Short presentation: Deviation of ergodic integrals for Locally hamiltonian flows

School on flat surfaces and interactions

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Locally Hamiltonian flows

Let M be a smooth compact connected orientable surface of genus $g \geq 1$. Let $X : M \to TM$ be a smooth vector field with finitely many vanishing points and $\psi_{\mathbb{R}}$ be a flow preserving the measure $d\mu = V(x,y)dx \wedge dy$ (V:positive, smooth). We consider the set of fixed points of the flow $\psi_{\mathbb{R}}$ (isolate and finite).

Locally Hamiltonian flows $\psi_{\mathbb{R}}$ is a (local) solution to the Hamiltonian equation

$$\frac{dx}{dt} = \frac{\frac{\partial H}{\partial y}(x,y)}{V(x,y)}, \quad \frac{dy}{dt} = -\frac{\frac{\partial H}{\partial x}(x,y)}{V(x,y)}$$

where H is the corresponding Hamiltonian function on U_σ.

A fixed point σ is called (perfect) saddle if there exists a (singular) chart (x,y) in the neighborhood of U_σ such that $d\mu = V(x,y)dx \wedge dy$ and $H(x,y) = \Im(x+iy)^{m_\sigma}$ for $m_\sigma \geq 2$.

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Deviation of ergodic integrals

By Birkhoff ergodic theorem, for a.e $x \in M$ and $f \in L^1(M)$

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f \; d\mu \right| \to 0, \quad \text{as } T \to \infty. \tag{1}$$

For any smooth observable $f : M \to \mathbb{R}$ we are interested in understanding the asymptotics of the growth of ergodic integrals (*Oscillations of Birkhoff integrals*).

- One can find an exponent $0 < \lambda < 1$ such that for $\int_M f \; d\mu = 0$,

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Historical remarks

- A. Zorich proved the deviation of Birkhoff sums for interval exchange maps ('97). i.e
$$\limsup_{n\to\infty} \frac{\log|S_n(f)(x)|}{\log n} \le \frac{\nu_2}{\nu_1}.$$
- Kontsevich and Zorich conjectured that there would be a full deviation spectrum
$$0 < \frac{\nu_g}{\nu_1} < \cdots < \frac{\nu_2}{\nu_1} < 1$$
with corresponding filtration of the space of smooth functions.
- Forni proved the deviation of ergodic integrals for area-preserving flows ('02) based on his work on cohomological equation for compact surface of higher genus ('97).
- Marmi-Moussa-Yoccoz ('05) / Marmi-Yoccoz ('16) solved cohomological equations for interval exchange maps.
- A.Bufetov ('14) proved the deviation of ergodic integrals for translation flows and its limit theorem.
- K. Frączek - C.Ulcigrai / Frączek - K ('24) proved the deviation of ergodic integrals for (non-degenerate / degenerate) locally Hamiltonian flows.
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