CENTRO DE CIENCIAS MATEMÁTICAS

Non extremely amenable subgroups of Big Mapping Class Groups.

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Throughout this talk, all considered surfaces will be connected, orientable, and (at least we talk about subsurfaces) without boundary.

We classify surfaces into two types according to their fundamental group. Let S be a surface, and $\pi_1(S)$ its fundamental group.

$$\text{We say that } S \text{ is of} = \begin{cases} \textbf{Finite type,} & \text{if } \pi_1(S) \text{ is finitely generated} \\ \textbf{Infinite type,} & \text{otherwise} \end{cases}$$

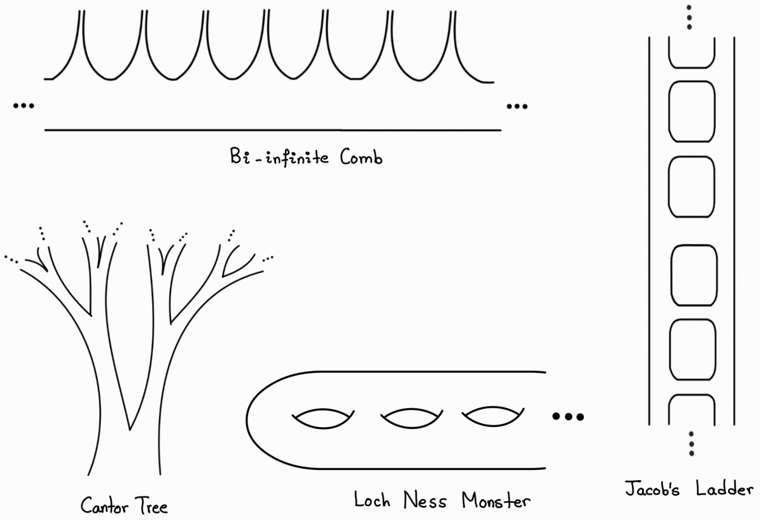
Let $E(S)$ and $E^g(S)$ denote the space of ends and the space of ends accumulated by genus of S respectively.

Theorem. (Kerékjártó-Richards)

The homeomorphism type of a surface S is determined by:

$$(g(S), E^g(S) \subset E(S))$$

And for any $A \subseteq B$ closed subsets of the Cantor Set, there exists a surface S with $E^g(S) = A$ and $E(S) = B$.


Bi-infinite Comb
Jacob's Ladder
Cantor Tree
Loch Ness Monster

Definition.

We define the **extended mapping class group** of S as

$$Map^{\pm}(S) := Homeo(S)/\text{Isotopy}.$$

And the **mapping class group** of S as

$$Map(S) := Homeo^{+}(S)/\text{Isotopy}.$$

$Map^{\pm}(S)$ and $Map(S)$ becomes topological group with the quotient topology arising from the compact-open topology on $Homeo(S)$ and $Homeo^{+}(S)$ respectively.

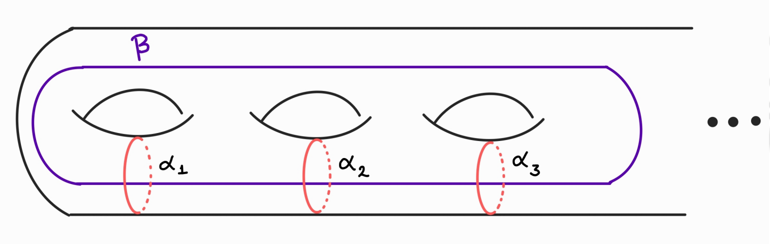
Definition. (The graph of curves)

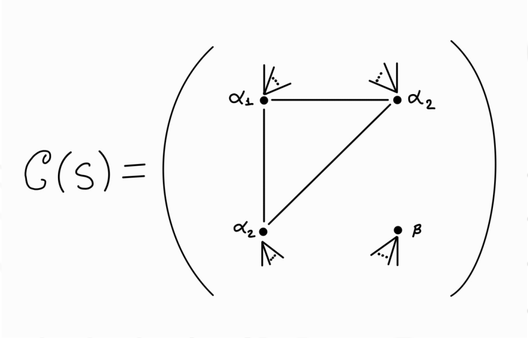
The graph of curves $\mathcal{C}(S)$ is the graph given by the following data:

Vertices. There is one vertex of $\mathcal{C}(S)$ for each isotopy class of essential simple closed curves in S.

Edges. There is an edge between any two vertices of $\mathcal{C}(S)$ that have disjoint representatives.

Let $Aut(\mathcal{C}(S))$ denote the group of simplicial automorphisms of $\mathcal{C}(S)$.


β
α1
α2
α3


C(S) =
α1
α2
α2
β

Theorem. (Hernández-Moralez-Valdez / Bavard, Dowdall, Rafi)

Let S an infinite-type surface, then

$$Map^{\pm}(S) \cong Aut(\mathcal{C}(S)).$$

Definition.

