Products of Strata, Invariant Subbundles, and Hodge Theory

(Bonus: Integrable Systems)

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$SL_2\mathbb{R}$-action and Hodge decompositions

On a stratum $\mathcal{H}(\mu)$ of flat surfaces, the $SL_2\mathbb{R}$-action induces the Kontsevich–Zorich cocycle on the complex *Hodge bundle* H of cohomology groups.
Problem: the (p, q)-components $H^{p,q}$ are rarely $SL_2\mathbb{R}$-invariant.

How does an $SL_2\mathbb{R}$-invariant subbundle E behave geometrically?

Theorem (Simion Filip, 2014)

E has a decomposition into $SL_2\mathbb{R}$-invariant components that are Hodge-orthogonal and respect Hodge structure:

$$E = \bigoplus E_i, \quad E_i = \bigoplus (E_i \cap H^{p,q}).$$

Main tools: Hodge theory (unsurprisingly), algebraic hulls, Fürstenberg theorems.

Products of strata

Studying products of strata is interesting.

Motivating example: weak mixing of (T, X, μ) is equivalent to ergodicity of $(T \times T, X \times X, \mu \times \mu)$.

Acting on $\mathcal{H}_n = \mathcal{H}(\mu_1) \times \cdots \times \mathcal{H}(\mu_n)$ by a product of $SL_2\mathbb{R}$ is easy. What if we want $G \subsetneq \prod_{i=1}^n SL_2\mathbb{R}$?

Work in progress (B., expected on arXiv in May 2024)

Results concerning Hodge decompositions of invariant subbundles can be generalized for the action of G if it projects surjectively on each $SL_2\mathbb{R}$ component.

(Fun fact under the hood: properties of algebraic hulls generalise too.)

Bonus: Integrable systems

The Neumann system.
(Particle on S^n under a force)

$$\begin{cases} \dot{x}_k(t) = y_k(t), \\ \dot{y}_k(t) = g(t) + a_k\big) x_k(t). \end{cases}$$

The KdV hierarchy.
(Shallow waves)

$$\partial_k U_1 = U'_{k+1} = \frac{1}{4} U'''_k - U_1 U'_k - \frac{1}{2} U'_1 U_k,$$
$$\partial_i U_1 = 0 \quad \text{for } i > n.$$

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Very different systems... have the same solutions!

Theorem (Mumford 1984 + B. 2024)

There is an explicit change of variables x_i that relates solutions of the n-stationary KdV hierarchy and solutions of the Neumann system on S^n. This change is a 2^{n+1}-sheeted branched covering.