tilde{\mu}_t\}_{t \in I}$  solves the Liouville equation (L)  $\Leftrightarrow$   $\{\tilde{\mu}_t\}_{t \in I}$  solves the characteristic equation (C).

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# Proof of the main results



## Proof of the main results

- ▶ Using measure theoretical techniques, we have :  
 ⇒ Almost sure existence of **unique global** solutions to (PFE) with a generalized global flow

$$\Phi_t = \Phi_t^f \circ \tilde{\Phi}_t.$$

- ▶ To get rid of almost sureness, we select a special choice of family of density matrices which is **coherent states** centered at initial data.
- ▶ It remains to check

$$\mu_t = (\Phi_t)_\# \mu_0, \quad \Phi_t = \Phi_t^f \circ \tilde{\Phi}_t.$$

- ▶ The important tool to do is the following link :

$$\mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1)) = \{(\Phi_t^f)_\# \tilde{\mu}_t, \tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_{\hbar}(t), \hbar \in (0, 1))\}$$

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**Thank you for your attention !**



# Slides for more details about the presentation



# Almost sure existence result

## Theorem 2 [Z. Ammari, M. Falconi and F. Hiroshima, 2022]

In a **separable Hilbert space**  $\mathcal{H}$ , consider the initial value problem (IPFE) with a vector field  $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$  **continuous and bounded on bounded sets**. Let  $I \ni 0$  be a bounded open interval and assume

(i)  $\exists t \in \mathbb{R} \rightarrow \mu_t \in \mathcal{P}(\mathcal{H})$  a weakly narrowly continuous solution to (L) satisfying

$$\int_I \int_{\mathcal{H}} \|v(t, u)\|_{\mathcal{H}} d\mu_t(u) dt < +\infty. \tag{Int}$$

(ii) **Uniqueness** of the solutions to (IPFE) over any  $I$ .

Then for  $\mu_0$ -almost all initial conditions  $x$  in  $\mathcal{H}$ , there exists a (unique) global strong solution (i.e.  $u(\cdot) \in C^1(\mathbb{R}, \mathcal{H})$ ) to (IPFE). Moreover, the set

$$\mathfrak{G} := \{x \in \mathcal{H} : \exists u(\cdot) \text{ a global strong solution of (IPFE) with } u(0) = x\},$$

is **Borel** subset of  $\mathcal{H}$  with  $\mu_0(\mathfrak{G}) = 1$  and for any time  $t \in \mathbb{R}$  the map

$$\begin{aligned} \Phi_t : \mathfrak{G} &\longrightarrow \mathfrak{G} \\ x &\longmapsto \Phi_t(x) = u(t). \end{aligned}$$

is **Borel measurable**.

## Existence of unique global solutions

- ▶ Apply Theorem 2 with the following choices :

- $\mu_t \equiv \tilde{\mu}_t$ ;
- $\mathcal{H} \equiv X^\sigma$ .

⇒ Almost sure existence of **unique global** solutions for (IPFE) with a generalized global flow  $\tilde{\Phi}_t$ .

- ▶ We have this equivalence : (PFE)  $\xrightleftharpoons[\Phi_t^f]{\Phi_{-t}^f}$  (IPFE)

⇒ Almost sure existence of **unique global** solutions to (PFE) with a generalized global flow

$$\Phi_t = \Phi_t^f \circ \tilde{\Phi}_t.$$

- ▶ To get rid of almost sureness, we select a special choice of family of density matrices which is **coherent states** centered at initial data.

## Probabilistic representation

The crucial tool that was used for constructing the above generalized global flow for (IPFE) is the following :

### Probabilistic representation

There exists  $\eta \in \mathcal{P}(X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma))$  satisfying :

(i)  $\eta(\mathcal{F}_I) = 1$  where

$$\mathcal{F}_I := \left\{ (u_0, u(\cdot)) \in X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) : u(\cdot) \text{ satisfies (IPFE) on } I \text{ with } u_0 \right\}$$

(ii)  $\tilde{\mu}_t = (e_t)_\# \eta, \quad \forall t \in I$ , where the map

$$\begin{aligned} e_t : X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) &\longrightarrow X^\sigma \\ (u_0, u(\cdot)) &\longmapsto e_t(u_0, u(\cdot)) = u(t). \end{aligned}$$

is the evaluation map.

