Entropic fluctuations in quantum two-time measurement framework

joint work in progress with T.Benoist, L.Bruneau, V.Jakšić, C.-A.Pillet

Annalisa Panati et Claude-Alain Pillet
CPT, Université de Toulon

16th meeting of GdR Quantum Dynamics , CY Advanced Studies, CY Cergy Paris Université, January 2024

Classical Non Equilibrium Statistical Mechanics

Classical System $(\mathcal{M}, \phi_t, \omega)$ dynamical system
ω reference state= probability measure on $\mathcal{M}$
$\omega_t = \omega \circ \phi_{-t}$

Relative entropy: $Ent(\nu|\mu) = -\nu(\log \frac{\mathrm{d}\nu}{\mathrm{d}\mu})$ (if ν and μ are absolutely continuous)

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Σ_t entropy production
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Σ_t can be viewed as a random variable on $(\mathcal{M},\ \omega)$
We denote its law by $\mathbb{P}_{\Sigma_t,\omega}$

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Statistical refinement of thermodynamics second law

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If $\mathbb{P}_{\Sigma_t,\omega}$ is the law of the random variable Σ_t on $(\mathcal{M},\omega)$ and $\bar{\mathbb{P}}_{\Sigma_t,\omega}(s) = \mathbb{P}_{\Sigma_t,\omega}(-s)$.
If ω if equilibrium state at temperature β, or a multi-thermal state and the system is T.R.I.

$$\frac{\mathrm{d}\bar{\mathbb{P}}_{\Sigma_t,\omega}}{\mathrm{d}\mathbb{P}_{\Sigma_t,\omega}}(s) = \mathrm{e}^{-s}$$

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$$\frac{\mathrm{d}\bar{\mathbb{P}}_{\Sigma_t,\omega}}{\mathrm{d}\mathbb{P}_{\Sigma_t,\omega}}(s) = \mathrm{e}^{-s}$$

This is called classical transient (ES) fluctuation relation

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$$\frac{\mathrm{d}\overline{\mathbb{P}}_{\Sigma_t,\omega}}{\mathrm{d}\mathbb{P}_{\Sigma_t,\omega}}(s) = \mathrm{e}^{-s}$$

equivalent to

$$\mathcal{F}^{cl}_{\omega,t}(\alpha) = \mathcal{F}^{cl}_{\omega,t}(1-\alpha)$$

with

$$\mathcal{F}^{cl}_{\omega,t}(\alpha) = \int e^{-\alpha s}\mathrm{d}\mathbb{P}_{\Sigma_t,\omega}(s) = \omega(e^{-\alpha\Sigma_t})$$

Classical non equilibrium statistical mechanics

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The related object cumulant generating function

$$e^{cl}_{\omega,t}(\alpha) = \frac{1}{t}\log \mathcal{F}^{cl}_{\omega,t}(\alpha)$$

is a well known object in large deviation theory.
Large time fluctuations can be described (via Gärtner-Ellis theorem) using properties of the limit

$$e^{cl}_{\omega,+}(\alpha) := \lim_{t\to\infty} e^{cl}_{\omega,t}(\alpha)$$

Classical Non Equilibrium Statistical Mechanics

Steady fluctuation relation

Classical steady(GC) fluctuation relation

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The initial state of the system is a *non-equilibrium steady state (NESS)* $\omega_{NESS} = \lim_{T\to\infty} \omega \circ \phi_{-T}$ (existence assumed).

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- for $t \to \infty$ the following is expected to hold under strong ergodic hypothesis

$$e^{cl}_{\omega_{NESS},+}(\alpha) = e^{cl}_{\omega,+}(\alpha)$$

proven in some (paradigmatic) models ; in [JPR11] advocated as Principle of Regular Entropic Fluctuation

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- *for t finite* $\mathcal{F}^{cl}_{T,t}(\alpha) \neq \mathcal{F}^{cl}_{\omega,t}(\alpha)$ unless $\omega = \omega_T$
- for $t \to \infty$ one has in many cases (for example on compact spaces)

$$e^{cl}_{\omega_T,+}(\alpha) := \lim_{t\to\infty} e^{cl}_{T,t}(\alpha) = e^{cl}_{\omega,+}(\alpha)$$

Principle of Regular Entropic Fluctuation equivalent to a limit exchange property

Quantization : transient fluctuation relation

Quantum case ?? Transient case

Attempt 1: "Naive quantization"

Mathematical setting: Reservoirs $\mathcal{R}_1, \mathcal{R}_2, \dots \mathcal{R}_M$ coupled directly or through a small system $\mathcal{S}$, $\dim \mathcal{H}_\mathcal{S} = N$

