Leading-order term expansion for the Teukolsky equation on subextremal Kerr black holes

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Classical field equations on a Kerr black hole

Equations of linearized gravity

$$D_{g_{M,a}} Ric(g) = 0$$

Equation of massless neutrino

$$\nabla^{AA'} \phi_A = 0$$

Maxwell's equations

$$\begin{cases} \nabla^{\mu} F_{\mu,\nu} = 0 \\ \mathrm{d}F = 0 \end{cases}$$

Scalar wave equation

$$\Box \phi = 0$$

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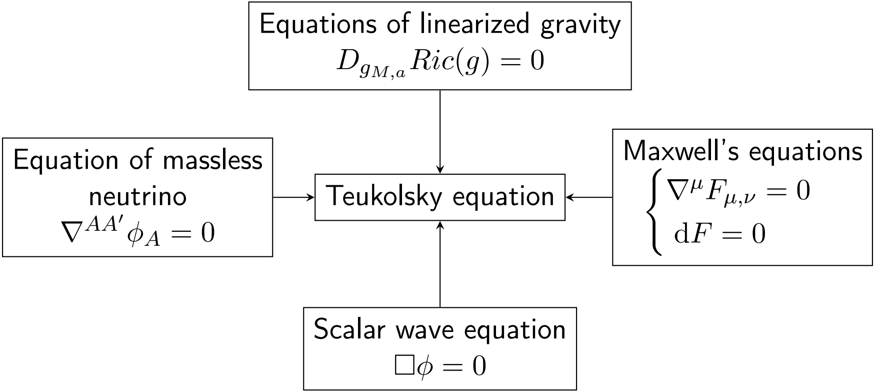

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Here we focus on the case $\Lambda = 0$.

Black hole solutions

Kerr solutions

Kerr (1963): $\mathcal{M} = \mathbb{R}_t \times (r_+, +\infty) \times \mathbb{S}^2$, metric $g_{M,a}$

Model for a rotating black hole.

Subextremal: $|a| < M$.

$$\begin{aligned} g_{M,a} :=& \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2}\, \mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\rho^2}\, \mathrm{d}t\, \mathrm{d}\phi - \frac{\rho^2}{\Delta_r}\, \mathrm{d}r^2 \\ & - \rho^2\, \mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}((a^2+r^2)^2 - a^2\Delta_r \sin^2\theta)\, \mathrm{d}\phi^2 \\ \Delta_r :=& a^2 + r^2 - 2Mr \\ \rho^2 :=& r^2 + a^2\cos^2\theta \end{aligned}$$

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- Asymptotically flat:
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- Contains trapped null geodesics.

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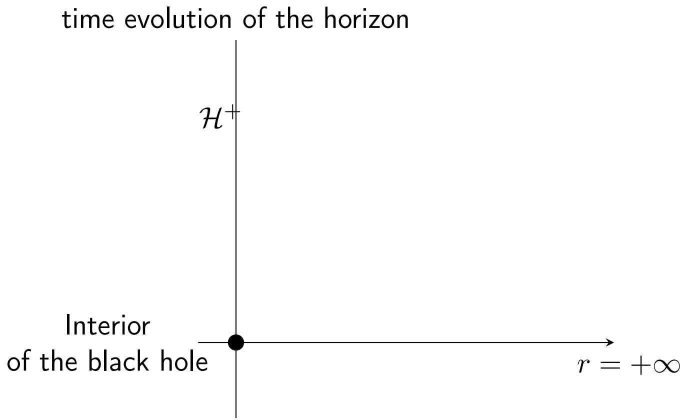

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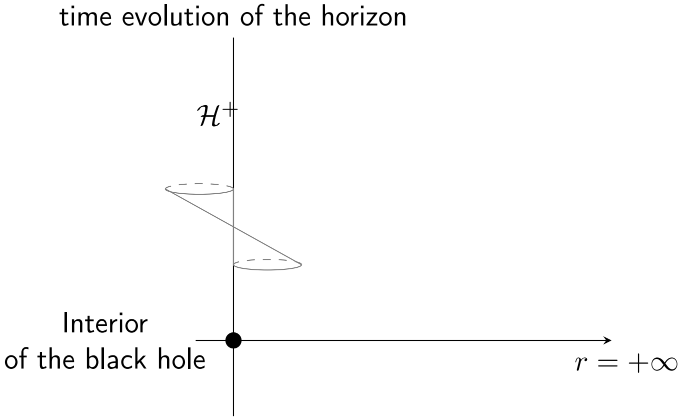