**Generalization** : Z. Ammari, S. Farhat and V. Sohinger "Almost sure existence of global solutions for general initial value problems."

## Global well-posedness of the particle-field equation

- ▶ Let  $u_0 = (z_0, \alpha_0) \in X^\sigma$  and consider the coherent vectors respectively in the particle and Fock spaces

$$W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi, \quad W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega$$

- ▶  $\psi(x) = (\pi\hbar)^{-dn/4} e^{-x^2/2\hbar} \in L^2(\mathbb{R}^{dn}, dx)$  is the **normalized gaussian** function on the particles.
- ▶  $\Omega$  is the **vacuum vector** on the Fock space.

Then, the following projection

$$C_\hbar(u_0) = \left| W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi \otimes W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right\rangle \left\langle W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi \otimes W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right|$$

gives rise to a family of **coherent states**.

- ▶ We have

$$\mathcal{M}(C_\hbar(u_0), \hbar \in (0, 1)) = \{\delta_{u_0}\} : \text{Dirac measure centered on } u_0$$

- ▶ Since  $u_0 \in X^\sigma$ , this implies

$$(C_\hbar(u_0))_\hbar \text{ satisfies } (Q_0) \text{ and } (Q_1).$$



Let  $u_0 \in X^\sigma$  and let  $\varrho_{\tilde{h}} = \mathcal{C}_{\tilde{h}}(u_0)$ .

- ▶ Apply Theorem A with the measure  $\tilde{\mu}_t$  to get the

GWP of (IPFE)  $\tilde{\mu}_0$ -almost all initial data in  $X^\sigma$

with a generalized global flow  $\tilde{\Phi}_t$ .

- ▶ We have also

$$\tilde{\mu}_0(\mathfrak{G}) = \delta_{u_0}(\mathfrak{G}) = 1.$$

This implies  $u_0 \in \mathfrak{G}$ .

- ▶ GWP of (PFE) with a generalized global flow

$$\Phi_t(u_0) = \Phi_t^f \circ \tilde{\Phi}_t(u_0),$$



## The classical limit : Validity of Bohr's correspondance

### Goal

To prove the second property :  $\mu_t = (\Phi_t)_\# \mu_0$ ,  $\Phi_t = \Phi_t^f \circ \tilde{\Phi}_t$ .

We have, by probabilistic representation, that

$$\tilde{\mu}_t = (\tilde{\Phi}_t)_\# \tilde{\mu}_0.$$

The important tool to do that is the following link :

$$\mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1)) = \{(\Phi_t^f)_\# \tilde{\mu}_t, \tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_{\hbar}(t), \hbar \in (0, 1))\}$$

= $\{\mu_t\}$  = $\{\tilde{\mu}_t\}$

This implies using the two boxes :

$$\begin{aligned} \mu_t &= (\Phi_t^f)_\# \tilde{\mu}_t = (\Phi_t^f \circ \tilde{\Phi}_t)_\# \tilde{\mu}_0 \\ &= (\Phi_t)_\# \tilde{\mu}_0 = (\Phi_t)_\# \mu_0 \end{aligned}$$

and where we have used  $\tilde{\mu}_0 = \mu_0$  as a consequence of

$$\tilde{\varrho}_{\hbar}(0) = \varrho_{\hbar}(0) = \varrho_{\hbar}.$$

## Globally defined quantum dynamical system

The **last part** of the Duhamel formula is **well-defined** :

$$\begin{aligned} & \text{Tr} \left( \frac{1}{\hbar} [\mathcal{W}(\xi), \hat{H}_k(s)] \tilde{\rho}_\hbar(s) \right) \\ &= \text{Tr} \left[ \underbrace{S^{-1} B_0(s, \hbar, \xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \mathcal{W}(\xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \tilde{\rho}_\hbar(s) S}_{\in \mathcal{L}^1(\mathcal{H})} \right] \\ &+ \hbar \text{Tr} \left[ \underbrace{S^{-1} B_1(\hbar, s, \xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \mathcal{W}(\xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \tilde{\rho}_\hbar(s) S}_{\in \mathcal{L}^1(\mathcal{H})} \right] \end{aligned}$$

- The second term in the last two lines is a consequence of Weyl-Heisenberg operator estimates ;
- The last term is a consequence of Assumption  $(Q_0)$  and  $(Q_1)$  together with equivalence between  $\hat{H}$  and  $\hat{H}_0$ .