Notation

Reservoirs $\mathcal{R}_1, \mathcal{R}_2, \dots \mathcal{R}_M$
$(\mathcal{O}_j, \tau_{\mathcal{R}_j,t}, \omega_{\beta_j})$, where ω_{β_j} is β_j KMS state for τ_t^j

Small system $\mathcal{S}$, $\dim \mathcal{H}_\mathcal{S} = N$
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Perturbed dynamics: $\tau_t(A)$ with generator $\delta_\omega = \delta_\omega + i[-, V]$

Annalisa Panati et Claude-Alain Pillet
CPT, Université de Toulon

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—attempted in work related litterature [Bochkov-Kuzovlev '70s-'80s])
—attempted in the '90, called "naive quantization"

Underlying idea in simplified setting : $\Sigma_t = -\beta H_{0,t} + \beta H_0 = S_t - S$ and consider its spectral measure on $\mathcal{H}$ and consider the spectral measure μ_{Σ_t}

From previous lecture:

$$\Sigma_t \text{ corresponds to } Ent(\omega_t|\omega) - Ent(\omega|\omega)$$

In the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated to ω;

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Remark: Σ_t is constructed as difference at time t and at time 0 of a given observable.

Quantization: transient fluctuation relation

Attempt 2:

Mathematical point of view: Phase space contraction

What is relevant is

$$\Sigma_t \text{ corresponds to } Ent(\omega_t|\omega)$$

In a GNS representation

$$\mathcal{F}_{\omega,t}(\alpha) = (\Omega_\omega, e^{-\alpha \log \Delta_{\omega_{-t}|\omega}}\Omega_\omega) = (\Omega_\omega, \Delta^{-\alpha}_{\omega_{-t}|\omega}\Omega_\omega)$$

In terms of cocycle

$$\mathcal{F}_{\omega,t}(\alpha) = \omega([D_{\omega_{-t}} : D_\omega]^\alpha)$$

leads to *fluctuation relation*!!!!!

Let $\mathbb{P}_{\omega,t}$ be such that

$$\mathcal{F}_{\omega,t}(\alpha) = \int e^{-\alpha s} \mathrm{d}\mathbb{P}_{\omega,t}.$$

What does $\mathbb{P}_{\omega,t}$ means?

Quantization: transient fluctuation relation

Attempt 2:

Physical point of view [Kurchan'00] Measurement has been neglected. Associate to S the *two-time measurement statistics* $\mathbb{P}_{\omega,t}$ defined as difference between two measurements

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this approach leads to *fluctuation relation*

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Physical point of view [Kurchan'00] Measurement has been neglected. Associate to S the *two-time measurement statistics* $\mathbb{P}_{\omega,t}$ defined as difference between two measurements
this approach leads to *fluctuation relation*

At the level of averages and variances, there is no difference!

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The success of TTM come with a price: unexpected phenomena (with no classical countepart) due to the invasive role of measurement [Benoist-P.Raquépas19, Benoist-P.Pautrat 20] and *now*

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Two-time measurement statistics

Full (Counting) Statistics [Lesovik, Levitov '93][Levitov, Lee,Lesovik '96]

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Confined systems: described by $(\mathcal{H}, H, \rho)$ $dim\mathcal{H} < \infty$
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Protocol:
- $t = 0$, we measure A (outcome a_j)
- evolve for time t
- measure again at time t (outcome a_k)

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$\mathbb{P}_{A,t}(\phi)$= probability of measuring a change in A equal to ϕ.

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Fact/Problem: *the measurement perturbes the state, the initial state reduces to* ρ_{am}

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Two-time measurement statistics -thermodynamic limit

This defines $\mathcal{F}^{(n)}_{\omega,t}(\alpha) = \int e^{-\alpha s} \mathrm{d}\mathbb{P}^{(n)}_{S,t}(s)$.

Theorem

It is possible to rewrite $\mathcal{F}^{(n)}_{\omega,t}(\alpha)$ in term of algebraic objects that survive the limit $n \to \infty$. Under standard hypothesis, $\lim_{(n)\to\infty} \mathcal{F}^{(n)}_{\omega,t}(\alpha) =: \mathcal{F}_{\omega,t}(\alpha)$ is well defined and correspond to the same formal expression.

