THE BOUNDARY OF LINEAR SUBVARIETIES
SCHOOL ON FLAT SURFACES AND INTERACTIONS - BORDEAUX 2024

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1. Strata, orbit cloures and period coordinates

The goal of these lecture notes is to explain recent developments on compactifications of orbit closures on strata of differentials. In this section we recall the necessary background before motivating the results on compactifications.

Let $g \geq 0$ be a genus and $\mu = (\mu_1, \ldots, \mu_n)$ be an integer partition of $2g-2$ (we allow negative entries). The stratum

$$\mathcal{H} = \left\{ (X, \omega) \,|\, X \text{ Riemann surface of genus } g, (\omega) = \sum_i \mu_i p_i \right\} /(\text{ iso.})$$

is a moduli space for Riemann surfaces and differentials with a fixed multiplicity of zeros.

Remark 1.1. A holomorphic differential form on K_X is the same as a global section of the holomorphic cotangent bundle K_X. The space of holomorphic 1-forms $H^0(X, K_X)$ is a g-dimensional complex vector space. Counted with multiplicity a holomorphic differential form always has $2g(X) - 2$ zeros.

Similarly, a meromorphic differential ω with poles of order m_i at p_i is a holomorphic section of $K_X(\sum_i m_i p_i)$, which has a $(g + \sum_i m_i - 1)$-dimensional space of global section. (Hint:Use Riemann-Roch).

1.1. **Equivalent points of view.** In the sequel we will mostly work complex-analytically but at various points it will be useful to keep the flat geometric view point in mind. For a holomorphic differential, the datum (X, ω) is equivalent to a *flat surface*, i.e. a compact complex manifold X of dimension 1 and a finite set of points Σ such that $X - \Sigma$ is equipped with an atlas of charts $\phi_i : U_i \to \mathbb{C}$ such that the transition functions $\phi_i \circ \phi_j^{-1}$ are given by linear functions, i.e.

$$\phi_i \circ \phi_j^{-1}(z) = z + c, c \in \mathbb{C}.$$

In the charts U_i the differential form ω can be written as

$$\omega = dz.$$

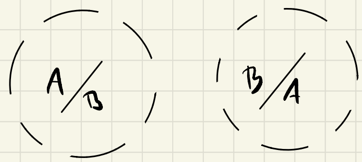
Since translations respect the flat metric $dz \wedge d\bar{z}$, the manifold $X - \Sigma$ admits a flat metric (i.e. a Riemannian metric with curvature zero).

Any translation surface can be represented by the following data: A polygon $P \subseteq \mathbb{C}$ together with gluing maps $z \mapsto z + c$ for the edges of the polygon. Different polygons can represent isomorphic translation surfaces, if one is obtained from another by a sequence of cutting and gluing operations.

"Meromorphic flat surfaces". In the course of these lecture notes we will often encounter "flat surfaces" corresponding to meromorphic differentials. These pictures might look strange at first glance, since they consist of "non-compact" polygons. The following pictures give examples of the variety of flat surfaces that one can encounter.

Meromorphic flat surfaces

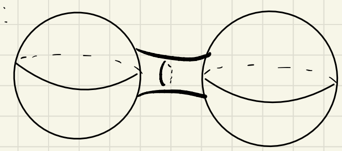
1)



$\in \mathcal{H}(1,1,-2,-2)$

copy of $\mathbb{C}$ with differential dz

topologically:



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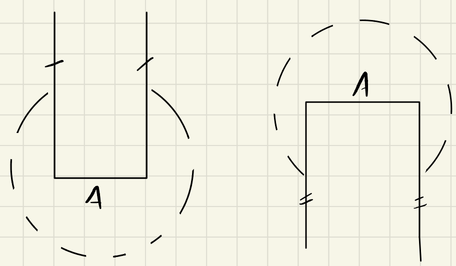
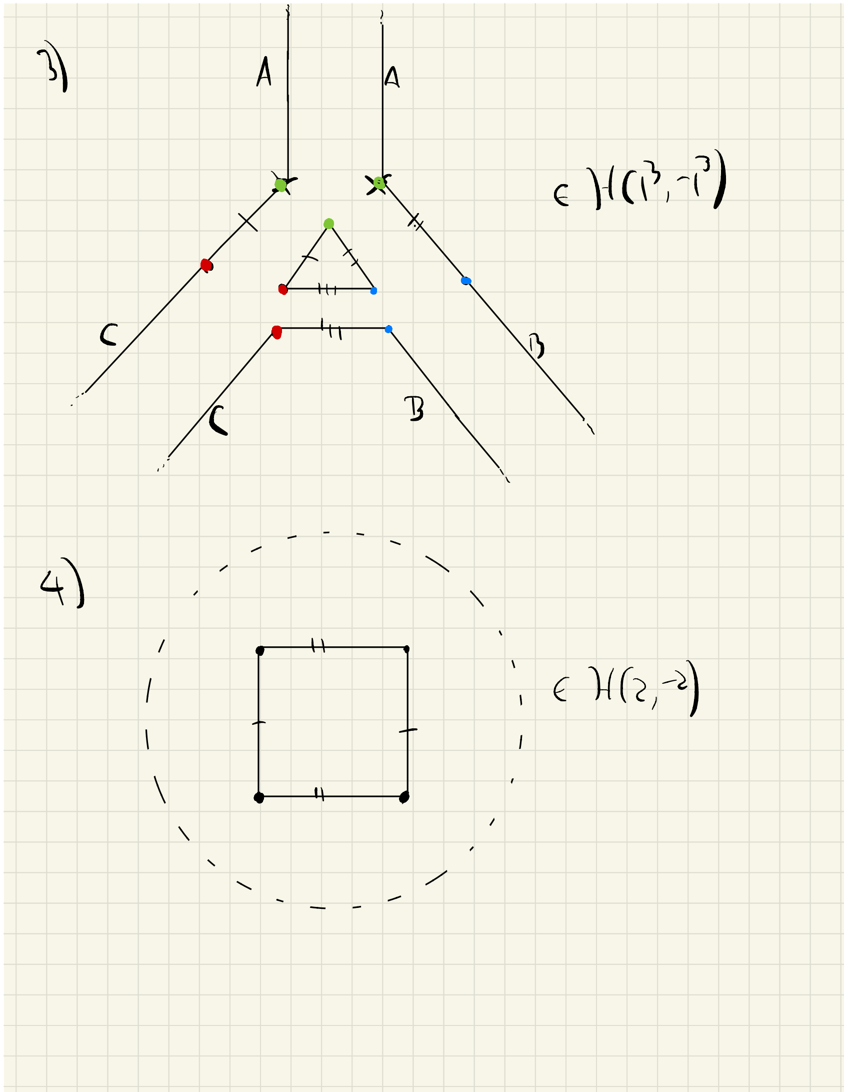


$\in \mathcal{H}(2,-2,-2)$

Here the residue around the two poles is non-zero



$\mathrm{GL}(2,\mathbb{R})$-action. The group $\mathrm{GL}(2,\mathbb{R})$ acts on the stratum $\mathcal{H}$ as follows. Given (X,ω), represent the flat surface as a polygon in the plane, with pairs of sides identified via a translation. An element $g \in \mathrm{GL}(2,\mathbb{R})$ acts linearly on $\mathbb{C} \simeq \mathbb{R}^2$ and keeps parallel sides parallel. Thus, the image $g \cdot P$ is again a polygon and we keep the same edge identification as for P.

