An induction on interval exchange transformations with flips (IETFs) to study bi-triangle tiling billiards

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Interval Exchange Transformations

Definition of an IET

Let $\mathcal{D}$ be a union of interval(s) and circle(s).

Definition

An ***interval exchange transfromation*** *(IET) on $\mathcal{D}$ is a bijection over $\mathcal{D}$ on itself, which is piecewise a translation.*

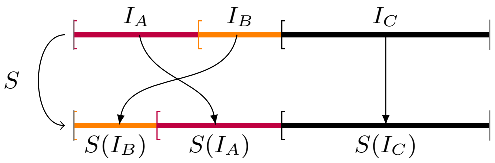

Figure: Example of an IET

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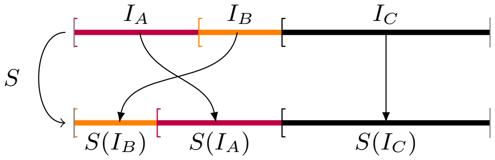


Figure: Example of an IET

An IET corresponds to the return map of the geodesic flow on some translation surface.

Definition of an IETF

Let $\mathcal{D}$ be a union of interval(s) and circle(s).

Definition

An ***interval exchange transformation with flips*** *(IETF) on $\mathcal{D}$ is a bijection over $\mathcal{D}$ on itself, which is piecewise continuous and obtained as the composition of a translation and an involution on each interval of continuity.*

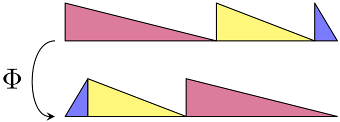


Figure: Example of an IETF

Generic behavior

Theorem (Keane - 1975)

Almost every IET without flip is minimal (that is, every of its orbits is dense).

Theorem (Nogueira - 1989)

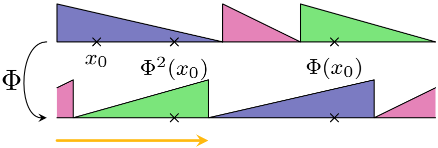
Almost every IET with flips admits a periodic point.

A specific class

We consider IETFs such that:

- The domain $\mathcal{D}$ is the union of circles with the same radius.
- All intervals are fliped.
- The permutation data has a certain type. For example:

$$\circ\begin{pmatrix} \overline{A} & \overline{B} & \overline{C} \\ \overline{A} & \overline{B} & \overline{C} \end{pmatrix}\circ$$



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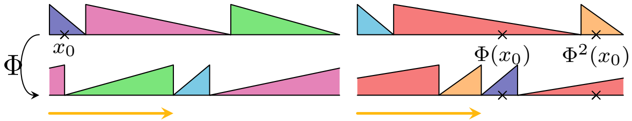

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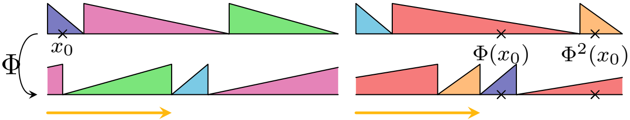

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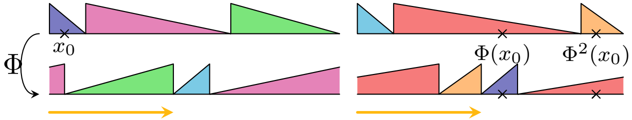

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We call them **tiling billiards IETFs**.

A result on this specific class

Theorem (J.+)

Let f be a tiling billiards IETF with permutation

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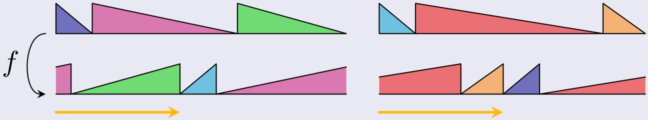

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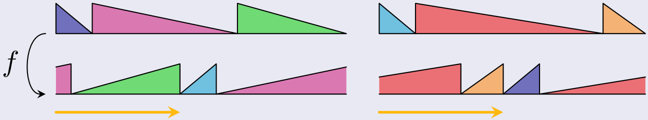

If f is minimal, then the lengths are the same on both circles.