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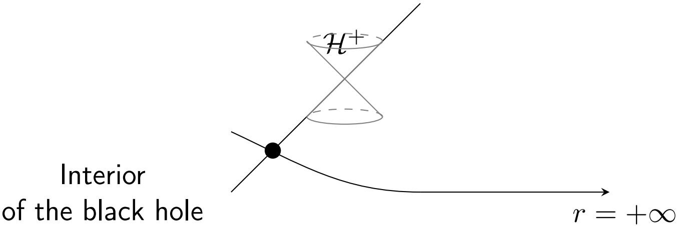

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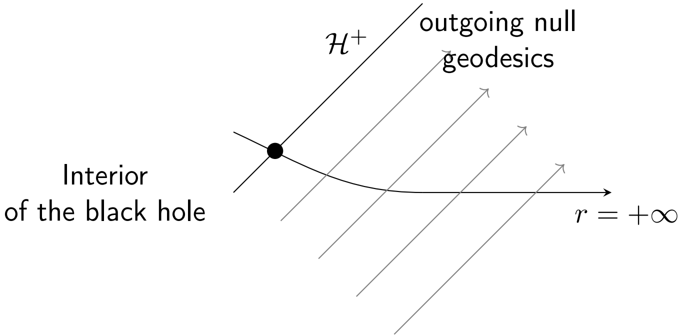

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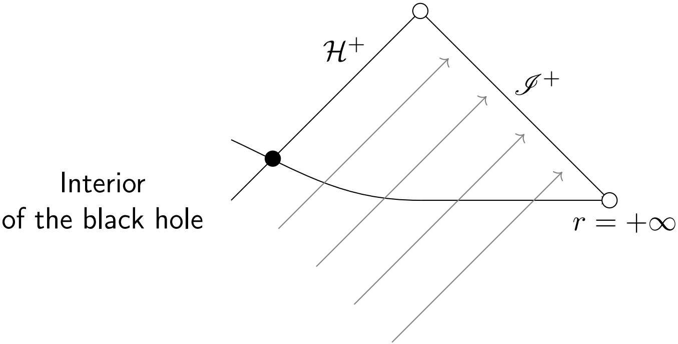

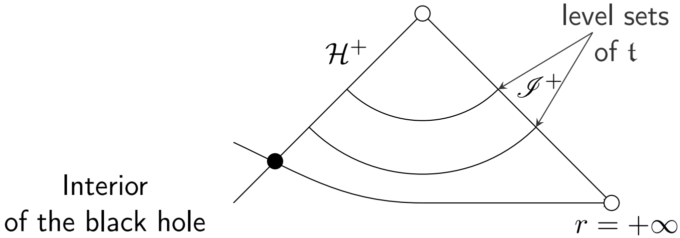

level sets
of $\mathfrak{t}$
$\mathcal{H}^+$
$\mathscr{I}^+$
Interior
of the black hole
$r = +\infty$

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$$\begin{aligned} \Psi_2 =& D_{g_{M,a}} W(\dot{g})(l, m, l, m) \\ \Psi_{-2} =& (r - ia\cos\theta)^4 D_{g_{M,a}} W(\dot{g})(n, \overline{m}, n, \overline{m}) \end{aligned}$$

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Proposition (Teukolsky)

$D_{g_{M,a}} Ric(\dot{g}) = 0 \Rightarrow T_s \Psi_s = 0$ for $s = \pm 2$.
T_s linear scalar operator, of order 2 with the same principal symbol as $\rho^2 \Box_{g_{M,a}}$.

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$$\begin{aligned} \mathrm{d}F &= 0 & \Psi_1 &:= F(l, m) \\ \mathrm{div} F &= 0 & \Psi_{-1} &:= (r - ia\cos\theta)^2 F(\overline{m}, n) \end{aligned}$$

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In coordinates:

$$T_s = \rho^2 \Box_{g_{M,a}} - \frac{2s(r-M)}{r^2 + a^2\cos^2\theta}\partial_r - 2s\left(\frac{a(r-M)}{\Delta_r} + \frac{i\cos\theta}{\sin^2\theta}\right)\partial_\phi - 2s\left(\frac{M(r^2-a^2)}{\Delta_r} - r - ia\cos\theta\right)\partial_t + \left(s^2\cot^2\theta - s\right)$$

Decay result

Theorem (M., 2023)

We consider a subextremal Kerr spacetime ($|a| < M$). We fix $s \in \frac{1}{2}\mathbb{Z}$. Let u_0, u_1 smooth and compactly supported on Σ_0. The solution u of the Cauchy problem

$$\begin{cases} T_s u = 0 \\ u(t=0) = u_0 \\ \frac{\partial}{\partial t} u(t=0) = u_1 \end{cases}$$

satisfies:

$$|u(r, \mathfrak{t}, \theta, \phi) - \mathfrak{p}_{u_0,u_1}(r, \mathfrak{t}, \theta, \phi)| \leq C r^{-1+} \mathfrak{t}^{-2-|s|+s-\epsilon} \left(\frac{\mathfrak{t}}{r} + 1\right)^{-1-s-|s|}$$

where $\epsilon > 0$.