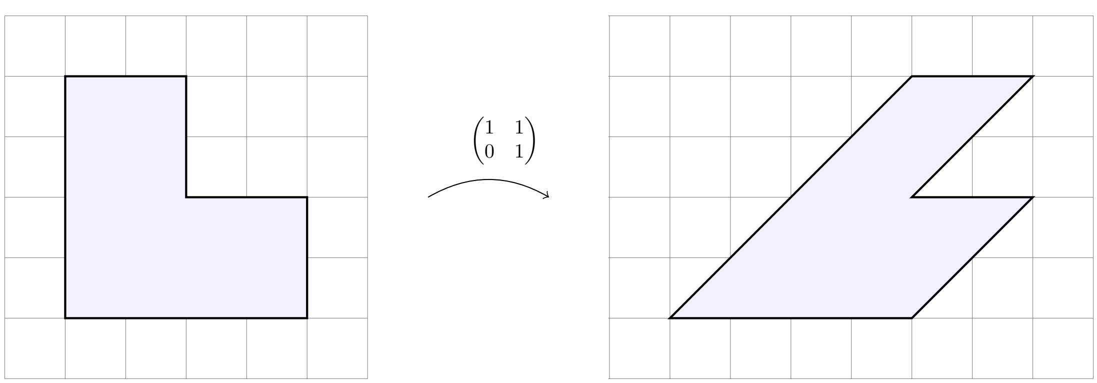


FIGURE 1. $\mathrm{GL}(2,\mathbb{R})$ acting on a translation surface

1.2. **Period coordinates.** Using the representation of translation surfaces as polygons in the plane, one can describe a natural coordinate system on $\mathcal{H}$. Let $(X, \omega) \in \mathcal{H}$ and choose a polygonal representation P. Every edge e of P either connects two different zeros of ω or is a closed loop. Thus we can consider e as an element of the relative homology group

$$H_1(X, \omega; \mathbb{Z}) := H_1(X - P(\omega), Z(\omega)); \mathbb{Z}).$$

In fact, the edges of P form a generating set. The length of e as a vector in $\mathbb{C}$ agrees with the integral

$$\int_e \omega.$$

By deforming the edges of the polygon P sligthly (and thus the complex numbers $\int_e \omega$), one can describe a neighborhood of (X, ω) in $\mathcal{H}$.

Theorem 1.2. *Let $(X, \omega) \in \mathcal{H}$. There exists a contractible neighborhood $U \subseteq \mathcal{H}$ containing (X, ω) such that the map*

$$U \to H^1(X - P(\omega), Z(\omega); \mathbb{C}), (X, \omega) \mapsto (\alpha \mapsto \int_\alpha \omega)$$

is a biholomorphism onto its image. By choosing a basis $\alpha_1, \ldots, \alpha_n$ one obtains a local biholomorphism

$$U \to \mathbb{C}^n, (X, \omega) \mapsto \left(\int_{\alpha_1} \omega, \ldots, \int_{\alpha_n} \omega \right).$$

Remark 1.3. Implicit in this theorem is a identification of homology groups

$$H^1(X - P(\omega), Z(\omega); \mathbb{Z}) \simeq H^1(X' - P(\omega'), Z(\omega'); \mathbb{Z})$$

for different points $(X, \omega), (X', \omega') \in U$. This can be achieved as follows. Over U there exists a complex manifold $\mathcal{C} \to U$ such that the fiber over (X, ω) is isomorphic to the curve X together with sections $\sigma_i : U \to \mathcal{C}$ such that the image $\sigma_i(X, \omega)$ is the i-th zero or pole of ω. The map $\mathcal{C} \to U$ is a proper submersion, thus by the Ehresmann fibration theorem the map $\mathcal{C} \to U$ is C^∞- fiber bundle. In particular, different fibers of $\mathcal{C} \to U$ are diffeomorphic. Furthermore, one can choose the diffeomorphisms so that they preserve the images of the section σ_i. The diffeomorphisms are not unique but different choices are homotopic, thus the induced map on homologies are the same.

A contractible open set U as in the theorem, is called a *period chart* and resulting local coordinates $\phi : U \to \mathbb{C}^n$ are called *period coordinates.*

The structure of $\mathrm{GL}(2, \mathbb{R})-$**orbit closusres.** The following foundational result describes the local and global structure of orbit closure for the $\mathrm{GL}(2, \mathbb{R})$-action.

Theorem 1.4 (Eskin-Mirzakhani-Mohammadi, Filip)**.** *Let $N = \overline{\mathrm{GL}(2, \mathbb{R}) \cdot (X, \omega)}$ be a $\mathrm{GL}(2, \mathbb{R})$-orbit closure. Then N is an algebraic subvariety of $\mathcal{H}$, which locally in period coordinates, is defined by linear equation with real coefficients.*

In the sequel we will work with the following, slightly more general, class of algebraic varieties.

Definition 1.5. *A linear subvariety $N \subseteq \mathcal{H}$ is an algebraic subvariety, which is defined by linear equations in local period coordinates.*

In the above definition the linear equations are allowed to have complex coefficients. If the linear equations can be chosen with real coefficients only, we say that N is $\mathbb{R}$-linear. As a consequence of Theorem 1.4, any orbit closure is an $\mathbb{R}$-linear subvariety. And conversely, any irreducible $\mathbb{R}$-linear subvariety is an orbit closure.

Motivation for studying closures of linear subvarieties. One of the big open problems in Teichmüller theory is the classification of $\mathbb{R}$-linear subvarieties.

Let N be an $\mathbb{R}$-linear subvariety. Then N is not compact, since once can use the $\mathrm{GL}(2,\mathbb{R})-$action to degenerate the surface. Thus the following gives a potential approach to classifying orbit closures.

(1) Construct a suitable compactification $\overline{\mathcal{H}}$ of the stratum, such that the boundary ∂H "looks like a stratum", i.e. still has period coordinates and a $\mathrm{GL}(2,\mathbb{R})$-action
(2) Take the boundary $\partial N = \overline{N} - N$ inside $\overline{\mathcal{H}}$
(3) For a suitable compactification, one can hope that the boundary ∂N of N is again $\mathrm{GL}(2,\mathbb{R})$-invariant and given by real linear equations in period coordinates
(4) The dimension of ∂N is smaller than N. So now one can proceed recursively and keep reducing the dimension of the $\mathbb{R}$-linear subvarieties

One possible "compactification" where such arguments can be carried out is the WYSIWYG partial compactification $\overline{\mathcal{H}}^{WY}$ developed by Mirzakhani-Wright [MW15] and Chen-Wright [CW19]. The WYSIWYG partial compactification $\overline{\mathcal{H}}^{WY}$ is a flat geometric construction, but not algebraic. The recursive approach has been used by Apisa-Wright [AW23] to classify orbit closures of large rank.

