Quantization of linear pseudo-Anosov maps

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Outline

1. Introduction
2. Anosov map quantization
3. Pseudo-Anosov quantization (Work in progress)

Introduction

Definition

Let (S, Σ) be a closed (orientable) surface with singularities. A homeomorphism $\phi : S \to S$ is said to be *pseudo-Anosov* map there exist a pair of transverse measured foliations $(\mathcal{F}^s, \mu_s), (\mathcal{F}^u, \mu_u)$ such that

$$\phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s) \qquad \phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u)$$

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Examples :

- $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ the Arnold cat map, acting on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

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- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^m$ composition of Dehn twists.

Theorem (A. Fathi, M. Shub 1979)

The dynamical system $(S, \phi, \mu_s \otimes \mu_u)$ is isomorphic to a Bernoulli shift. In particular, a pseudo-Anosov map is strongly mixing.

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For any square integrable functions $f, g \in L^2(S, d\mu)$,

$$C_{f,g}(n) := \langle f, g \circ \phi^n \rangle \underset{n \to +\infty}{\to} \langle f, 1 \rangle \langle g, 1 \rangle.$$

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Question : can we know the speed of this convergence ?
Note that the study of the Fourier transform $\widehat{C}_{f,g}(\xi) = \sum_{n \in \mathbb{N}} e^{in\xi} C_{f,g}(n)$ is still pertinent for linear Anosov maps.

Definition

ϕ is said to have a Ruelle spectrum $\Lambda = (\lambda_i)$ if, for any $\varepsilon > 0$, the correlation function has an asymptotic expansion of the form

$$\int_S f \cdot g \circ \phi^n = \sum_{|\lambda_i| \geq \varepsilon} \sum_j \lambda_i^n n^j c_i(f, g) + o(\varepsilon^n)$$

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Remark :

- Λ can be reduced to 1 (Anosov map on the torus).

Theorem (F. Faure, S. Gouezel, E. Lanneau 2018)

Let ϕ be a linear pseudo-Anosov map on a genus g compact surface $(S\Sigma)$, with dilatation factor λ. Besides 1, let $\{\mu_1, \ldots, \mu_{2g-2}\}$ be the spectrum of ϕ^* in $\mathrm{Span}([dx], [dy])^\perp$ in $H^1(S)$. Then, for any i and $n \in \mathbb{N}^*$, $\lambda^{-n}\mu_i$ is a Ruelle resonance of multiplicity n.

Can the λ_i's be seen as eigenvalues of some operator ?

Prequantum line bundle

Let (S,ω) be a symplectic manifold and $\phi : S \to S$.
Under "nice" quantization hypotheses, we can consider a circle bundle $U(1) \to M \xrightarrow{\pi} S$ above S, and $A \in C^\infty(M, \Lambda^1(TM))$ a connection such that $\pi^*\omega = dA$. so that we can lift(non uniquely) $\phi : S \to S$ to $\widetilde{\phi} : M \to M$ so that :

- $\pi \circ \widetilde{\phi} = \phi \circ \pi$,
- $\widetilde{\phi}$ is equivariant : $\forall \theta \in U(1), \forall x \in M, \widetilde{\phi}(e^{i\theta}x) = e^{i\theta}\widetilde{\phi}(x)$,
- $\widetilde{\phi} \circ A = A$.

For $N \in \mathbb{N}$, we define the space of functions of N-Fourier mode

$$C^\infty_N(M) = \{u \in C^\infty(M) \mid \forall p \in M, \forall \theta \in \mathbb{R}, u(e^{i\theta}p) = e^{iN\theta}u(p)\}$$

In fact, this space naturally identifies with $C^\infty(S, L^{\otimes N})$ where L is an Hermitian complex line bundle, called the *prequantum line bundle*.
We also introduce the *quantum Hilbert space*

$$\mathcal{H}_N = \{u \in C^\infty_N(M) \mid \ u \text{ is invariant along a vertical foliation } \}.$$

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Then,

Theorem (S. De Bièvre, A. Bouzouina 1998)

On $\mathbb{T}^2$, the space $\mathcal{H}_N$ is a a Hilbert space of dimension N, with a basis given by Fourier modes supported on vertical cycles based at $\frac{k}{N}$ for $k = 0, \dots, N-1$.