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And that expression is same as before!!!

$$\mathcal{F}_{\omega,t}(\alpha) = (\Omega_\omega, e^{-\alpha \log \Delta_{\omega_{-t}|\omega}}\Omega_\omega) = (\Omega_\omega, \Delta^{-\alpha}_{\omega_{-t}|\omega}\Omega_\omega)$$

In other words

$\mathcal{F}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{ttm}}_{\omega,t}(\alpha)$ obtained though two-time measurement protocol

$\mathcal{F}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{psc}}_{\omega,t}(\alpha)$ obtained in the spirit of quantum phase space contraction

Quantum steady entropic functional: our proposal

How to quantize steady FR? Difficulties:

- A measurement would destroy the steady state
- steady state exists only in the thermodynamic limit
- in the thermodynamic limit, non-normality of the steady state (to the initial state)

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Two time measurement framework (start with finite dimensional approximation $\dim \mathcal{H} = n$)
– start with ω initial state as in the transient case
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– start with ω initial state as in the transient case
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– let the system evolve for time t
– perform the measurement at an unknown very large time $T + t$
This defines

$$\mathbb{P}^{(n)}_{T,t}, \quad \mathcal{F}^{(n)}_{\omega_T,t}(\alpha) = \int e^{-t\alpha s} \mathrm{d}\mathbb{P}^{(n)}_{T,t}(s)$$

Quantum steady entropic functional: our proposal

Proposition (Thermodynamic limit (T.Benoist, L.Bruneau, V.Jakšić, A.P., C.-A.Pillet '23))

It is possible to rewrite $\mathcal{F}^{(n)}_{\omega_T,t}(\alpha)$ in term of algebraic objects that survive the limit. Under standard hypothesis, $\mathcal{F}_{\omega_T,t}(\alpha) := \lim_{n\to\infty} \mathcal{F}^{(n)}_{\omega_T,t}(\alpha)$ *is well defined correspond to the same formal expression.*

$$\mathcal{F}_{\omega_T,t}(\alpha) = \lim_{R\to\infty} \int_0^R \omega_T(\zeta_\theta[D_{\omega_{-t}} : D_\omega]^\alpha)\mathrm{d}\theta$$

with $\zeta_\theta(-)$ modular dynamic associated to ω (free dynamics up to a temperature time scale factor)

Recall:

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Definition

$$\mathcal{F}_{\omega_{NESS},t}(\alpha) := \lim_{T\to\infty} \mathcal{F}_{\omega_T,t}(\alpha)$$

Quantum steady entropic functional: our proposal: results

Theorem (direct coupling, no S, T.Benoist, L.Bruneau, V.Jakšić, A.P., C.-A.Pillet '23)

Let assume the reservoirs are coupled directly (no S); assume the dynamics $\tau^0_{\mathcal{R},t}$ is ergodic. Then

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Consequences:

- $\mathcal{F}_{\omega_{NESS},t}(\alpha) = \mathcal{F}_{\omega,t}(\alpha)$
- If the symmetry true for $\mathcal{F}_{\omega,t}(\alpha)$, then for the steady functional $\mathcal{F}_{\omega_{NESS},t}(\alpha)$ also satisfy the symmetry AT FINITE TIME t;
- if $e_{\omega_{NESS},+}(\alpha), e_{\omega,+}(\alpha)$ exist, they are equal.

Quantum steady entropic functional: our proposal: results

Theorem (coupling through $\mathcal{S}$,
T.Benoist, L.Bruneau, V.Jakšić, A.P., C.-A.Pillet '23)

Let assume the reservoirs are through a small system $\mathcal{S}$; assume the dynamics $\tau^0_{\mathcal{R},t}$ is ergodic. Then

$$e_{\omega_{NESS},+}(\alpha) = e_{\omega,+}(\alpha)$$

Remark

In both theorems:

1. *No additional hypothesis on the perturbed dynamics; strong coupling allowed.*
2. *Underlying mechanism: invasive measurement role; memory erasing effect of first measurement; stability of the fluctuation*
3. *General proof, with algebraic methods (no need for resonance analysis model by model)*
4. *We are able to cope with NESS in the thermodynamic limit.*

Proof in abstract terms

Consider $(\mathcal{O}, \tau_t, \omega)$ with ω a τ_t invariant.
$(\mathcal{O}, \tau_t, \omega)$ is ergodic iff for any ω normal state ν and any $A \in \mathcal{O}$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \nu(\tau_t(A))\mathrm{d}t = \omega(A)$$

Apply this property to

$$\mathcal{F}_{\omega_T,t}(\alpha) = \lim_{R\to\infty} \frac{1}{R} \int_0^R \omega_T(\zeta_\theta[D_{\omega_{-t}} : D_\omega]^\alpha)\mathrm{d}\theta$$