Details

Related results on Teukolsky

Ma-Zhang ('21): case $|a| \ll M$, $s = \pm 1, \pm 2$.
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$$\mathfrak{p} = \mathfrak{t}^{-3-2|s|} \frac{(2|s|+2)\left(\frac{\mathfrak{t}}{r}\right)^{2+|s|+s} + 2(|s|-s+1)\left(\frac{\mathfrak{t}}{r}\right)^{1+|s|+s}}{\left(\frac{\mathfrak{t}}{r}+2\right)^{2+|s|+s}} F_{u_0,u_1}$$

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- We can assume H^N regularity with N large instead of smooth.
- We get a precise decay estimate (without leading order term) when initial data only have inverse polynomial decay.

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- Inverse Fourier transform:

$$v(\mathfrak{t}) = \int_{\mathbb{R}+iC} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\,\mathrm{d}\sigma$$

Formally

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4. Regularity of $R(\sigma)$ near zero. Principal term in the development $\Leftrightarrow$ highest order singularity (here at zero).

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$\|R(\sigma)\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \leq C$ when $|Re(\sigma)| \to +\infty$, $\Im(\sigma)$ bounded.

small parameter: $h = \frac{1}{|\sigma|}$, $\sigma = h^{-1}z$

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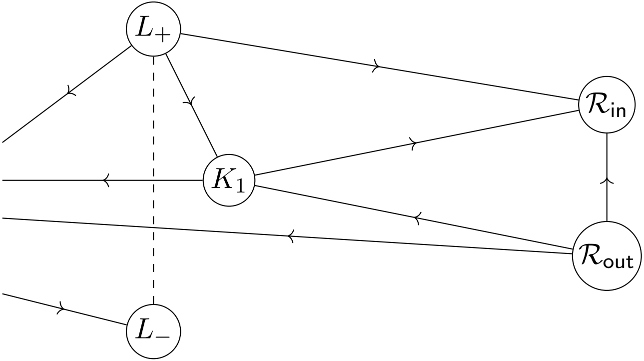
$u = \sum_{i=1}^{N} \chi_i(x, hD)u$, χ_i microlocalizers.

- Detailed study of the semiclassical Hamiltonian flow.



2: High frequency bound (continuation)

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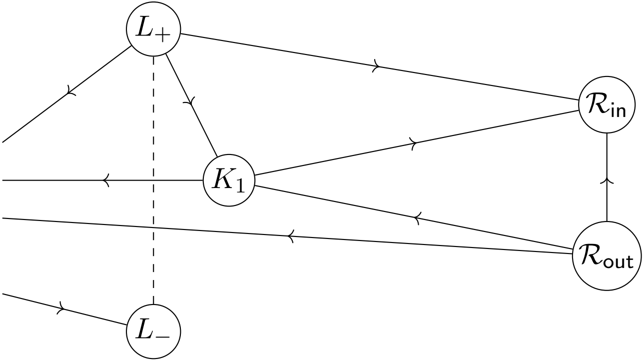
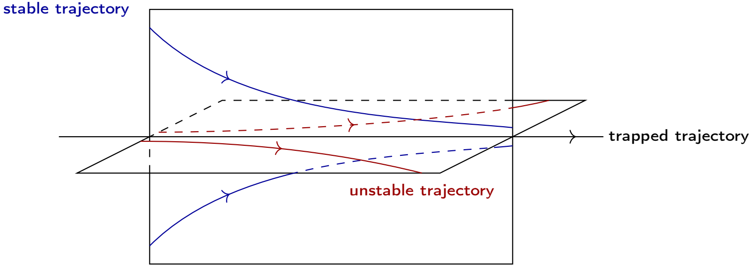


- Semiclassical version of the radial points estimates (Vasy '13, '19)

- Estimates at the normally hyperbolic trapped set (Wunsch-Zworski '14, Dyatlov '14)



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Thank you for your attention !