The goal of these lecture notes is to describe an alternative, algebraic compactification $\overline{\mathcal{H}}$, where the boundary of $\mathbb{R}$-linear subvarieties have (almost) all the desired properties to allow a recursive approach. In the end this will recover the results about the WYSIWYG compactification.

2. An extended example

To get an idea of how the boundary of an orbit closure looks like, we will discuss a simple example of how flat surfaces in an orbit closure can degenerate to the boundary. This is an informal discussion, which hopefully helps to build the intuition. In particular, we will not define thoroughly what it means to degenerate or for a sequence of flat surfaces to converge.

Consider the stratum $\mathcal{H} = \mathcal{H}(1,1)$. An example of a surface in $\mathcal{H}$ can be found using the slit torus construction as follows. Consider two elliptic curves E_1 and E_2 and choose two slits of the same length wich are identified by a translation. Topologically, the resulting surface is a connected sum of two tori.

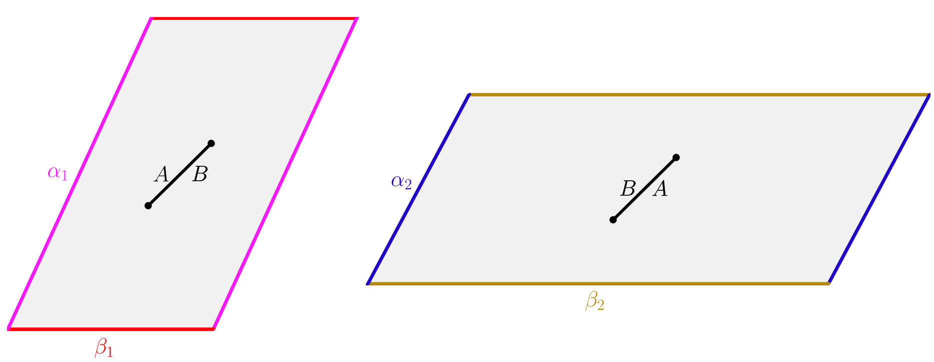


Figure 2. A flat surface in $\mathcal{H}(1,1)$

Define the subvariety

$$N = \{(X,\omega) \in \mathcal{H} \,|\, \exists\, f : X \xrightarrow{2:1} E, g(E) = 1, f^*dz = \omega\} \subseteq \mathcal{H}.$$

consisting of double covers of an elliptic curve E, where the differential is the pullback of the unique (up to scaling) holomorphic one-form on E.

For example, one can find an element in N by forcing both elliptic curves E_1 and E_2 in Figure 2 to be the same. The resulting double covering is then obtained just by projecting onto one parallelogram.

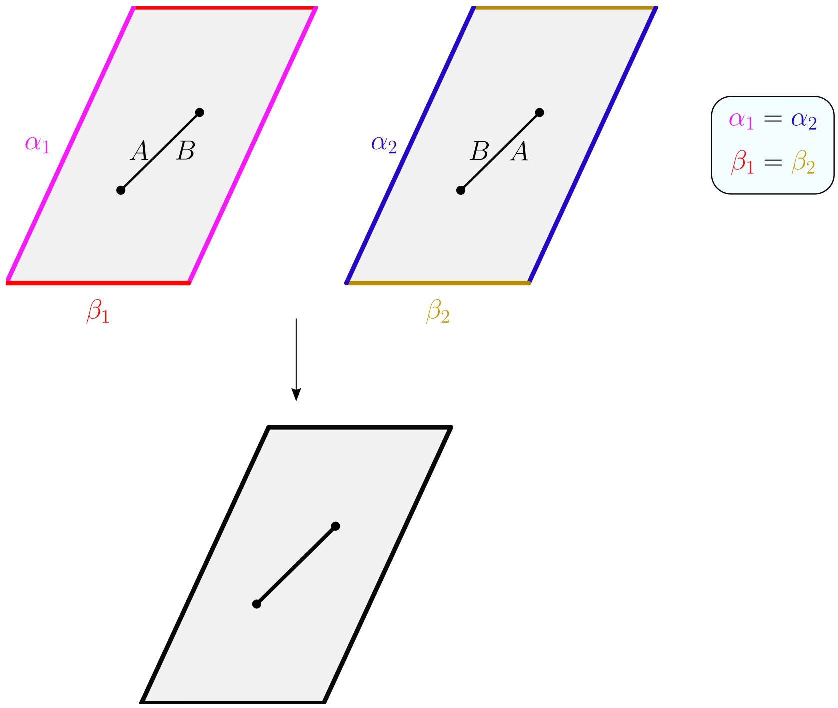


FIGURE 3. A flat surface in $\mathcal{H}(1,1)$

We claim that N is a $\mathbb{R}$-linear subvariety of dimension 3, locally defined by the condition

$$\omega \in f^* H^1(E, p, q; \mathbb{C}),$$

where p, q are the two branch points of $f : X \to E$. Equivalently, for any two paths α, β in $H_1(X, Z(\omega); \mathbb{Z})$ such that

$$f_*\alpha = f_*\beta,$$

we have

$$\int_\alpha \omega = \int_\alpha f^* dz = \int_{f_*\alpha} dz = \int_{f_*\beta} dz = \int_\beta f^* dz \int_\beta \omega.$$

Thus in local period coordinates, N is contained in the union of all subspace $f^* H^1(E, p, q; \mathbb{C})$. Note that these spaces are defined over $\mathbb{Z}$, in particular there are only countably many such subspaces in period coordinates.

We can be more precise about the linear equations. Choose a relative homology basis $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma$ such that α_i forms a symplectic homology basis for N and γ is represented by the slit. Then the condition

$$\int_{\alpha_1} \omega = \int_{\alpha_2} \omega, \int_{\beta_1} \omega = \int_{\beta_2} \omega$$

defines a 3-dimensional $\mathbb{Q}$-subspace containing (X, ω).

Exercise 2.1. N is a 3-dimensional $\mathrm{GL}(2, \mathbb{R})$-invariant subvariety of $\mathcal{H}$.

The following criterion for linearity can be useful:

Proposition 2.2. *Suppose N is an irreducible, algebraic variety of dimension at least d. Suppose there exists a number field k such that for any point $(X, \omega) \in N$ there exists a linear subspace $V \subseteq H^1(X, Z(\omega); \mathbb{C})$ defined over k and of dimension at most d, such that*

$$[\omega] \in V.$$

Then N is a linear subvariety, defined by k-linear equations.

Proof. (Sketch) N is contained in the union of all countable many subspaces defined over k. In particular, N has dimension at most d. Furthermore, N has locally only finitely many irreducible components, so it agrees with a finite union of such. ■

For a more detailed proof, see [MMW17, Thm. 5.1].

We now describe several ways of degenerating surfaces in N.

