On the stable norm of flat surfaces

Pablo Montealegre
Le Teich, 2024

Let S be a closed surface equipped with a (possibly singular) Riemannian metric.

Classical question: How many are there closed geodesics inside a homology class $h \in H_1(S, \mathbb{Z})$ whose length is less than a positive number x ?

⟶ Phillips – Sarnak 87

Let S be a closed surface equipped with a (possibly singular) Riemannian metric.

Classical question: How many are there closed geodesics inside a homology class $h \in H_1(S, \mathbb{Z})$ whose length is less than a positive number x?

→ Phillips – Sarnak 87

Orthogonal question: How many are there homology classes on S that admit a closed geodesic representative whose length is less than a positive number x?

The stable norm

S closed surface with a (singular) Riemannian metric.

Definition:

- For $h \in H_1(S, \mathbb{Z})$, set:

$$\|h\|_S := \min \{ \ell ength(\gamma),\ \gamma \text{ multicurve},\ [\gamma] = h \}$$

- Extend to $H_1(S, \mathbb{R})$ by linearity and density.

Goal n°1: Compute the stable norm on examples.

Counting problem

Definition: A class $h \in H_1(S, \mathbb{Z})$ is simple if it is minimized by a connected simple curve.

Set $P_S(x) := \#\{h \text{ simple}, \|h\|_S \leq x\}$.

Goal n°2: Understand the asymptotics of P_S.

What is known

1. **Flat tori**, Babenko 88

Explicit computations, Euclidean norm, $p(x) \sim cst \cdot x^2$

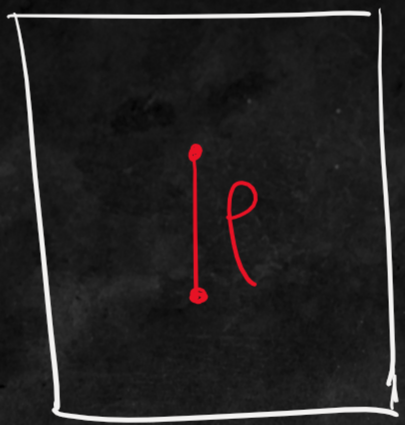
2. **Hyperbolic punctured tori**, McShane - Rivin 95

Explicit, $p(x) \sim cst \cdot x^2$

3. **Translation surfaces**, Masur 86, Eskin, Veech, Zorich...

- $\exists c_1, c_2 > 0$ such that $c_1 \cdot x^2 \leq p(x) \leq c_2 \cdot x^2$
- For almost every translation surface, $p(x) \sim cst \cdot x^2$

I) Flat slit torus

X : 

- Vertical slit of length $0<\ell<1$
- $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^2$

  $R \longleftrightarrow (m,n)$

I) Flat slit torus

X :

- Vertical slit of length $0<\rho<1$
- $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^2$

$$k \longleftrightarrow (m,n)$$

<u>Definition:</u> The Farey sequence of order $k \in \mathbb{Z}$ is the ordered set:

$$F_k = \left\{ \frac{p}{q},\ p\wedge q = 1,\ q \leq k \right\}$$

<u>Ex:</u> $F_3 = \left\{ \dots, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \dots \right\}$

Theorem: (M)

Let $L = \lfloor 1/\rho \rfloor$. The unit ball B_X of the stable norm of X has a vertex in the direction (m, n) if and only if either:

- $\frac{n}{m} \in F_L$, and in that case $\|m, n\|_X = \|m, n\|_2$

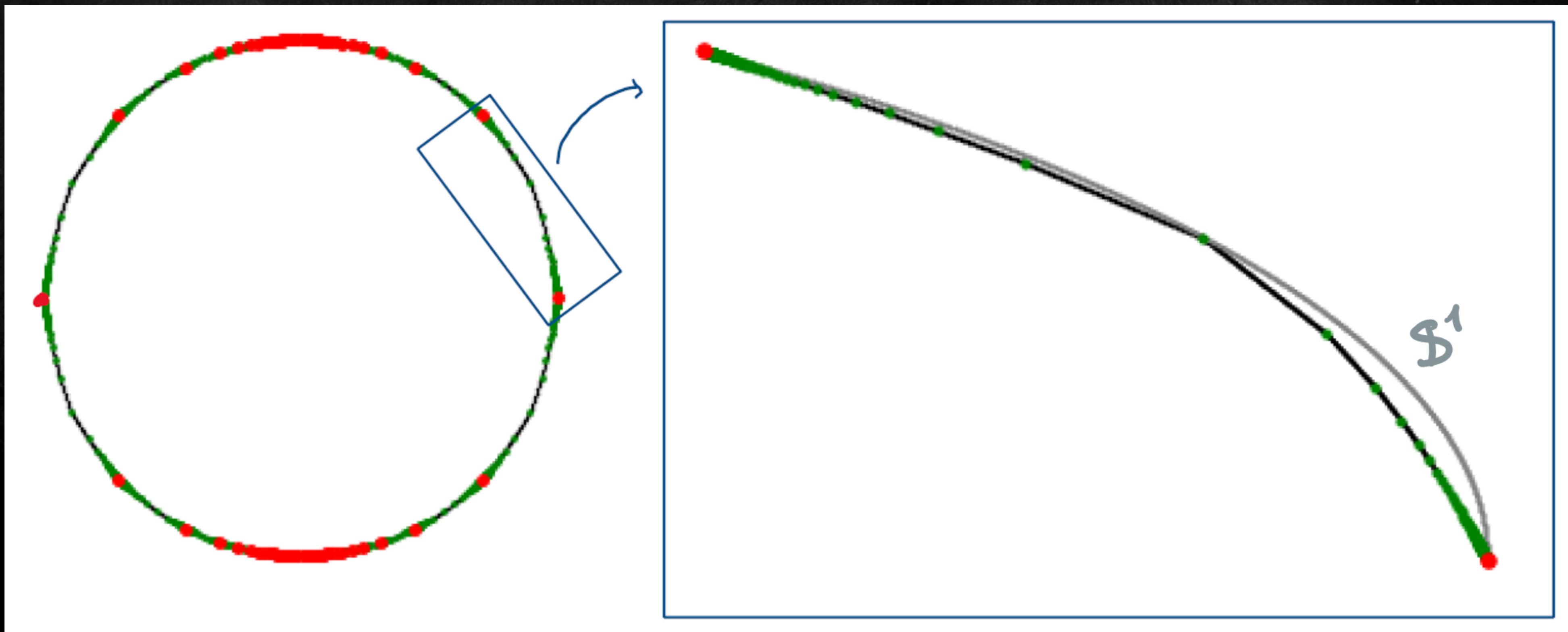
or

- $\frac{n}{m} \notin F_L$ but $\frac{a}{b} \in F_L$ or $\frac{c}{d} \in F_L$ where a/b and c/d are the Farey parents of n/m, and $\|m, n\|_X = \sqrt{b^2 + (a+\rho)^2} + \sqrt{d^2 + (c-\rho)^2} > \|m, n\|_2$

In every other direction, B_X has a flat.

Theorem: (M)

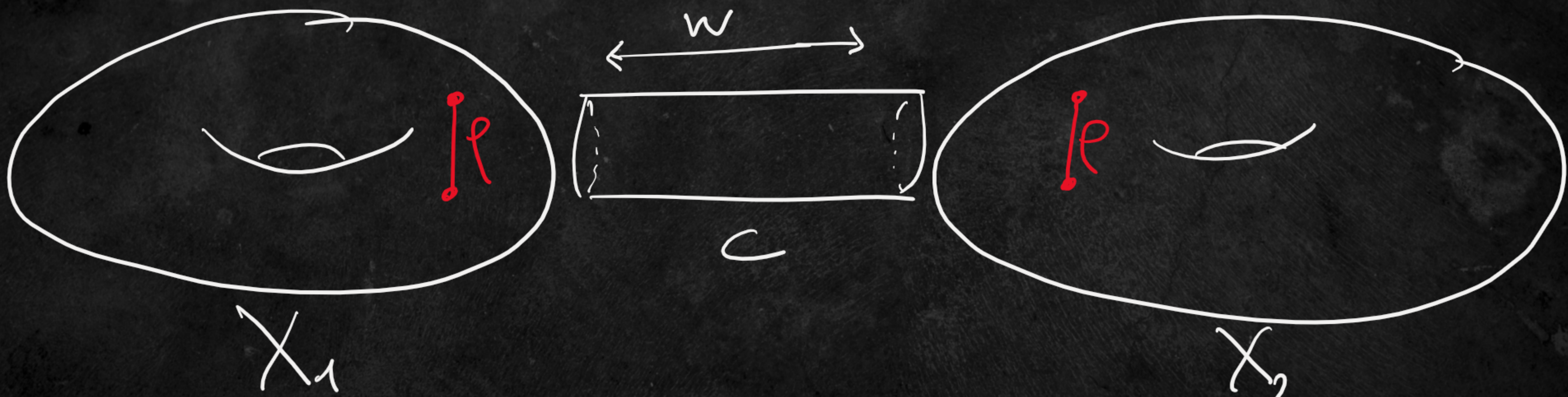
- Explicit formula for $\|m, n\|_X$, depending on ρ and on the position of $\frac{n}{m}$ in the Farey sequence.
- Vertices of B_X:
  - Primary: $\|m, n\|_X = \|m, n\|_2$
  - Secondary: $\|m, n\|_X > \|m, n\|_2$