The next step is to pass to the limit in the Duhamel formula as  $\hbar$  tends to zero. So that, we prove that there exists a subsequence  $(\hbar_\ell)_{\ell \in \mathbb{N}}$  such that

$$\mathcal{M}(\tilde{\rho}_{\hbar_\ell}(t), \ell \in \mathbb{N}) = \{\text{Singelton}\}$$

# A single Wigner measure for all times

## Proposition. (Wigner measure for all times)

Let  $(\varrho_{\hbar})_{\hbar}$  be a family of density matrices satisfying  $(Q_0)$  and  $(Q_1)$ . Then, for any sequence  $(\hbar_n)_{n \in \mathbb{N}}$  such that  $\hbar_n \xrightarrow{n \rightarrow \infty} 0$ , we can extract a subsequence  $(\hbar_\ell)_{\ell \in \mathbb{N}}$  such that  $\hbar_\ell \xrightarrow{\ell \rightarrow \infty} 0$  and a family of Borel probability measures  $(\tilde{\mu}_t)_{t \in \mathbb{R}}$  such that for all  $t \in \mathbb{R}$ ,

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t); \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

Moreover, for any compact interval, there exists  $C > 0$  such that for  $t \in J$ :

$$\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_t(u) \leq C.$$

- To prove the above proposition, we have to use the following result [Z. Ammari, F. Nier (2008)]

Let  $(\varrho_{\hbar})_{\hbar \in (0,1)}$  satisfies :  $\exists C > 0, \forall \hbar \in (0,1), \text{Tr}[\varrho_{\hbar} (\hat{p}^2 + \hat{q}^2 + \hat{N}_{\hbar})] < C$ .  
 Then :  $\forall \hbar_n \xrightarrow{n \rightarrow \infty} 0, \exists \hbar_\ell \xrightarrow{\ell \rightarrow \infty} 0; \mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbb{N}) = \{\mu\}$ .

## Sketch of the proof :

Recall that

$$\exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar} d\Gamma(\omega^{2\sigma})] \leq C_0, \quad (Q_0)$$

$$\exists C_1 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar} (\hat{q}^2 + \hat{p}^2)] \leq C_1. \quad (Q_1)$$

- Let  $(\varrho_{\hbar})_{\hbar \in (0,1)}$  satisfies  $(Q_0)$  and  $(Q_1)$ . Then, the family of states

$$(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$$

satisfy  $(Q_0)$  and  $(Q_1)$  uniformly for any  $t \in \mathbb{R}$  in every arbitrary compact time interval.

Indeed, we have the following inequalities with some  $C_1, C_2, C_3 > 0$

$$\text{Tr}[\tilde{\varrho}_{\hbar}(t) d\Gamma(\omega^{2\sigma})] \leq C_1 \text{Tr}[\varrho_{\hbar} (d\Gamma(\omega^{2\sigma}) + 1)] e^{C_2|t|} \leq C_3.$$

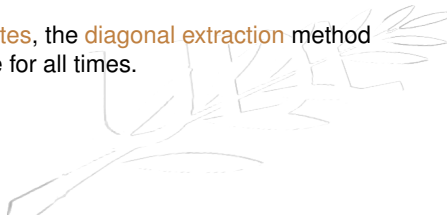
$$\text{Tr}[\tilde{\varrho}_{\hbar}(t) (\hat{p}^2 + \hat{q}^2)] \leq C_1 \text{Tr}[\varrho_{\hbar} (\hat{H}_0 + \hat{p}^2 + \hat{q}^2 + 1)] e^{C_2|t|} \leq C_3,$$

- ▶ For each **fixed**  $t_0 \in \mathbb{R}$  :

$$\mathcal{M}(\tilde{\rho}_{\hbar_\ell}(t_0); \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_0}\}, \quad \int_{X^0} \underbrace{\|u\|_{X^\sigma}^2}_{=p^2+q^2+\|\alpha\|_{G^\sigma}^2} d\tilde{\mu}_{t_0}(u) \leq C.$$

For all  $\mu \in \mathcal{M}(\rho_{\hbar}, \hbar \in (0, 1))$ , we have the implications below