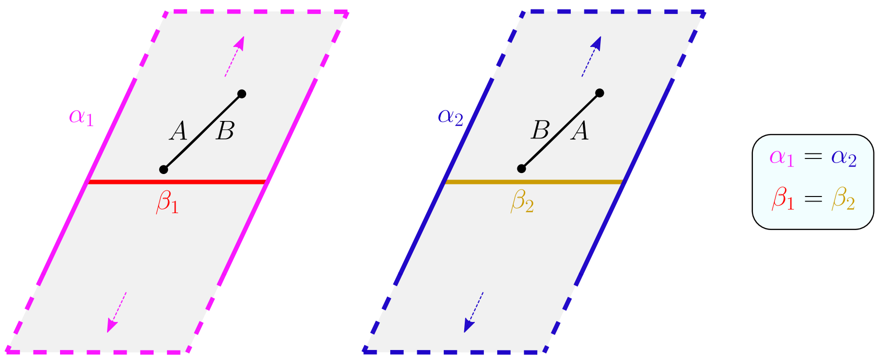


FIGURE 4. Degenerating the surface in N by stretching cylinders

1. Case: Stretch the periods over α_1 and α_2 to ∞. The flat picture is that of two infinite cylinders (where for each cylinder the points at infinity are glued together to form a nodal Riemann surface) glued together along a slit.

Exercise 2.3. (Local model for simple poles) Let $C := \{z \in \mathbb{C} \,|\, 0 \leq \mathrm{Re}(z) \leq 1$ equipped with the differential form dz. Consider the corresponding flat surfaces obtained by identifying both sides of the strip via $z \mapsto z + 1$. Show that the differential acquires a simple pole at the points at infinity with residue ± 1.

The limit X of this degeneration is a meromorphic differential in the stratum $\mathcal{H}(1^2, -1^4)$. If we draw the limits of α_i on the nodal surface X, then α_i crosses the node and we can not interpret the integral of the limit differential over α_i as a period on X. On the other hand, the linear equation

$$\int_{\beta_1} \omega = \int_{\beta_2} \omega$$

persists in the limit and it becomes a linear equation on the residues of the limiting differential with simple poles.

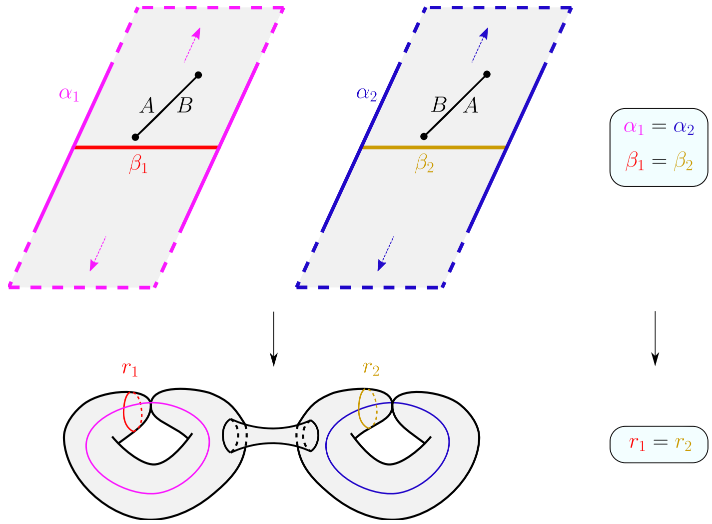


FIGURE 5. The limit surface after stretching α_i

Upshot: The limit differential (X, ω) is a nodal surface with a meromorphic differential with simple poles at the nodes. The periods of (X, ω) satisfy linear equations among periods, which are obtained by taking limits of periods. Linear equations crossing nodes with simple poles disappear. We will see that this happens in general.

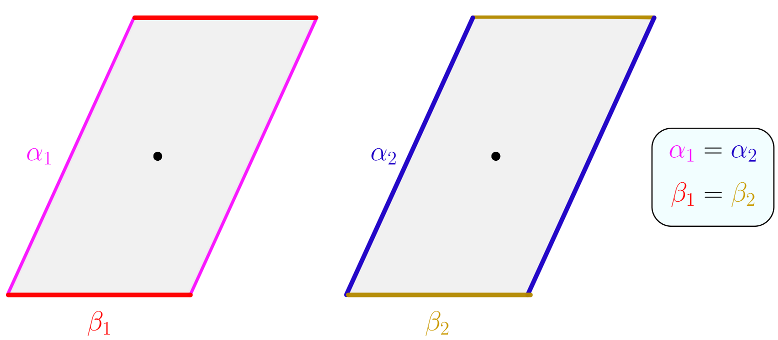


FIGURE 6. The WYSIWYG limit after shrinking the slit

2. Case: Shrink the slit to the zero. For the next degeneration, we fix the elliptic curves and only let the length of the slit go to zero. In this case, the first guess is that the limit is just a union of two elliptic curves connected at a fixed point (the limit of the slit).

In fact, this is the limit, when considered in the WYSIWYG partial compactification $\overline{\mathcal{H}}^{WY}$. Note that the two linear equations

$$\int_{\alpha_1} \omega = \int_{\alpha_2} \omega, \int_{\beta_1} \omega = \int_{\beta_2} \omega$$

persist in the limit.

Remark 2.4. The WYSIWYG partial compactification $\overline{\mathcal{H}}^{WY}$ was constructed by Mirzakhani-Wright, Chen-Wright, where it was also shown that the closure of a $\mathbb{R}$-linear subvariety N is again $\mathbb{R}$-linear and the linear equation defining the boundary are obtained by taking limits of linear equations on N. We will be able to recover these results from the results in this lecture series.)

The multi-scale limit. We now introduce a new way of taking limits, which still remembers the information of the slit and which will lead to the definition of the multi-scale compactification of strata. The main observation is that different parts of the surface behave differently along the degeneration. There are parts of the surface which stay of constant size (the two parallelograms). And there are parts which go to zero (the slit). To obtain information about the slit in the limit, one rescales the family of differentials so that the length of the slit stays constant. This pushes the parallelogram defining E_1 and E_2 to infinity in the limit. In the end, the result is two copies of the complex plane (which we think of $\mathbb{P}^1$ with a point at infinite), glued together along the slit.

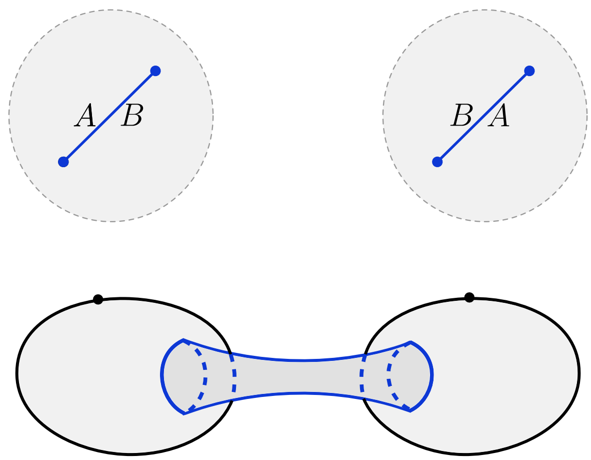


FIGURE 7. The limit of the slit, obtained by rescaling the slit to constant size

Then we combine this with the WYSIWYG limit and glue the two points at infinity to the limits of the slit. The final result is a nodal Riemann surface with 3 components: Two components of genus 1 connected to each other by a $\mathbb{P}^1$.