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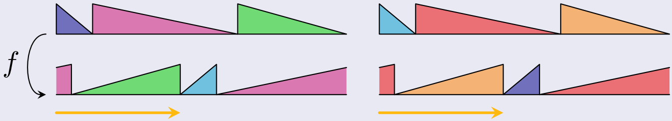

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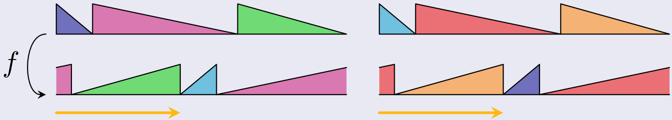
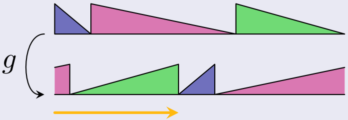


In this case, the IETF 

is minimal.

Tiling billiards

Triangle tiling billiards

Theorem (Baird-Smith,Davis,Fromm,Iyer and Hubert,Paris-Romaskevich)

For any triangle, for almost every initial direction, the trajectory is either periodic or at bounded distance from a line.

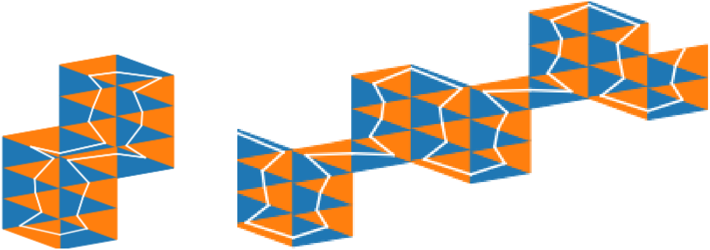


Figure: The two generic types of trajectories

Triangle tiling billiards

Theorem (Baird-Smith,Davis,Fromm,Iyer and Hubert,Paris-Romaskevich)

For any triangle, for almost every initial direction, the trajectory is either periodic or at bounded distance from a line.

Definition

We call a trajectory ***chaotic*** *if it is neither periodic nor at a bounded distance from a line.*

Studying tiling billiards with IETFs

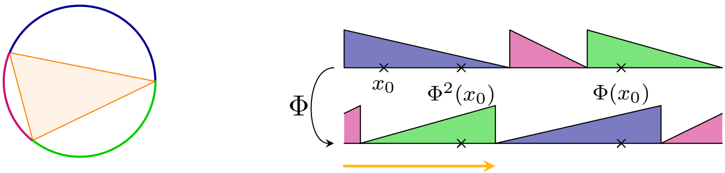


Figure: An IETF corresponding to a triangle tiling billiard.

A trajectory corresponds to the orbit of a point.

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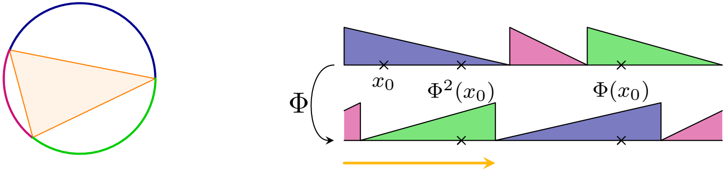

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(drift) periodic trajectory $\Leftrightarrow$ periodic orbit

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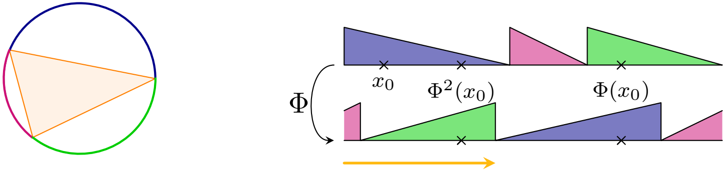


Figure: An IETF corresponding to a triangle tiling billiard.

A trajectory corresponds to the orbit of a point.

(drift) periodic trajectory	$\Leftrightarrow$	periodic orbit
chaotic trajectory	$\Leftrightarrow$	minimal IETF

Bi-triangle tiling billiards

Figure: Trajectories of a bi-triangle tiling billiard and corresponding IETF

Bi-triangle tiling billiards

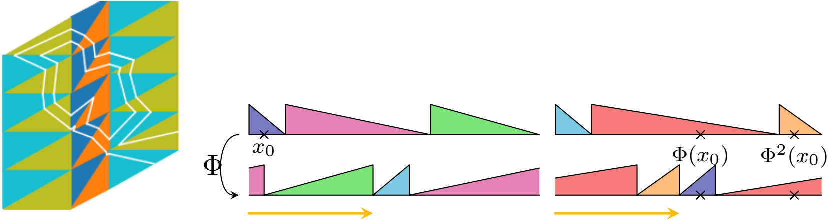


Figure: Trajectories of a bi-triangle tiling billiard and corresponding IETF

Strategy for the proof

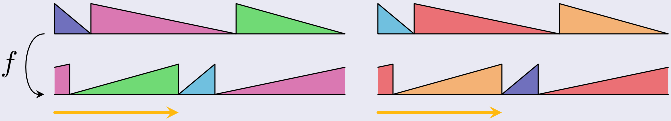
Recall of the statement

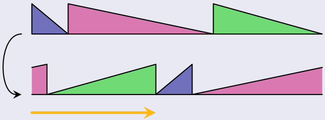
Theorem (J.+)