A topological group G is **amenable** if for every continuous action $G \curvearrowright X$ on a compact space X there exists $\mu : \mathcal{B}(X) \to [0,1]$ a $G-$invariant measure.

Definition.

A topological group G is called **extremely amenable** if for every action $G \curvearrowright X$ on a compact space X has a fix point, ie, there existe $x \in X$ such that $g \cdot x = x$ for each $g \in G$.

Theorem (Kechris, Pestov, Todorcevic)

Let F be a countable Fraïssé structure and $G = Aut(F)$, the following are equivalent:

1. G is extremely amenable.
2. $Age(F)$ consists of rigid structures and has the Ramsey Property.

Let S be an infinite-type surface and Γ be a closed subgroup of $Map^{\pm}(S)$.

We can use a process known as **Fraïsséfication** with respect to Γ to enrich the structure of $\mathcal{C}(S)$.

We obtain a new structure call $\mathcal{C}_{\Gamma}(S)$ with the following properties:

1. $\mathcal{C}_{\Gamma}(S)$ is a countable Fraïssé structure.
2. $\Gamma \cong Aut(\mathcal{C}_{\Gamma}(S))$

Theorem. (Hernández, M, Morales, Navarro, Pérez, Ramos, Randecker)

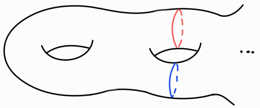
Let S be an infinite type surface and $\Gamma \leq Map^{\pm}(S)$ be a closed subgroup. If there exists an essential simple closed curve α on S and $g \in \Gamma$ such that $\{\alpha, g(\alpha)\}$ bounds (modulus isotopy) a finite-type subsurface of S, then $Age(C_\Gamma(S))$ does not have the Ramsey property.

Corollary.

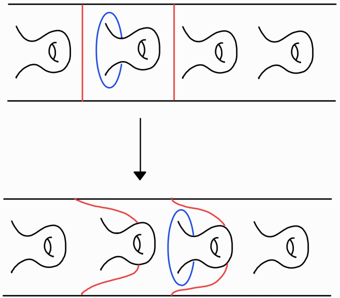
Let S be an infinite type surface and $\Gamma \leq Map^{\pm}(S)$ be a closed subgroup. If there exists an essential simple closed curve α on S and $g \in \Gamma$ such that $\{\alpha, g(\alpha)\}$ bounds (modulus isotopy) a finite-type subsurface of S, then Γ is not extremely amenable.

Let S be a infinite-type.

- If S has genus at least two, then $Map(S)$ and $Map^{\pm}(S)$ are not extremely amenable.



- If S has at leas two ends accumulates by genus, then $PMap(S)$ is not extremely amenable.



Work in progress

Let S be an infinite-type surface and $\Gamma \leq Map^{\pm}(S)$ be a closed subgroup. If there exists a multicurve $\mathcal{C}$ on S and $g \in \Gamma$ such that

$$\mathcal{C} \cup \{g(\alpha) : \alpha \in \mathcal{C}\}$$

bounds (modulus isotopy) a finite-type subsurface, then $\mathcal{C}_{\Gamma}(S)$ has no the Ramsey property and particullary, Γ is not extremely-amenable

References

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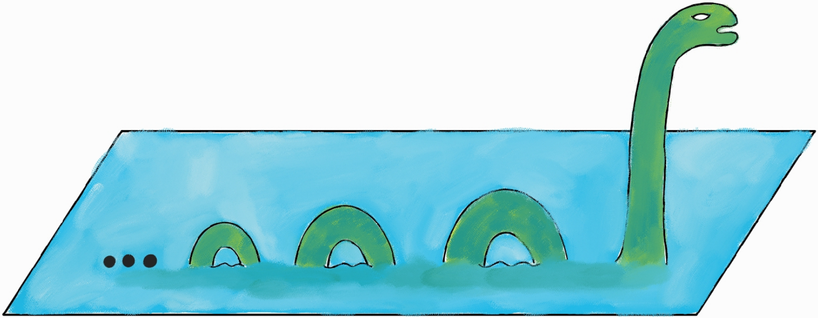
Thank You



We can think in the curve graph $\mathcal{C}(S)$ as the set of vertex with a relation that say when two vertices are joined by an edge.

Given $\Gamma \leq Map^{\pm}(S)$ a closed subgrup we can add relations to $\mathcal{C}(S)$ to enrich its structure.

For each $n \in \mathbb{N}$. Let R_n be the relation in $\mathcal{C}(S)^n$ given by

$$(a_1, \cdots, a_n) R_n (b_1, \cdots, b_n) \iff \exists h \in \Gamma : h(a_i) = b_i, \text{ for } 1 \leq i \leq n.$$

We denote $\mathcal{C}_\Gamma(S)$ as the curve graph $\mathcal{C}(S)$ together with the relations $\{R_n\}_{n \in \mathbb{N}}$.