Anosov maps

Let M be a smooth manifold, and $\phi : M \to M$ be a diffeomorphism. ϕ is said to be *Anosov* if, for some metric g, there exists $\lambda > 1$, and there is a ϕ-invariant decomposition of TM :

$$T_xM = E^s(x) \oplus E^u(x),$$

such that, for any $x \in M$,

$$\|d_x f_{|E^s}\| \leq \lambda^{-1},$$
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Example : the Arnold cat map on $\mathbb{T}^2$.

Anosov map quantization

Theorem (F. Faure, 2007)

For ϕ a Anosov symplectic map on $\mathbb{T}^2$, the space $\mathcal{H} = L^2(\mathbb{T}^2, L)$ of L^2 sections of $L \to \mathbb{T}^2$ decomposes as $\mathcal{H} = \mathcal{H}_N \otimes L^2(\mathbb{R})$, and the transfer operator as $\phi^* = \widehat{\phi} \otimes \phi^{(2)}$.
The operator $\widehat{\phi}$ is unitary, and $\phi^{(2)}$ is unitary equivalent to a trace class operator, which spectrum is given by

$$\exp(-\lambda_n), \quad \text{where } \lambda_n = \lambda\left(n + \frac{1}{2}\right), n \in \mathbb{N}.$$

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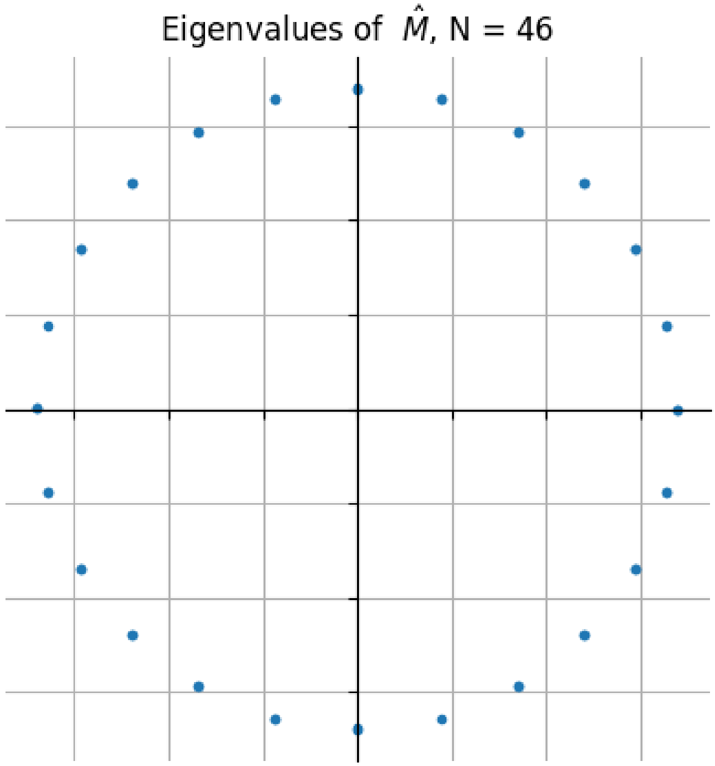
The proof uses ideas from quantum theory (representation of Heisenberg group).

What does this quantum spectrum look like ?