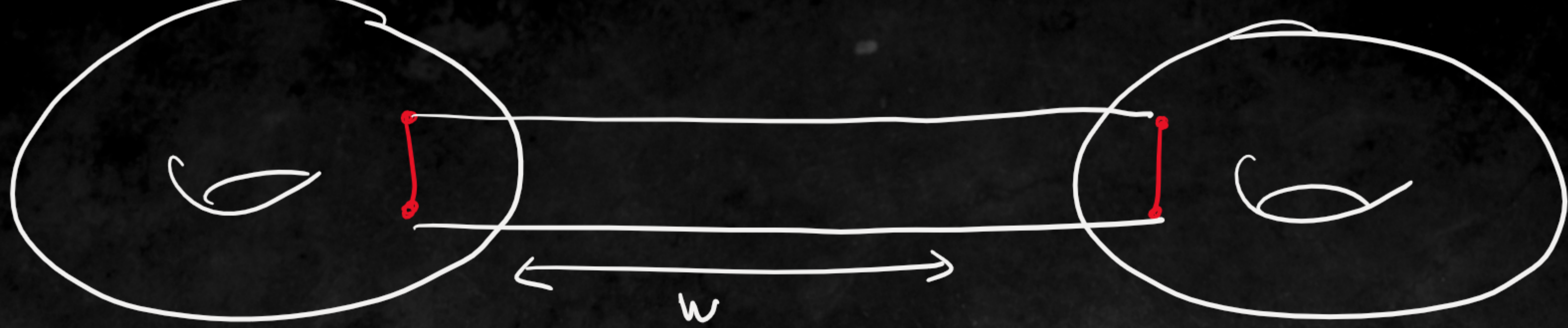
II) Glueing split tori

Let - X_1, X_2 be two identical copies of X.

- C be a flat cylinder of width w.



Let $\Sigma = X_1 \cup C \cup X_2$ be the glued surface.

Σ : 

- Σ is a genus 2 half-translation surface, with 4 singularities of cone angle 3π.
- $H_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}^4 \simeq H_1(X_1, \mathbb{Z}) \oplus H_1(X_2, \mathbb{Z})$

$$(m, n, p, q) \longleftrightarrow (m, n) \oplus (p, q)$$

Theorem: (M)

If the cylinder is long enough, i.e if $w > \rho$,
then $\|m, n, p, q\|_\Sigma = \|m, n\|_{X_1} + \|p, q\|_{X_2}$

→ Goal n°1

III) Counting simple homology classes

Recall: Given a surface S,

- $h \in H_1(S, \mathbb{Z})$ simple $\Longleftrightarrow$ h is minimized by a simple curve.
- $p_S(x) = \#\{ h \text{ simple}, \|h\|_S \leq x \}$
- For translation surfaces, p is quadratic.

What about half-translation surfaces?

III) Counting simple homology classes

Recall: Given a surface S,

- $h \in H_1(S, \mathbb{Z})$ simple $\Leftrightarrow$ h is minimized by a simple curve.
- $p_S(x) = \# \{ h \text{ simple}, \|h\|_S \leq x \}$
- For translation surfaces, p is quadratic.

What about half-translation surfaces?

Theorem: (M)

On Σ, if $w > \rho$ we have

$$p_\Sigma(x) \underset{x \to +\infty}{\sim} 8 \sum_{b=1}^{\lfloor 1/\rho \rfloor} \frac{\varphi(b)}{b} \cdot x \ln(x) + O(x)$$

where φ is Euler's totient function.

→ Goal n°2

Thank you