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$$\mathcal{F}_{\omega_T,t}(\alpha) = \lim_{A\to\infty}\frac{1}{A}\int_0^A (\Omega_T, e^{i\theta L_0}\Delta^\alpha_{\omega_t|\omega}e^{-i\theta L_0}\Omega_T)\mathrm{d}\theta$$
$$= (\Omega, e^{iTL_\infty}1_{\{0\}}(-\beta L_0)e^{-itL_\alpha}\Omega)$$

Direct coupling: If the dynamics on $\mathcal{R}$ is ergodic, 0 is a simple eigenvalue for L_0 and $1_{\{0\}}(L_0) = |\Omega\rangle\langle\Omega|$

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Coupling through a small system $\mathcal{S}$: If the dynamics on $\mathcal{R}$ is ergodic, 0 is a simple eigenvalue for L_0 and $\ker(L) = \ker(L_\mathcal{S}) \otimes \Omega_\mathcal{R}$; equality is attained in in the long time limit $t \to \infty$

Summing up so far

- we have introduced classical entropic functional and we have review the quantization in the transient case
- we have introduced a proposal for quantum steady (GC) entropic functional $\mathcal{F}_{\omega_{NESS},t}(\alpha)$
- we have shown a "memory erasing effect"/ stability due to the first measurement ; this can be interpreted as stability in the quantum case of the fluctuations
- we have shown $e_{\omega_{NESS},+}(\alpha) = e_{\omega,+}(\alpha)$ under very minimal ergodicity hypothesis
- direct measurement on infinitely extended or (very large) reservoir is an idealization. Are we able to write similar (less general) results ins the framework of an indirect measurement using resonance theory ? YES, next subject.

Indirect Measurment: Ancilla State Tomography

The Entropic Ancilla State Tomography (EAST)

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ancilla = spin 1/2
$(\mathfrak{H}_a, H_a, \rho_a)$ with $\mathfrak{H}_a = \mathbb{C}^2$
Denote by $v_\pm$ the eigenvectors of the Pauli matrix σ_z associated to the eigenvalues ± 1. ρ_a a density matrix such that $\langle v_+, \rho_a v_- \rangle \neq 0$.

Consider the extended system
$\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2$
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We introduce family of Hamiltonians (for each α)
$\widehat{H}_\alpha = \mathrm{e}^{\frac{\alpha}{2} \log \omega \otimes \sigma_z} (H \otimes \mathbb{1}) \, \mathrm{e}^{-\frac{\alpha}{2} \log \omega \otimes \sigma_z}$

$P := |v_-\rangle\langle v_+|$, a simple computation gives

$$\mathrm{tr}(\mathrm{e}^{-\mathrm{i}t\widehat{H}_\alpha} \widehat{\nu} \mathrm{e}^{\mathrm{i}t\widehat{H}_\alpha} \mathbb{1} \otimes P) = \langle v_+, \rho_a v_- \rangle \mathcal{F}_{\omega,t}(\alpha)$$

Indirect Measurement: Ancilla State Tomography

In the general case:
$\widehat{\mathcal{O}} := \mathcal{O} \otimes \mathcal{O}_{\mathbb{C}^2}$
$\widehat{\tau}^t = \tau^t \otimes \mathrm{id}$ $\quad \widehat{\tau}^t_\alpha$ perturbed dynamics corresponding to $\widehat{H}_\alpha$

$$\widehat{\omega}(\widehat{\tau}^t_\alpha(\mathbb{1} \otimes P)) = \langle v_+, \rho_a v_- \rangle \mathcal{F}^{\mathrm{ancilla}}_{\omega,t}(\alpha).$$

$$\mathcal{F}^{\mathrm{ancilla}}_{\omega,t}(\alpha) := \omega\left(([D\omega_{-t} : D\omega])^{\frac{\bar{\alpha}*}{2}} [D\omega_{-t} : D\omega])^{\frac{\alpha}{2}}\right)$$
$$= \omega\left([D\omega_{-t} : D\omega]^{\alpha})\right) = \mathcal{F}_{\omega,t}(\alpha)$$

Indirect Measurment: Ancilla State Tomography

If the intial state is ω the multi-thermal state

$$\mathcal{F}^{\mathrm{ttm}}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{ancilla}}_{\omega,t}(\alpha)$$

But for general ν, in particular if $\nu = \omega_T$ the functionals

$$\mathcal{F}_{\omega_T,t}(\alpha) \neq \mathcal{F}^{\mathrm{ancilla}}_{\omega_T,t}(\alpha)$$

This indirect measurement protocol is not "invariant" if with respect of the time T.