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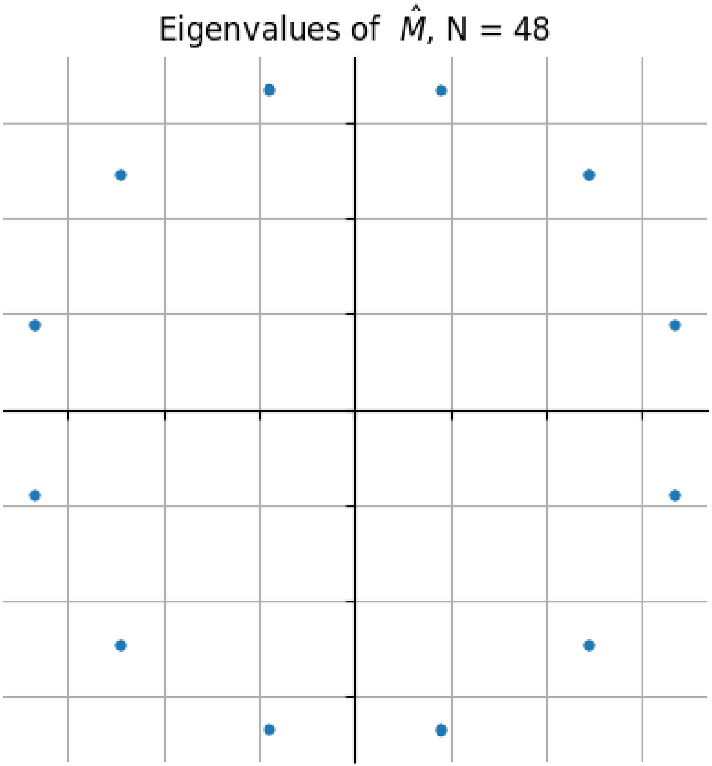

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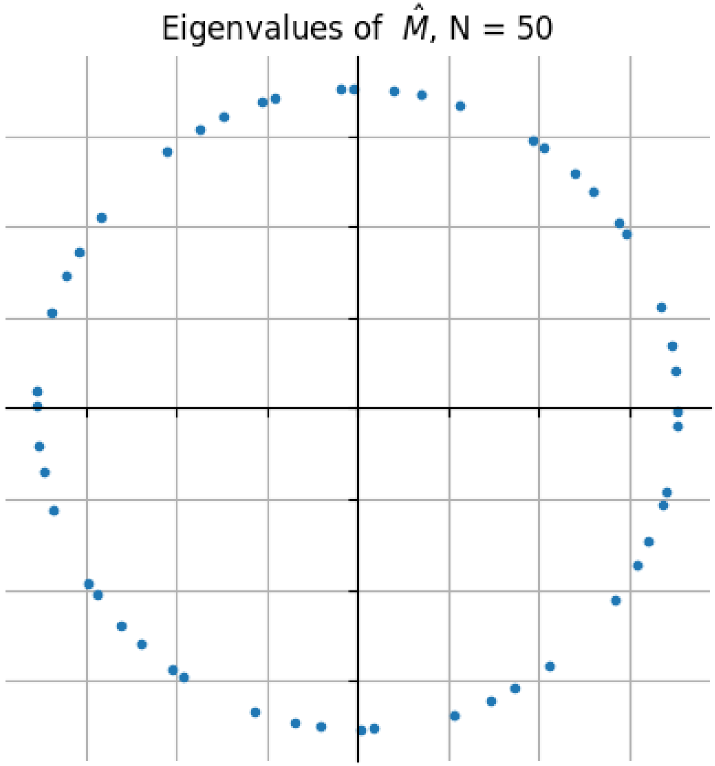