Let f be a tiling billiards IETF with permutation

$$\left[\begin{pmatrix}\overline{A_1} & \overline{B_1} & \overline{C_1}\\ \overline{A_2} & \overline{B_1} & \overline{C_1}\end{pmatrix}, \begin{pmatrix}\overline{A_2} & \overline{B_2} & \overline{C_2}\\ \overline{A_1} & \overline{B_2} & \overline{C_2}\end{pmatrix}\right].$$

If f is minimal, then the lengths are the same on both circles.



Moreover the IETF g  *is minimal.*

Outline

We define an induction and a condition (C) such that:

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Hence, if f is minimal, then we can process the induction on f for ever.

- If we can process the induction for ever, then the lengths on the two circles are the same.

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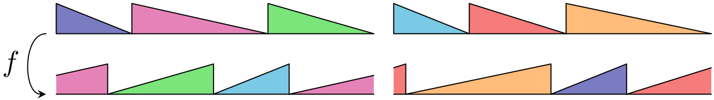
□

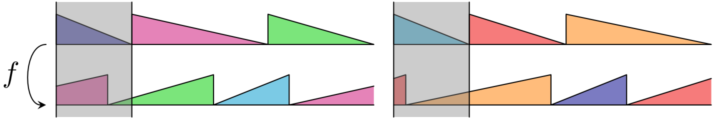
The induction on an example

We study the first return map of the IETF on a well chosen interval.



The induction on an example

We study the first return map of the IETF on a well chosen interval. Here, $l(A_1) + l(A_2) < l(B_1) + l(B_2)$, $l(C_1) + l(C_2)$.



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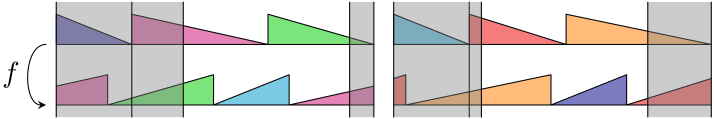

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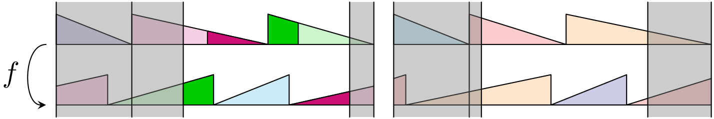

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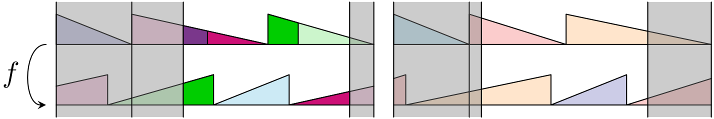

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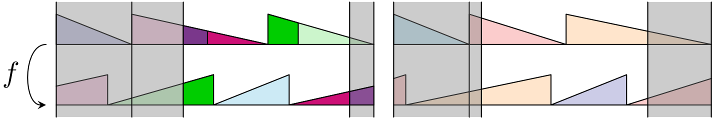

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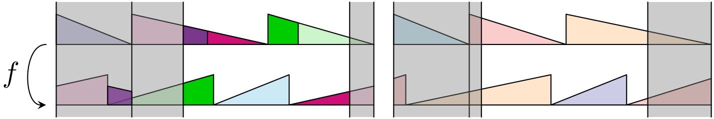

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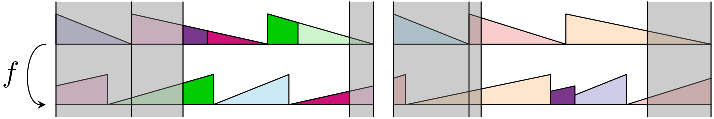