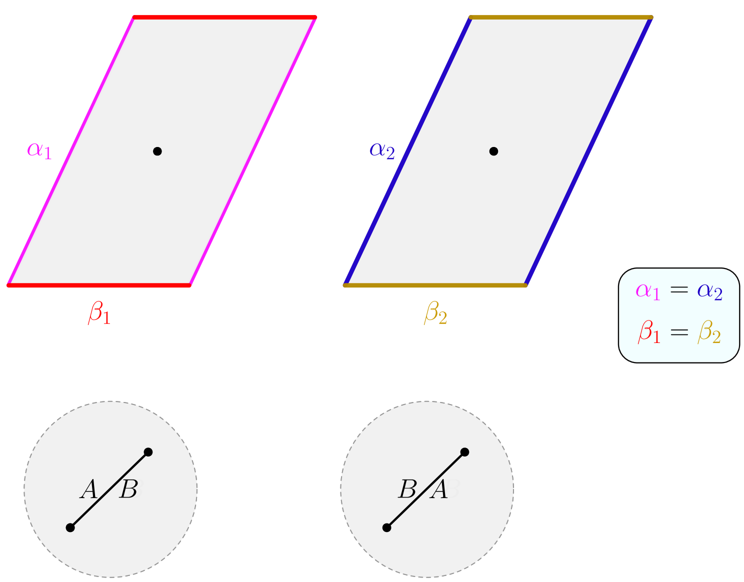


FIGURE 8. The multi-scale limit is obtained by gluing both pictures together

We want to organize the components of the limit, depending on the speed at which periods vanish. The top level records all pieces of constant size, in this case the two elliptic components.

The $\mathbb{P}^1$ component corresponds to parts of the surface that go to zero (the slit), which we record at level -1. The *multi-scale limit* is the resulting nodal surface equipped with the limit differentials, together with the information of which component appears at which level.

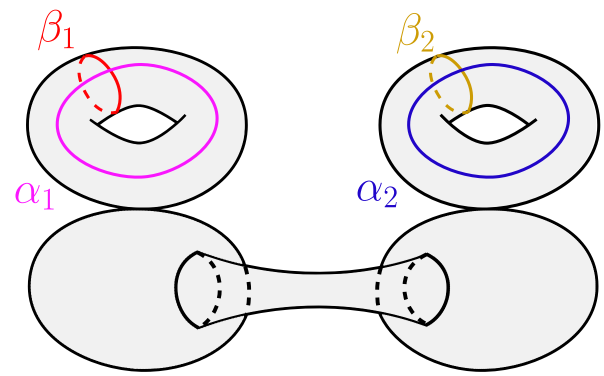


FIGURE 9. The resulting multi-scale limit is a flat surface in the stratum $\mathcal{H}(0) \times \mathcal{H}(0) \times \mathcal{H}(1^2, -1^4)$

In general, different pieces of the surface can move at vastly different rates and there can be many levels in the limit. The top level always corresponds to constant size. The level -1 and below contain everything that shrinks to zero. Level -2 corresponds to the pieces that shrink faster than pieces on level -1 and so on. We will make this idea precise later using multi-scale differentials.

Summary. We see that the boundary of N satisfies the following properties along the degenerations we discussed.

(1) The limit surface satisfies linear equations among its periods.
(2) Linear equations persist in the limit along a degeneration, unless they cross nodes with simple poles.
(3) There are no linear equations relating periods from different levels, i.e. the boundary has a product structure corresponding to different levels.

It turns out that these observations hold for the boundary of any linear subvariety inside the multi-scale compactification of strata. Before we can make this precise, we need to define the multi-scale compactification in detail.

3. Background on the Deligne-Mumford compactification of moduli space

Before addressing degenerations of differential forms, we start by degenerating the underlying Riemann surface only.

Definition 3.1. *A n-marked stable curve X is a finite union of marked Riemann surfaces X_i and finite set $\Sigma = \{q_e^+, q_e^-, e \in E(X)\} \subseteq X$ of nodes with q_e^+ glued to q_e^- such that*

- *the marked points on X are disjoint from Σ,*
- *each component X_i of genus zero has at least three marked points or nodes.*

The combinatorics of how the different components of X interact is recorded in the dual graph.

Definition 3.2. *Let X be a stable curve. The dual graph Γ of X is a graph with a vertex v_i for each irreducible component X_i. Two vertices v_i and v_j are connected by an edge if the corresponding components X_i and X_j are connected by a node. Additionally, we decorate each vertex v_i with the integer $g_i = g(X_i)$ and for every marked point on X_i we add a half-leg (an edge with starting point but no end point). The genus $g(X)$ is defined by*

$$g(X) := \sum_{i=1}^{n} g(X_i) + \dim H_1(\Gamma).$$

Exercise 3.3. Find all dual graphs of stable curves of genus 2 with one and two marked points.

The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ is a moduli space for n-marked stable curve of genus g, up to isomorphism. The space $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic variety (orbifold) of dimension $3g-3+n$. There is a stratification of $\overline{\mathcal{M}}_{g,n}$ by dual graphs. Let

$$\overline{\mathcal{M}}_\Gamma := \overline{\{X \in \overline{\mathcal{M}}_{g,n} \,|\, \Gamma(X) = \Gamma\}}$$

be the closure of the locus of curves with dual graph Γ. Then

$$\operatorname{codim} \overline{\mathcal{M}}_\Gamma = |E(\Gamma)|.$$

Remark 3.4. Topologically a degeneration of Riemann surfaces is obtained by pinching a collection of homology cycles, called *vanishing cycles*. If $\widetilde{X}$ is a degeneration of X, then there exists a collection of edges $\widetilde{E}$ in the dual graph of $\Gamma(\widetilde{X})$ such that $\Gamma(X)$ is obtained by contracting the edges in $\widetilde{E}$.

Remark 3.5. The topology on $\overline{\mathcal{M}}_{g,n}$ can be defined as follows. A sequence $(X_n)_n$ converges to Y if there exists an exhaustion Y_n of $Y - \Sigma$ by compact sets and conformal maps $g : Y_n \to X_n$.

3.1. **Level graphs and multi-scale differentials.**

Definition 3.6. *A* level graph $\overline{\Gamma}$ *is a stable graph* Γ *together with a level function* $\ell : V(\Gamma) \to \{0, -1, \dots, -L\}$ *and a prong order* $\kappa : E(\Gamma) \to \mathbb{N}_{\geq 0}$ *subject to the following condition:*

- **(Horizontal nodes have prong** 0**)** *Let* e *be an edge connecting two vertices* v, v'*. Then* $\kappa(e) = 0$*, if and only if* v, v' *have the same level, i.e.* $\ell(v) = \ell(v')$*.*

If $\kappa(e) > 0$*, then we let* v_e^+ *be the vertex with higher level and* q_e^+ *the corresponding node. We define* v_e^-, q_e^- *analogously. An edge* e *with* $\kappa(e)$ *is called a vertical edge/node. If* $\kappa(e) = 0$*, then* e *is called a* horizontal node *and we make an arbitrary choice of which vertex is* $v_e^\pm$ *and let* $q_e^\pm$ *be the corresponding preimages of the nodes.*

Definition 3.7. *Let* $\mathcal{H} = \mathcal{H}(\mu), \mu = (\mu(p_1), \dots, \mu(p_n))$ *be a stratum and* $\overline{\Gamma}$ *be a level graph. For any vertex* $v \in V(\overline{\Gamma})$*, the partition* μ_v *is determined by the following:*