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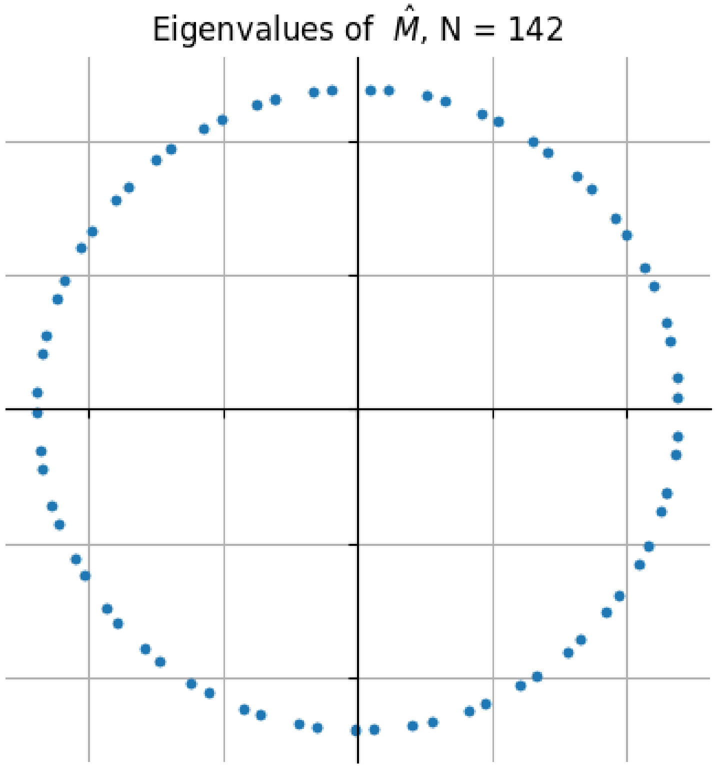

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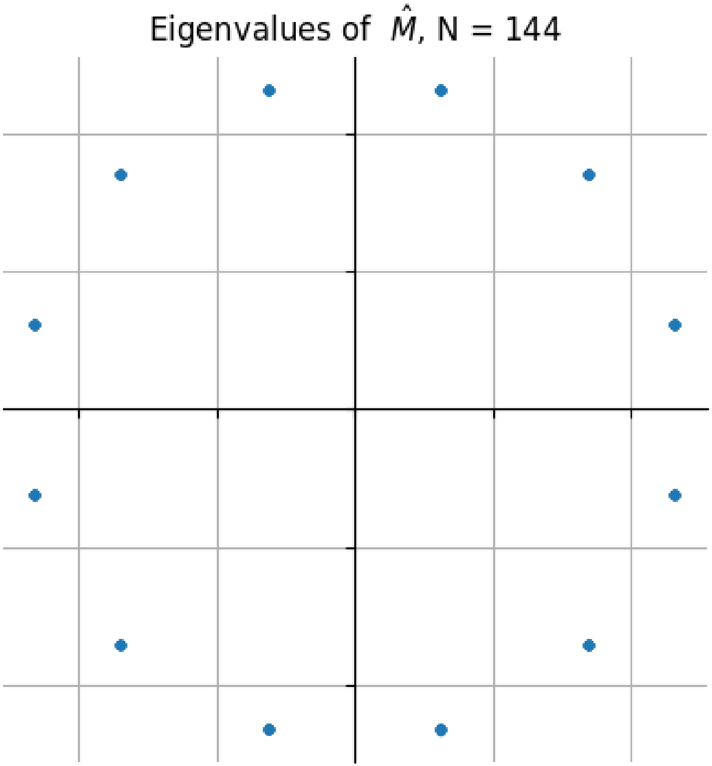

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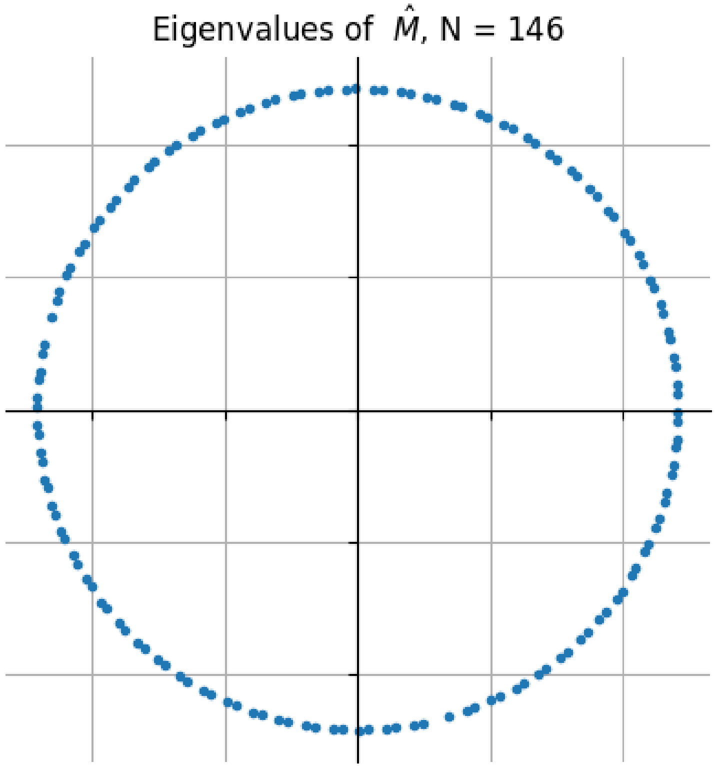