Is the protocol "stable"? Meaning that in the limit $t \to +\infty$ the corresponding cumulant generating function will be the same?

Indirect Measurment: Ancilla State Tomography

Answer is : YES! .... but not in general

For ω_T finite, true under additional hypothesis

For ω_{NESS} expected under additional chaotic property of the dynamics, we can prove it for Spin-Fermion/boson model. One can argue for Ancilla ST, the situation is parallel to the classical case.

Annalisa Panati et Claude-Alain Pillet
CPT, Université de Toulon

Indirect Measurment: Ancilla State Tomography, results

General results:

(AnV) For some $\vartheta > 0$ the map

$$\mathbb{R} \ni s \mapsto \varsigma_\omega^\theta(V) \in \mathcal{O}$$

has an analytic extension to the complex strip
$\mathfrak{S}(\vartheta) = \{z \in \mathbb{C} | \operatorname{Re} z - 1\backslash 2| < \vartheta\}$.

Theorem (T.Benoist, L.Bruneau, V.Jakšić, A.P., C.-A.Pillet '24)

Suppose that **(AnV)** *holds with* $\theta > \frac{1}{2}$. *Set*

$$C_T := \mathrm{e}^{|T|(\|\varsigma_\omega^{-\mathrm{i}\frac{1}{2}}(V)\|+\|V\|)} \quad D_T := \mathrm{e}^{-|T|(\|\varsigma_\omega^{\mathrm{i}\frac{1}{2}}(V)\|+\|V\|)}.$$

Then for any $\alpha \in (-\vartheta, \vartheta)$,

$$D_T \mathcal{F}_{\omega,t}(\alpha) \leq \mathcal{F}^{\mathrm{ancilla}}_{\omega_T,t}(\alpha) \leq C_T \mathcal{F}_{\omega,t}(\alpha).$$

Consequence (stability of the measurement protocol for finite T):

$$e_{\omega,+}(\alpha) = e^{\mathrm{ancilla}}_{\omega_T,+}(\alpha)$$

Indirect Measurment: Ancilla State Tomography: results

Model dependent results

Theorem (T.Benoist, L.Bruneau, V.Jakšić, A.P., C.-A.Pillet '24)

For the Spin-Fermion. Suppose that **(AnV)** *holds with* $\theta > \frac{1}{2}$ *and some additional regularity hypothesis. Then for any* $\alpha \in (-\vartheta, \vartheta)$*,*

$$e_{\omega_{NESS},t}(\alpha) = e_{\omega,t}(\alpha).$$

A comment on entropic functionals at $T > 0$

For $T = 0$, i.e. with respect to ω

$$\mathcal{F}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{psc}}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{ttm}}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{ancilla}}_{\omega,t}(\alpha)$$

$$\mathcal{F}^{\mathrm{psc}}_{\omega,t}(\alpha) = \mathcal{F}^{\mathrm{ttm}}_{\omega,t}(\alpha) = \omega\left(([D\omega_{-t} : D\omega])^{\alpha}\right)$$

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For $T > 0$, i.e. with respect to ω_T

$$\mathcal{F}^{\mathrm{psc}}_{\omega_T,t}(\alpha) \neq \mathcal{F}^{\mathrm{ttm}}_{\omega_T,t}(\alpha) \neq \mathcal{F}^{\mathrm{ancilla}}_{\omega_T,t}(\alpha)$$

$$\mathcal{F}^{ttm}_{\omega_T,t}(\alpha) = \lim_{R\to\infty}\int_0^R \omega_T(\zeta_\theta[D_{\omega_{-t}} : D_\omega]^{\alpha})\mathrm{d}\theta$$

$$\mathcal{F}^{ancilla}_{\omega_T,t}(\alpha) = \omega_T(([D\omega_{-t} : D\omega])^{\frac{\bar{\alpha}*}{2}}[D\omega_{-t} : D\omega])^{\frac{\alpha}{2}})$$

$$\mathcal{F}^{psc}_{\omega_T,t}(\alpha) = \omega_T([D_{\omega_{-t}} : D_\omega]^{\alpha})$$

A comment on entropic functionals at $T > 0$

Equality are restored in the large time limit under some conditions

If reservoir are ergodic and (AnV) is verified

$$e_{\omega,+}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}^{\mathrm{ttm}}_{\omega_T,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}^{\mathrm{ancilla}}_{\omega_T,t}(\alpha)$$

Thank you for your attention