Some Real Rel Trajectories in $\mathcal{H}(1,1)$ that are not Recurrent

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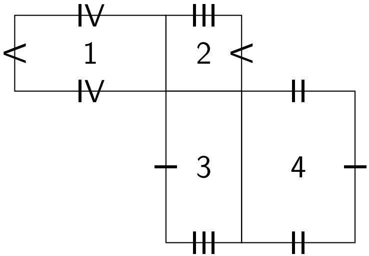
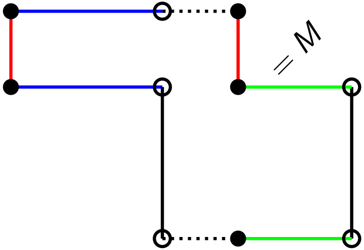
March 2024

- A translation surface M is a collection of polygons $\{P_1, \ldots, P_l\}$ whose edges are glued with a family of translations of the form $z \mapsto z + c$.
- Collections of objects are strata $\mathcal{H}(r_1, \ldots, r_k)$ with $2g - 2 = \sum_{i=1}^{k} r_i$.

Example in $\mathcal{H}(1, 1)$





Rel foliation

$T_N\mathcal{H}(r_1,\ldots,r_k) = H^1(S,\{r_1,\ldots,r_k\};\mathbb{R}^2)$

Let $Res : H^1(S,\{r_1,\ldots,r_k\};\mathbb{R}^2) \to H^1(S;\mathbb{R}^2)$ be the restriction map.
Define $\mathcal{R} := ker(Res)$.
It integrates to a foliation of $\mathcal{H}(r_1,\ldots,r_k)$: the **Rel** foliation.

Other names: Kernel, Isoperiodic, Absolute Period
The leaves have real dimension $2(k-1)$.

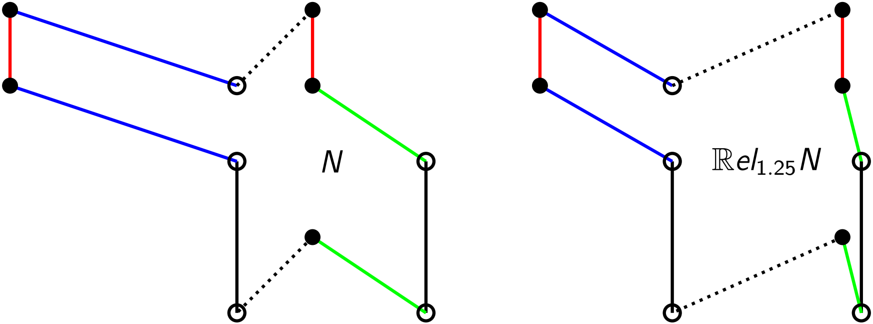
Moreover, $Z = \mathcal{R} \cap H^1(S,\{r_1,\ldots,r_k\};\mathbb{R}_x)$ defines the **real Rel** flow(s).

In $\mathcal{H}(1,1)$, the leaves of this subfoliation have dim $= 1$.

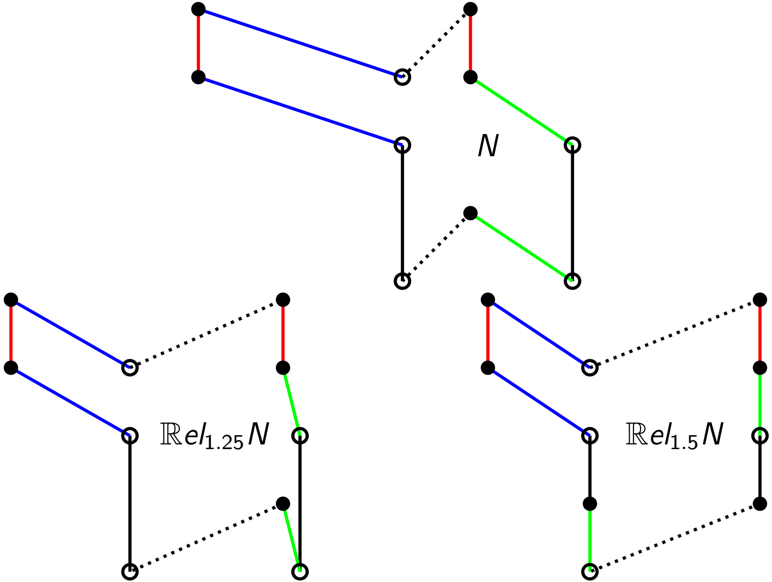
Real rel flow

Deforms $N \in \mathcal{H}(1,1)$ so that the relative positions of • and ∘ change horizontally.