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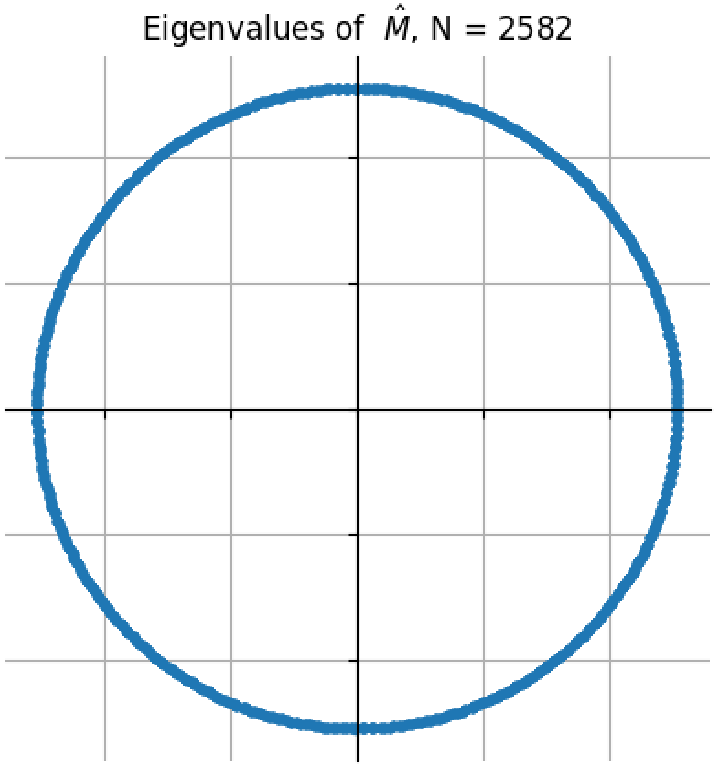

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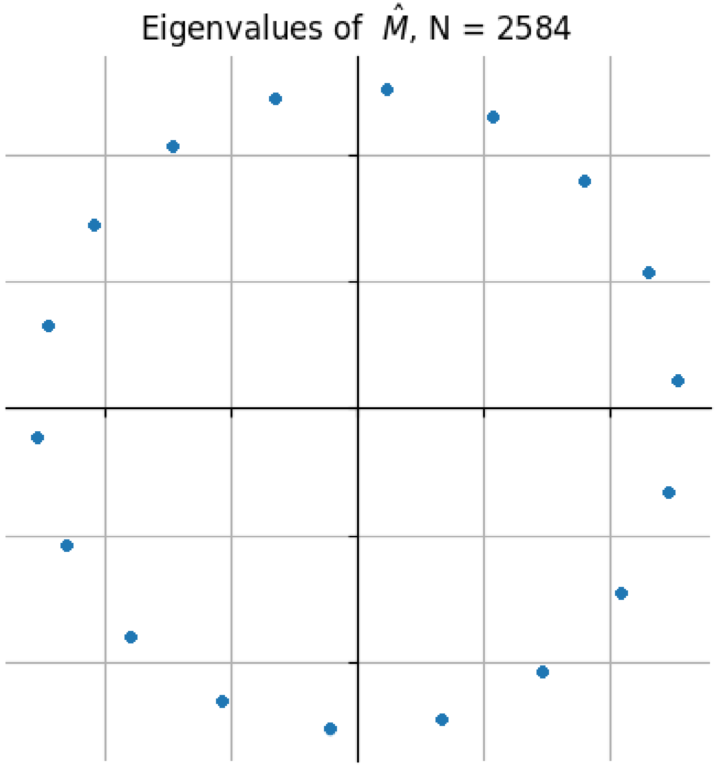