(1) μ_v *contains* $\mu(p_i)$ *if the half-leg corresponding to the marked point* p_i *is adjacent to* v*,*
(2) For every node e *and every preimage* $q_e^\pm$ *adjacent to* v*, the partition* μ_v *contains an entry* $\kappa(e) - 1$ *if* q_e^+ *is adjacent to* v *and* $-\kappa(e) - 1$ *if* q_e^- *is adjacent to* v*.*

For every level i *of* $\overline{\Gamma}$ *we define*

$$\mathcal{H}_i := \prod_{v:\ell(v)=i} \mathcal{H}(\mu_v).$$

Definition 3.8. *Fix a stratum* $\mathcal{H}$*. A* multi-scale differential (X, ω) *for* $\mathcal{H}$ *compatible with a level graph* $\overline{\Gamma}$ *consists of a stable marked curve with underlying dual graph* Γ *and a differential*

$$\omega \in \mathcal{H}_0 \times \prod_{i<0} \mathbb{P}\mathcal{H}_i.$$

Additionally, a multi-scale differential has to satisfy additional linear equations among its residues and has to have a prong matching. See below for more details on these additional conditions.

Suppose (X, ω) is a multi-scale differential. We write ω_v for the differential on the irreducible component X_v and let $Z(\omega_v), P(\omega_v)$ be the zeros and poles, respectively, of ω_v on X_v. Note that every zero or pole is either a marked point or a node.

For any level i, we let

$$X_{(i)} := \bigcup_{v:\ell(v)=i} X_v$$

be the subsurface of level i. Let $\omega_{(i)}$ be the restriction of ω to $X_{(i)}$ and $Z(\omega_{(i)}), P(\omega_{(i)})$ the zeros and poles at level i.

Residue conditions. A multi-scale differential has to satisfy the following linear equations among residues:

(1) **Matching residues at horizontal nodes** If e is a horizontal node, then

$$\operatorname{res}_{q_e^+} \omega + \operatorname{res}_{q_e^-} \omega = 0.$$

(2) **Global residue condition** For every level i and every connected component Y of

$$X_{>i} = \bigcup_{v:\ell(v)>i} X_v$$

that does not contain a marked point with a prescribed pole, let $e_1, \dots, e_n$ be the set of nodes connecting Y to components with level $< i$. Then

$$\sum_{k=1}^{n} \operatorname{res}_{q_{e_i}^-} \omega = 0.$$

Prongs. Given a differential ω choose a chart such that

$$\omega = z^k dz.$$

A prong is a tangent vector $\pm\eta\partial_z$ where $\eta^{k+1} = 1$. Up to sign, there are $|k+1|$ prongs and a prong matching is a bijection between prongs at two preimages of a node. For us it mostly matters that a choice of prong matchings is a finite datum. For counting problems it matter how many prong matchings there are and this is computable. Since we are mostly interested in the local structure of the moduli space of multi-scale differentials the prong matching won't play much of a role.

The moduli space of multi-scale differentials.

Definition 3.9. *The moduli space of multi-scale differentials $\overline{\mathcal{H}}$ is the space of all multi-scale differentials corresponding to the stratum $\mathcal{H}$.*

Theorem 3.10 ([BCGGM19])**.** *The moduli space of multi-scale differentials $\overline{\mathcal{H}}$ is a smooth algebraic orbifold. The boundary is a normal crossing divisor, which is stratified by level graphs.*

The boundary of $\overline{\mathcal{H}}$ is stratified by level graphs, analogous to the stratification of $\overline{\mathcal{M}}_{g,n}$ by stable graphs.

Definition 3.11. *Let $\overline{\Gamma}$ be a level graph and*

$$D_{\overline{\Gamma}} := \{(X,\omega) \in \overline{\mathcal{H}} \,|\, \omega \text{ compatible with } \overline{\Gamma}\}.$$

Then $D_{\overline{\Gamma}}$ is a subvariety of codimension $\ell(\overline{\Gamma}) - 1 + |h(\overline{\Gamma})|$, where $\ell(\overline{\Gamma})$ is the number of levels of $\overline{\Gamma}$ and $h(\overline{\Gamma})$ the set of horizontal nodes.

How can we describe a neighborhood of a point in $D_{\overline{\Gamma}}$? A point in $D_{\overline{\Gamma}}$ consists of a collection of differentials ω_v each contained in a stratum $\mathcal{H}(\mu_v)$. The partition μ_v is determined by the original partition μ and the level graph $\overline{\Gamma}$. Thus $D_{\overline{\Gamma}}$ looks like a product of strata and has natural period coordinates. Since in lower levels the differentials are projectivized we have to use projectivized period coordinates there.

Proposition 3.12. *Let $D_{\overline{\Gamma}}$ be a boundary stratum of $\overline{\mathcal{H}}$. Then $D_{\overline{\Gamma}}$ has local period coordinates modeled on a linear subspace of*

$$H^1(X_{(0)}, \omega_{(0)}; \mathbb{C}) \times \prod_{i<0} \mathbb{P}H^1(X_{(i)}, \omega_{(i)}; \mathbb{C}).$$

Here $X_{(i)}$ is the subsurface consisting of irreducible components of level i and $\omega_{(i)}$ the restriction of ω to $X_{(i)}$.

Remark 3.13. As before we are ignoring residue equations and prong-matchings here. Inside $\mathcal{H}_i$ there is a subspace $\mathcal{H}_i^{\mathcal{R}}$ consisting of all multi-scale differentials satisfying the global residue conditions and matching residues at horizontal nodes. Since residues are part of the period coordinates, this means passing to a linear subspace in period coordinates (defined by linear equations with rational coefficients).

Choosing a prong-matching means passing to a finite cover which does not change the local structure. We also ignore issues arising from the stack structure of the boundary. These delicate issues are dealt with in great detail in [CMZ20].

Remark 3.14. In $\overline{\mathcal{M}}_{g,n}$ a degeneration of a Riemann surface is obtained by pinching a collection of vanishing cycles. Hence if a Riemann surface with dual graph Γ degenerates to one with dual graph Γ', the graph Γ is obtained by contracting a set of edges.

A similar description is still valid for $\overline{\mathcal{H}}$. Any degeneration creates new levels and new horizontal nodes. If $\overline{\Gamma}'$ is the level graph of a degeneration of surfaces with level graph $\overline{\Gamma}$, then $\overline{\Gamma}$ is obtained by choosing a set of horizontal edges F^{hor} and a set of levels I and contracting all of the horizontal edges in F^{hor} and *any* vertical edge that connects vertices between level i and $i+1$ for some $i \in I$

Exercise 3.15. Find all level graphs in strata of genus 2 and their degenerations.

4. Closures of orbit closures

Theorem 4.1 ([Ben22]). *Let $D_{\overline{\Gamma}} \subseteq \overline{\mathcal{H}}$ be a boundary stratum and $N \subseteq \mathcal{H}$ a linear subvariety. Then the boundary*

$$\partial N_{\overline{\Gamma}} := \partial N \cap D_{\overline{\Gamma}}$$

is a linear subvariety (in the period coordinates of the boundary). Furthermore, $\partial N_{\overline{\Gamma}}$ is a level-wise product, i.e.

$$\partial N_{\overline{\Gamma}} = \prod V_i,$$

where each $V_i \subseteq H^1(X_{(i)}, \omega_{(i)}; \mathbb{C})$ are linear subspaces.

