Semisimplicity of the Kontsevich-Zorich cocycle for products of strata

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The KZ cocycle

The Moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g has the natural real (complex) *Hodge bundle* $H^1_{\mathbb{R}}$ ($H^1_{\mathbb{C}}$) of first cohomologies $H^1(S, \mathbb{R})$ ($H^1(S, \mathbb{C})$). This bundle can be pulled back to the Hodge bundle $E_{\mathbb{R}}$ ($E_{\mathbb{C}}$) over the space of flat surfaces $\mathcal{H}_g$. Identifying its lattices $H^1(S, \mathbb{Z})$ ($H^1(S, \mathbb{Z} \otimes i\mathbb{Z})$), one obtains the *Gauss–Manin connection* on this bundle. Parallel transport of the cohomology classes along the connection gives the *Kontsevich–Zorich cocycle* G^{KZ}.

For a surface $(M, \Sigma, \omega) \in \mathcal{H}(\mu)$, where $\mu = (\mu_1, \dots, \mu_n)$ gives the degrees of zeroes of ω, denote by $\gamma_1, \dots, \gamma_{2g}$ the basis of the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. The KZ cocycle is given by the linear action of $SL_2\mathbb{R}$ on *the periods*

$$\int_{\gamma_1} \omega, \; \dots, \; \int_{\gamma_{2g}} \omega.$$

The Hodge bundle

A *weight* n *Hodge structure on* $E_{\mathbb{C}}$ is a decomposition of the complexification

$$E_{\mathbb{C}} = \bigoplus_{p+q=n} E_{\mathbb{C}}^{p,q} \quad \text{such that} \quad E_{\mathbb{C}}^{p,q} = \overline{E_{\mathbb{C}}^{q,p}} \;\; \forall p+q=n.$$

The *Hodge filtration* is

$$F_i = \bigoplus_{p \geq i} E_{\mathbb{C}}^{p,q}.$$

The *Weil operator* acts by multiplication by $\sqrt{-1}^{\,p-q}$ on $E_{\mathbb{C}}^{p,q}$.
The *polarization* is a non-degenerate bilinear form $I(\ ,\ )$ such that $I(Cx, y)$ is symmetric and positive-definite.

The KZ cocycle is a *variation of Hodge structures* over the stratum.

The invariant subbundles

Let μ be an ergodic $SL_2\mathbb{R}$-invariant measure. A subbundle V is $SL_2\mathbb{R}$-*invariant* if it is measurable and invariant under the parallel transport along a.e. $SL_2\mathbb{R}$-orbit.

Problem: $E_{\mathbb{C}}^{p,q}$ are rarely $SL_2\mathbb{R}$-invariant.

How do Hodge decompositions of $SL_2\mathbb{R}$-invariant subbundles look?

Applications: affine invariant submanifolds (linear equations in local period coordinates), rigidity, zero Lyapunov exponents, etc.

Theorem (**Deligne semisimplicity**. Simion Filip, 2014)

E has a decomposition into $SL_2\mathbb{R}$-invariant components that are Hodge-orthogonal and respect Hodge structure:

$$V = \bigoplus V_i, \quad V_i = \bigoplus (V_i \cap E_{\mathbb{C}}^{p,q}).$$

Products of strata

Studying products of strata is interesting.

Motivating examples:
(1) Weak mixing of $(T,\, X,\, \mu)$ is equivalent to ergodicity of $(T \times T,\, X \times X,\, \mu \times \mu)$.
(2) The Rel foliation.

Acting on $\mathcal{H}_n = \mathcal{H}(\mu_1) \times \cdots \times \mathcal{H}(\mu_n)$ by a product of $SL_2\mathbb{R}$ is easy. What if we want $G \subsetneq \prod_{i=1}^{n} SL_2\mathbb{R}$?

If G projects on each of its factors surjectively but does not decompose into a product, then G is a conjugate of the diagonal subgroup:

$$G = \{g \times h_2^{-1} g h_2 \times h_n^{-1} g h_n \mid g \in SL_2\mathbb{R},\ \ h_2,\, \ldots,\, h_n \text{ are fixed}\}.$$

Products of strata

Theorem (B., expected on arXiv in May 2024)

Results concerning Hodge decompositions of invariant subbundles can be generalized for the action of $G \subsetneq \prod_{i=1}^{n} SL_2\mathbb{R}$ on $\mathcal{H}_n$ if G projects surjectively on each $SL_2\mathbb{R}$ component.

Main ideas:

(a) Relationship between the GM connection (flat structure) and the *Hodge connection* (direct sum of connections of the decomposition) $\Rightarrow$ sections flat along $SL_2\mathbb{R}$ orbits have flat (p, q)-components;

(b) The *algebraic hull* of the KZ cocycle is reductive (any invariant subspace has a complement);

(c) Decompose the subbundle and construct the variation of Hodge structures "by hand".

Thank you for your attention!