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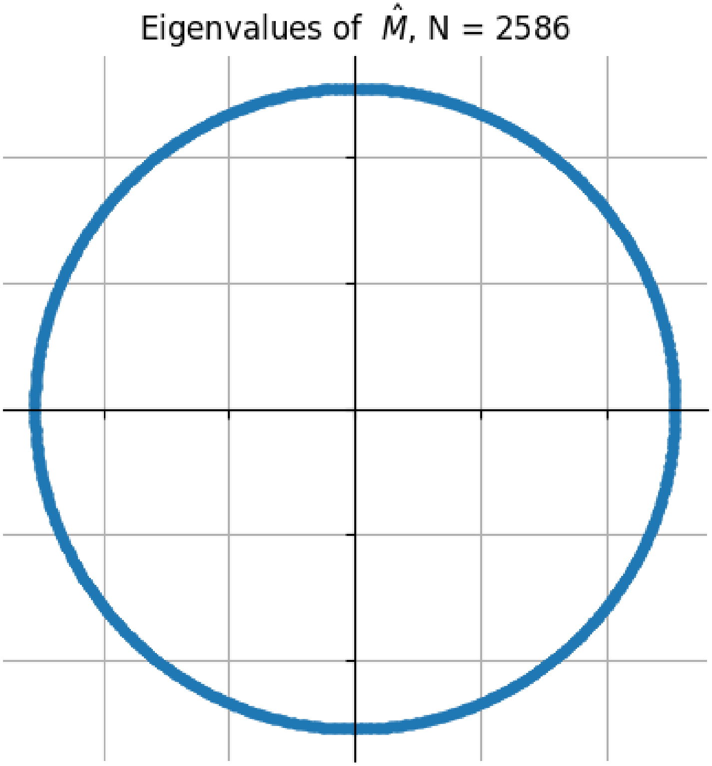

What is happening ? (scars !)

Every quantum map $\widehat{\phi_N} : \mathcal{H}_N \to \mathcal{H}_N$ has a quantum period $p(N)$, such that $\widehat{\phi}_N^{p(N)} = e^{i\varphi(N)} Id$.

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- $p(N)$ depends on N in an erratic way. It is known that for "almost all" integers, $p(N) \geq \sqrt{N}$.
- But there exists infinitely many integers N_i such that the period is "short" : $p(N_i) = 2\frac{\log N_i}{\lambda} + \mathcal{O}(1)$.

The work in progress (joint with F. Faure and E. Lanneau)

$\phi : (S, \Sigma) \to (S, \Sigma)$ pseudo-Anosov acting on a translation surface, for example $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

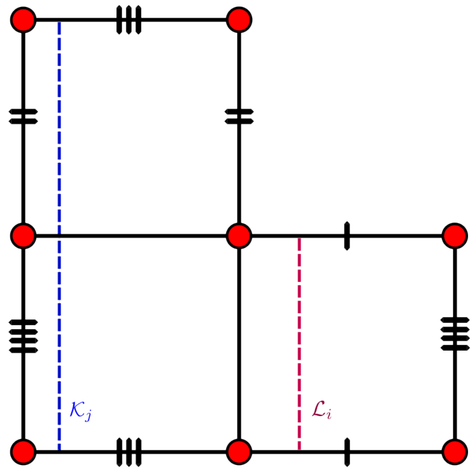

Figure 1: 3 square tiled surface, with a vertical foliation

Expected results

The Ruelle spectrum is known and related to the action in homology (cf F. Faure, S. Gouezel and E. Lanneau theorem).
Can we write it as the prequantum spectrum of some operator that is decomposed as $\phi = \widehat{\phi} \otimes \phi^{(2)}$? Do we have scars ?

Thank you for your attention !