$\mathbb{R}el_t : \mathcal{H}(1,1) \to \mathcal{H}(1,1)$



Real rel flow



Some things about Rel

- Calta '04 noticed that one can push affine measures by real Rel to obtain more $U-$invariant measures.
- The Rel leave of the Arnoux-Yoccoz surface is dense in $\mathcal{H}(g-1, g-1)$, for $g \geq 3$. Hopper-Weiss '18
- The real Rel trajectory of Arnoux-Yoccoz is divergent *H-W '18*
- Rel leaves can be dense in some affine manifolds if they have Property P. *Florent Ygouf '22*
- There are dense real Rel orbits in every connected comp. of any stratum. *Winsor '22*
- Ergodicity real Rel and Rel: *Chaika-Weiss '23*

Theorem (O. '24)

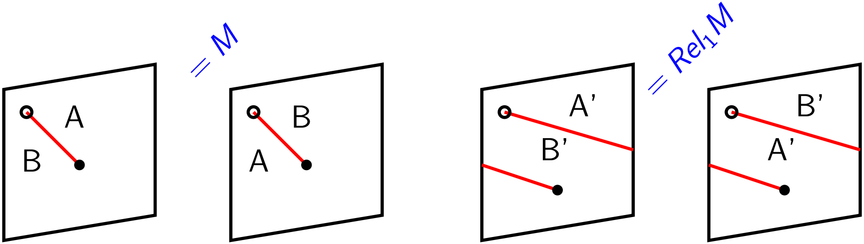
There are real Rel trajectories in $\mathcal{H}(1,1)$ *that are non-recurrent and not divergent.*

$N \in \mathcal{H}(1,1)$ is recurrent if $\exists t_i \to \infty$: $\lim Rel_{t_i} N = N$.

$$\lim_{t_i \to \infty} Rel_{t_i} N = N.$$

Double covers

Let $\mathcal{E} \subset \mathcal{H}(1,1)$ branched double covers of tori:



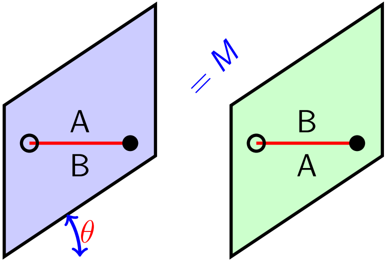
Rel trajectories in $\mathcal{E}$ are determined by the location of the singularities and the horizontal flow of the underlying torus.

- periodic orbits
- $\overline{\{Rel_t(M)\}} \sim \mathbb{T}^2$
- divergent orbits (the singularities collapse)

What about *Rel* on $\mathcal{H}(1,1)\backslash\mathcal{E}$

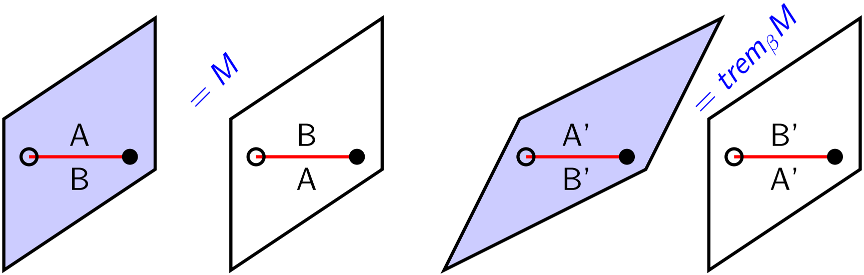
Fix $M \in \mathcal{E}$. Lebesgue is invariant w.r.t horizontal flow on M. It could be ergodic. But if it is not, for example:

For θ irrational, Lebesgue restricted to one of the copies of the tori is an ergodic measure.