- ▶  $\text{Tr}[\rho_{\hbar}(\hat{p}^2 + \hat{q}^2)] \leq C \Rightarrow \int_{X^0} (p^2 + q^2) d\mu(u) \leq C;$
  - ▶  $\text{Tr}[\rho_{\hbar} \hat{N}_{\hbar}] \leq C \Rightarrow \int_{X^0} \|\alpha\|_{L^2}^2 d\mu(u) \leq C;$
  - ▶  $\text{Tr}[\rho_{\hbar} d\Gamma(\omega^{2\sigma})] \leq C \Rightarrow \int_{X^0} \|\alpha\|_{G^\sigma}^2 d\mu(u) \leq C.$
- ▶ We use the above **localization estimates**, the **diagonal extraction method** and the **prokhorov's theorem** to prove for all times.



## Convergence of the interacting terms

Let  $\varphi(k) := 2\pi i k \cdot q_j F_j(k)$ , we have for  $u = (p, q, \alpha) \in X^0$

$$\left| \text{Tr}[a_{\hbar_\ell}^* (e^{-2\pi i k \cdot \hat{q}_j} \tilde{F}_j(\hbar_\ell, k)) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_\ell}(s)] - \int_{X^0} \langle \alpha, e^{-2\pi i k \cdot q_j} \varphi(\cdot) \rangle e^{Q(\xi, u)} d\tilde{\mu}_s(u) \right|$$

$$+ \left| \text{Tr}[a_{\hbar_\ell}^* (e^{-2\pi i k \cdot \hat{q}_j} \varphi(\cdot)) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_\ell}(s)] - \int_{X^0} \langle \alpha, e^{-2\pi i k \cdot q_j} \varphi(\cdot) \rangle e^{Q(\xi, u)} d\tilde{\mu}_s(u) \right| \dots (2) \rightarrow 0$$

- ▶ (1) goes to zero as  $\ell \rightarrow \infty$  by lebesgue dominated convergence theorem.
- ▶ (2) goes to zero as  $\ell \rightarrow \infty$  by exploiting the following convergence for all  $\varphi \in L^2(\mathbb{R}_k^d)$  :

$$\lim_{\ell \rightarrow \infty} \text{Tr} \left[ a_{\hbar_\ell} (e^{-2\pi i k \cdot \hat{q}_j} \varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] = \int_{X^0} \langle e^{-2\pi i k \cdot q_j} \varphi, \alpha \rangle_{L^2(\mathbb{R}_k^d)} e^{Q(\xi, u)} d\mu(u)$$

$$\lim_{\ell \rightarrow \infty} \text{Tr} \left[ a_{\hbar_\ell}^* (e^{-2\pi i k \cdot \hat{q}_j} \varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] = \int_{X^0} \langle \alpha, e^{-2\pi i k \cdot q_j} \varphi \rangle_{L^2(\mathbb{R}_k^d)} e^{Q(\xi, u)} d\mu(u)$$

# Equivalence between characteristic and Liouville equation

$$\mathcal{M}(\tilde{\varrho}_{\tilde{n}_\ell}(t), \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

## Lemma [Regular properties of the Wigner Measure $\tilde{\mu}_t$ ]

The Wigner measures  $(\tilde{\mu}_t)_{t \in \mathbb{R}}$  extracted in above arguments satisfy

- (i)  $\tilde{\mu}_t(X^\sigma) = 1$  i.e.  $\tilde{\mu}_t$  concentrates on  $X^\sigma$ .
- (ii)  $\mathbb{R} \ni t \mapsto \tilde{\mu}_t \in \mathcal{P}(X^\sigma)$  is **weakly narrowly continuous**.

## Lemma [Continuity, integrability and boundedness]

Assume  $(C_0)$  and  $(C_1)$  are satisfied.

Then, the vector field  $v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$  is **continuous** and **bounded** on bounded subsets of  $\mathbb{R} \times X^\sigma$ . Moreover, for any bounded open interval  $I$ ,

$$\int_I \int_{X^\sigma} \|v(t, u)\|_{X^\sigma} d\tilde{\mu}_t(u) dt < +\infty. \tag{Int}$$

## Equivalence between Liouville equation and Characteristic equation

$\{\tilde{\mu}_t\}_{t \in I}$  solves the Liouville equation (L)  $\Leftrightarrow$   $\{\tilde{\mu}_t\}_{t \in I}$  solves the characteristic equation (C).