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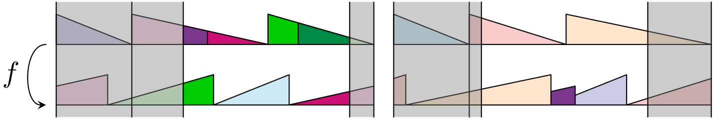

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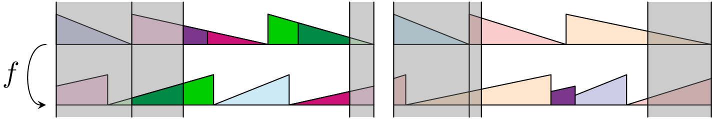

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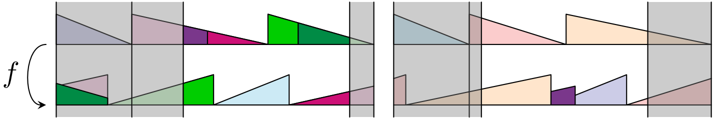

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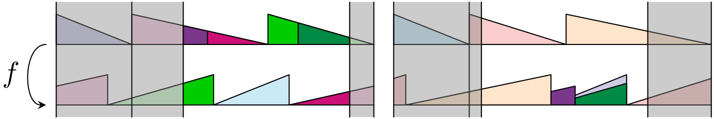

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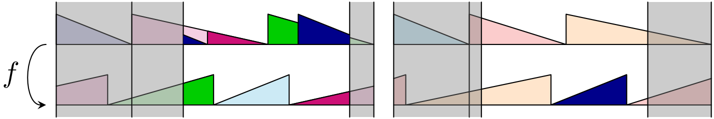

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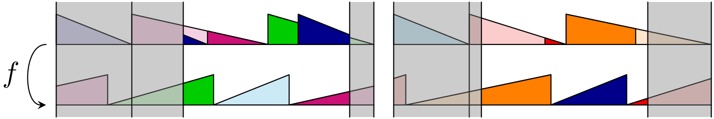

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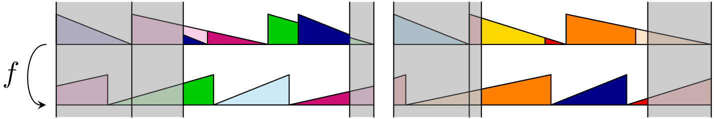

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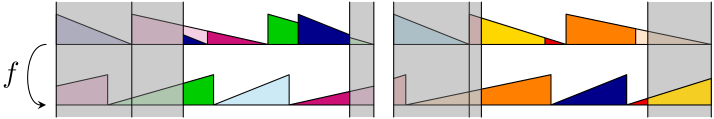

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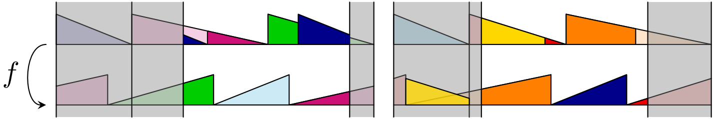

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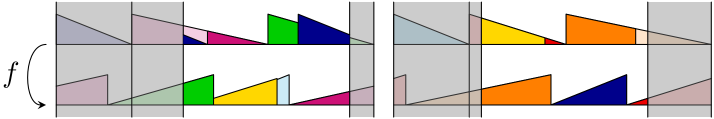

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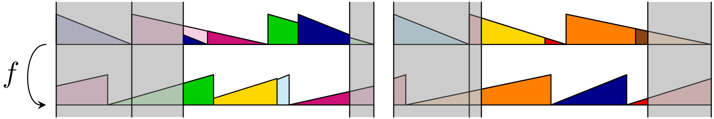

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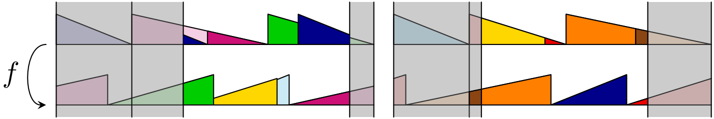

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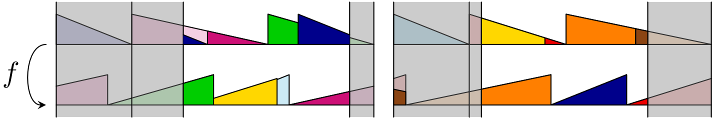

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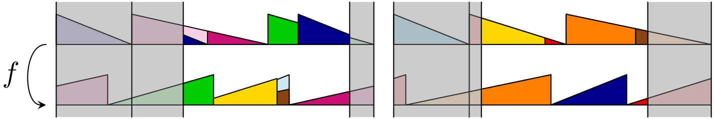