What about *Rel* on $\mathcal{H}(1,1)\backslash\mathcal{E}$

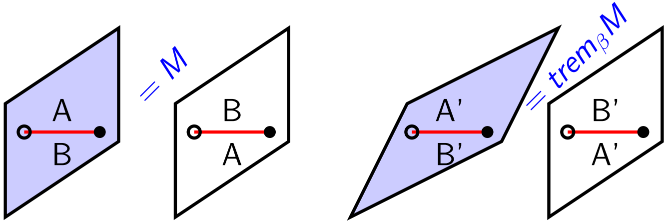
We can use a deformation of translation surfaces that commutes with *Rel*. Tremors:



Tremor

If μ is an inv. measure for the horizontal flow, we can see it as the product of a transverse measure β and $\mathrm{d}x$.
Like a product: $\mu = \mathrm{d}x \otimes \beta$.
Moreover we can assume $\beta \in H^1(S, \{p_1, p_2\}, \mathbb{R})$.
Thus $trem_\beta M$ is the solution of some ODE. The local coord. of it are of the form:

$$\int_\gamma \mathrm{d}x_{trem_\beta M} = \int_\gamma \mathrm{d}x_M + \beta(\gamma) \qquad \int_\gamma \mathrm{d}y_{trem_\beta M} = \int_\gamma \mathrm{d}x_M$$

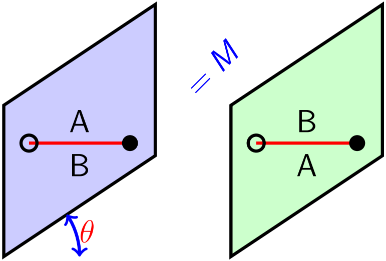


Recurrence

$N \in \mathcal{H}(1,1)$ is recurrent if $\exists t_i \to \infty$ such that

$$\lim_{t_i \to \infty} Rel_{t_i} N = N.$$

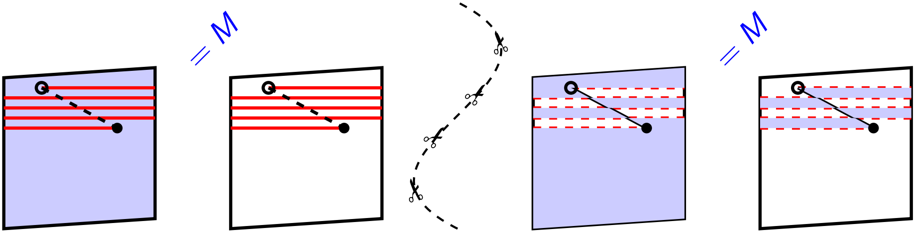
For aperiodic surfaces glued along a horizontal slit, we have the following:



Theorem

Let $M \in \mathcal{E}$ as above. The surface $trem_\beta M \in \mathcal{H}(1,1) \backslash \mathcal{E}$ is recurrent if and only if $\theta \notin \mathbb{Q}$ is well approximable.

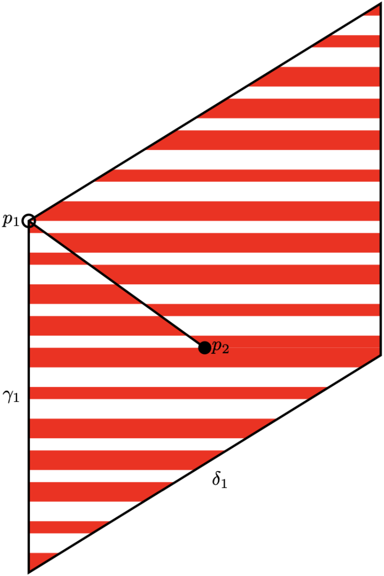
When the slit is **very** long, it is possible to represent the same surface with a shorter slit:



By doing this, some **area exchange** happens.

A key to understand how close $Rel_t trem_\beta M$ is from $trem_\beta M$ by controlling this area exchange.

Area exchange



When $\theta \notin \mathbb{Q}$ is badly approximable there are constants $0 < c < C < 1$

$$c < \frac{|\textit{Area red} - \textit{Area white}|}{\textit{Total area}} < C$$

independently of the length of the slit.