Informally the last part of the theorem can be stated as: Linear equations don't mix levels. Note that this theorem does not contain any information on whether a given boundary stratum is empty or not. It only says that if a boundary component is non-empty then it is defined by linear equations among residues.

In fact, we can be much more precise and identify the linear equations defining ∂N as limits of linear equations defining N.

For this, we need a way of comparing homology groups of a multi-scale differential with the homology groups of nearby surfaces. We start with an informal description: The stratum is locally modeled on the cohomology groups

$$H^1(X, \omega; \mathbb{C}) = H^1(X - P(\omega), Z(\omega); \mathbb{C}).$$

Thus a linear relation

$$\int_\alpha \omega = 0$$

can be thought of as a homology cycle

$$\alpha \in H_1(X, \omega; \mathbb{C})$$

such that $[\omega] \in \mathrm{Ann}(\alpha)$.

Let Y be a smooth Riemann surface and X be a stable curve, which is obtained by pinching a collection of vanishing cycles Λ. Let $\overline{\Gamma}$ be a level graph on the dual graph of X. Given a homology class α on Y, represent it by a path on Y. And consider the image under the map that pinches the vanishing cycles. We are interested in the limit of periods

$$\int_\alpha \omega.$$

when the differential ω converges to a multi-scale differential.

How can we understand a sequence of differentials converging to a multi-scale differentials? After removing the vanishing cycles, Y is cut into several connected components. Each component Y_v can be identified with a vertex of the dual graph of X and thus be assigned a level $\ell(v)$. For each level i, there exists constants $c_i(Y)$, depending on the degeneration, such that

$c_i(Y)\omega_{(i)}(Y)$ converges to $\omega_{(i)}(X)$. Furthermore, the ratios

$$\frac{c_{i-1}(Y)}{c_i(Y)}$$

converge to zero.

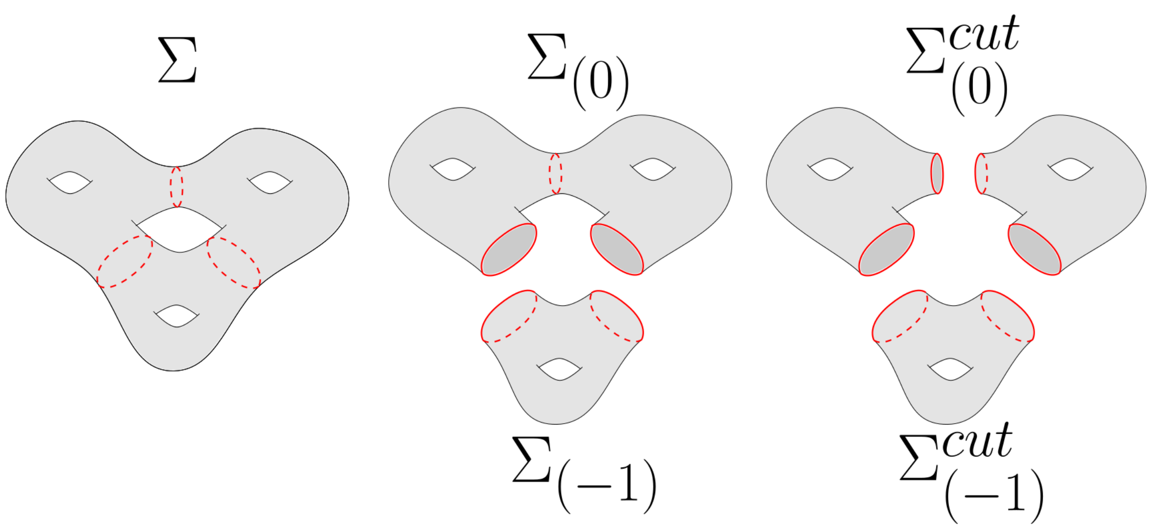


FIGURE 10. Cutting along the vanishing cycles decomposes Y into componentss

To see what will happen to periods $\int_\alpha \omega(Y)$ in the limit, we first consider a few special cases.

1. Case: α can be represented by a path in top level, but cannot be homotoped into a lower level.
The differential ω converges to zero on the lower level compared to the top level. Thus in the limit only the parts of α the surface in the top level survive.

2.Case:α can be homotoped into a lower level. In that case the integral

$$\int_\alpha \omega$$

goes to zero. But remember that there are rescaling coefficients $c_{-1}(t)$ such that $c_{-1}(t)\omega$ converges to $\omega_{(-1)}$. In particular if the linear equation

$$\int_\alpha \omega = 0 = \int_\alpha c(t)\omega$$

is satisfied, then in the limit it becomes

$$\int \omega_{(-1)} = 0.$$

3.Case: α crosses a horizontal node. In this case the integral

$$\int_\alpha \omega$$

diverges and we do not take the limit. (The differential locally near the nodes looks like $\frac{r}{z}dz$ so that the period blows up when integrated through the node.)

4.1. **Limits of periods: The level filtration.** For each vanishing cycle $\lambda = \lambda_e$ let λ° be a small open neighborhood of λ that deformation retracts onto λ. Let $\partial\lambda^\circ = \lambda_e^+ \sqcup \lambda_e^-$ be the the boundary of λ° (with opposite orientations).

In particular let $\Lambda^\circ = \sqcup_e \lambda_e^\circ$. The surface $Y - \Lambda^\circ$ is a compact Riemann surface with boundary that can be embedded into X. In particular, we choose λ_e^+ to be in the component of X with a higher level, if e is a vertical node.

Note that $Y - \Lambda^{ver,\circ}$ decomposes into several connected component. Let Y_v^{hor} be the connected component contained in X_v.

By further removing the thickened horizontal vanishing cycles Λ^{hor} we then obtain the surfaces Y_e.

Let $Y_{(i)} = \sqcup_{v:\ell(v)} Y_v$ be the subsurface of level i. We define the subsurface $Y_{<i}$ of level below i similarly.

The first step is to define a restriction map to level 0.

$$\alpha_0 : H_1(X, \omega; \mathbb{Z}) \to H_1(X_{(0)}, \omega_{(0)})$$

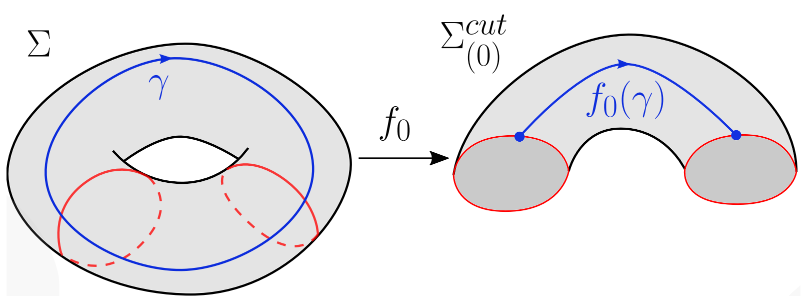


FIGURE 11. The specialization map onto level 0

Informally, the map $\alpha_{(0)}$ is defined by representing a homology class γ be a sum of smooth paths and cutting off the path at any point that crosses a vanishing cycles connecting to lower level. The end result is a relative cohomology class (relative to the new boundary components and the marked zeros) Note that at this point we do not take care of horizontal nodes yet. We define $W_0 = H_1(X, \omega; \mathbb{Z}), W_1 = \ker \alpha_{(0)}$. Note that W_0/W_1 is isomorphic to the cohomology of the surface $Y_{(0)}$. Note that $Y_{(0)}$ still contains the horizontal vanishing cycles of level 0. In the next step we want to cut along those. We define $W_0^\perp \subseteq W_0/W_1$ to be the subspace of cycles that do not intersect any horizontal cycle in level 0.