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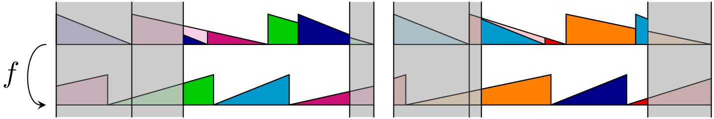

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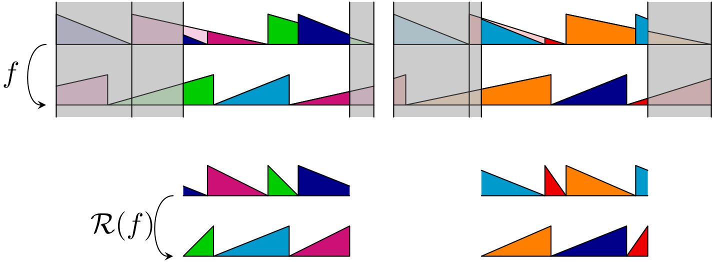

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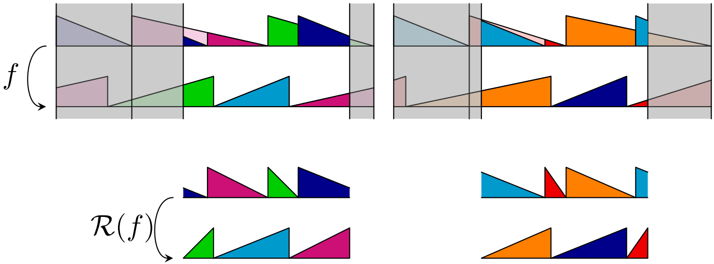
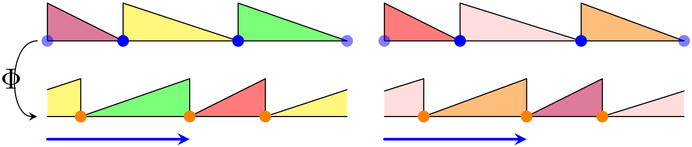


$$\left[\left(\circ\begin{matrix}\overline{A_1} & \overline{B_1} & \overline{C_1}\\ \overline{A_2} & \overline{B_1} & \overline{C_1}\end{matrix}\circ\right), \qquad \left(\circ\begin{matrix}\overline{A_2} & \overline{B_2} & \overline{C_2}\\ \overline{A_1} & \overline{B_2} & \overline{C_2}\end{matrix}\circ\right)\right].$$

Thank you!

The technical condition for the induction



Link with tiling billiards

Changing the frame of reference

Theorem (Baird-Smith, Davis, Fromm, Iyer)

Let **P** *be a polygon inscribed in a circle. In a (generalized)* **P***-tiling billiard, the trajectory is always subtended by the same chord of the circumcircle.*

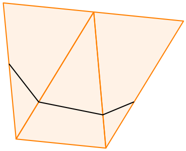


Figure: The parameter τ of the trajectory

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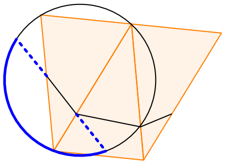

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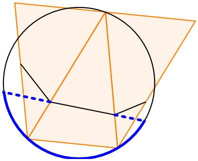

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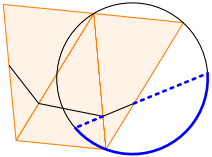

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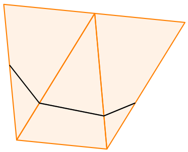

Figure: Changing the frame of reference

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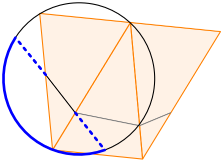
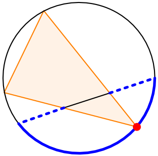

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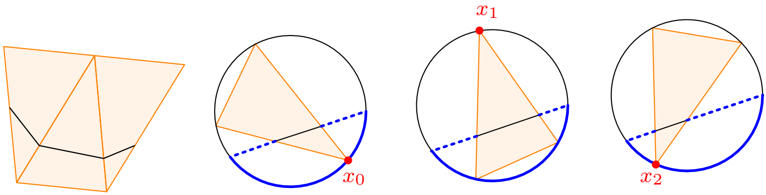

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