Let $Y_{(0)}^{cut}$ be the Riemann surface with boundary obtained by cutting $Y_{(0)}$ along all horizontal vanishing cycles. We want to define a map

$$\beta_0 : W_0^\perp \to H_1(Y_{(0)}^{cut}, \omega_{(0)}; \mathbb{Z})$$

as follows. Represent a homology class in $W_0^\perp$ by a sum of paths not intersecting the horizontal vanishing cycles of level 0. And restrict this class to $Y_{(0)}^{cut}$. This map is not well-defined, the ambiguity is the difference $\lambda_e^+ - \lambda_e^-$.

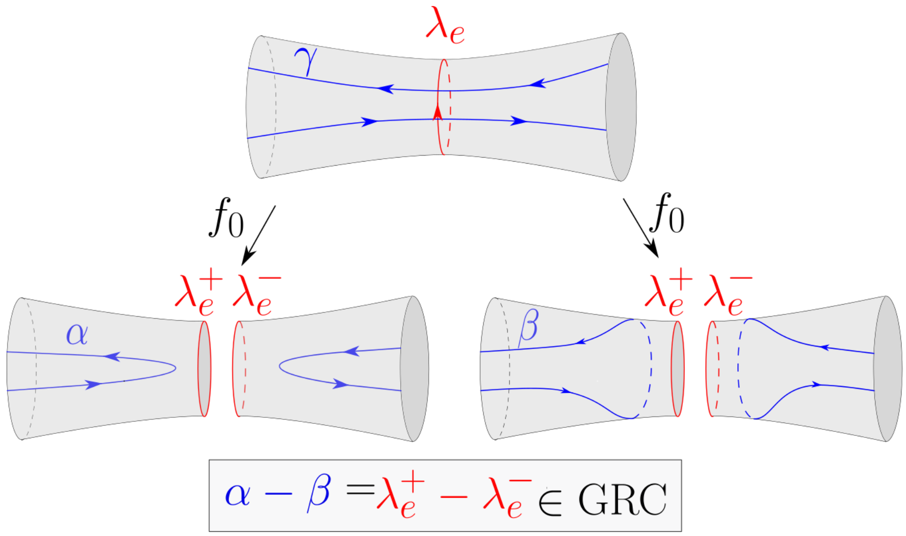


FIGURE 12. The restriction map is not well-defined

We thus define the residue subspace $\mathcal{R}_0$ to be the span $\lambda_e^+ - \lambda_e^-$, where e ranges over all horizontal nodes in level 0. We thus get a well-defined map

$$\beta_0 : W_0^\perp \to H_1(Y_{(0)}^{cut}, \omega_{(0)}; \mathbb{Z})/\mathcal{R}_0.$$

One can check that β_0 is an isomorphism. Note that the dual of $H_1(Y_{(0)}^{cut}, \omega_{(0)}; \mathbb{Z})/\mathcal{R}_0$ can be identified with the local model for $\overline{\mathcal{H}}$ in level 0. We now proceed with the deeper levels. Let W_1 be the kernel of α_0. Every class in W_{-1} can be represented by a cycle completely supported in the subsurface $Y_{(-1)}$. That way we can try to define a map

$$W_{-1} \to H_1(Y_{(\leq -1)}, \omega_{(\leq -1)}; \mathbb{Z}),$$

but as before there is ambiguity. It turns out the ambiguity is exactly obtained by the global residue condition. If we let $\mathcal{R}_{-1}^{GRC}$ the subspace generated by the cycles in the GRC condition we get a map

$$\alpha_{-1} : W_{-1} \to H_1(Y_{(-1}, \omega_{(-1)}; \mathbb{Z})/\mathcal{R}^{GRC}.$$

Afterward we still need to take care of the horizontal nodes. As before, set $W_{-1}^\perp \subseteq W_{-1}/\ker \alpha_{-1}$ the cycles with zero intersection against horizontal vanishing cycles in level -1. There is a well-defined map

$$\beta_{-1} : H_1(Y_{(-1)}^{cut}, \omega_{(-1)}; \mathbb{Z})/\mathcal{R}_{-1},$$

where $Y_{(-1)}^{cut}$ is obtained from $Y_{(-1)}$ by cutting along horizontal vanishing cycles and $\mathcal{R}_{-1}$ defined by the global residue conditions as well as the residue subspace defined by $\lambda_e^+ - \lambda_e^-$ for horizontal vanishing cycles in level -1.

Upshot: Proceeding in the same way for the remaining levels, one obtains a filtration

$$W_{-L} \subseteq \ldots \subseteq W_0,$$

with restriction maps

$$\alpha_i : W_i \to H_1(Y_{(i)}, \omega_{(i)}; \mathbb{Z})/\mathcal{R}_i^{GRC},$$

with subspaces $W_i^\perp \subseteq W_i/W_{i-1}$ and isomorphisms

$$\beta_i : W_i^\perp \to H_1(Y_{(i)}^{cut}, \omega_{(i)}; \mathbb{Z})/\mathcal{R}_i \simeq H_1(X_{(i)}, \omega_{(i)}; \mathbb{Z})/\mathcal{R}_i.$$

4.2. **Limits of linear equations.** Suppose $(X, \omega) \in \partial N$ is a boundary point of an orbit closure and choose a nearby point $(Y, \eta) \in N$.

In local period coordinates near (Y, η) the linear subvariety N is defined by the vanishing of some linear equations, in other words

$$N = \mathrm{Ann}(V),$$

where V is a subspace $V \subseteq H_1(Y, \eta; \mathbb{C})$.

For every level i of (X, ω) we define

$$V_i = V \cap W_i \subseteq W_i$$

and $V_i^\perp := [V_i] \cap W_i^\perp \subseteq W_i / W_{i-1}$.

Theorem 4.2 ([Ben22])**.** *Suppose N is a linear subvariety, $(X, \omega) \in \partial N \cap D_{\overline{\Gamma}}$ is a boundary point and $(Y, \eta) \in N$ is a nearby point such that $N = \mathrm{Ann}(V)$ in local period coordinates.*

Then $\partial N \cap D_{\overline{\Gamma}}$ is a linear subvariety, which in local period coordinates is defined as

$$\partial N \cap D_{\overline{\Gamma}} = \mathrm{Ann}(V_0^\perp) \times \prod_{-L}^{-1} \mathbb{P}(\mathrm{Ann}(V_i^\perp))$$

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