The Exterior Calculus Discrete De Rham complex

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NEMESIS

New generation methods
for numerical simulations

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References for this presentation

- Finite Element Exterior Calculus [Arnold et al., 2006], [Arnold, 2018]
- Finite Element Systems [Christiansen and Gillette, 2016], [Christiansen and Hu, 2018]
- Virtual element complexes [Beirão da Veiga et al., 2016], [Beirão da Veiga et al., 2018]
- Discrete de Rham complexes [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023]
- Bridges VEM–DDR [Beirão da Veiga et al., 2022]
- Polytopal Exterior Calculus (PEC) [Bonaldi et al., 2023]
- PEC on manifolds [Droniou et al., 2024] → see M. Hanot's talk
- C++ open-source implementation available in the `HArDCore` library.

Outline

Outline

Two model problems: Stokes

- With $\Omega \subset \mathbb{R}^3$ connected, $\nu > 0$, and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads: Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(local equilibrium)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\textstyle\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

Two model problems: Magnetostatics

- For $\mu > 0$ and $\boldsymbol{J} \in \operatorname{\mathbf{curl}} \boldsymbol{H}(\operatorname{\mathbf{curl}};\Omega)$, the magnetostatics problem reads:
  Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu \boldsymbol{H} - \operatorname{\mathbf{curl}} \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\operatorname{\mathbf{curl}} \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\operatorname{\mathbf{curl}};\Omega) \times \boldsymbol{H}(\operatorname{div};\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \operatorname{\mathbf{curl}} \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{\mathbf{curl}};\Omega),\\
\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)
\end{aligned}
$$

Outline

De Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.

De Rham complex

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- Complex: image of an operator included in kernel of the next one.
- Key properties, depending on the topology of Ω and providing stability of PDE models:

$$\text{no “tunnels”}\ (b_1 = 0) \implies \mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl} \quad \text{(Stokes in curl-curl)}$$

$$\text{no “voids”}\ (b_2 = 0) \implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics, Stokes)}$$

De Rham complex

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- Key properties, depending on the topology of Ω and providing stability of PDE models:

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$$\text{no "voids" } (b_2 = 0) \implies \mathrm{Im}\,\textbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics, Stokes)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\textbf{curl}/\mathrm{Im}\,\textbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}/\mathrm{Im}\,\textbf{curl}$$

- **Emulating these properties is key for stable discretizations**

Outline

Crash course on alternating forms I

Select $k \in \mathbb{N}$.

- *k*-alternating form: $\omega : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ multilinear and fully antisymmetric;

$$\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k).$$

- $\mathrm{Alt}^k(\mathbb{R}^n)$ space of k-alternating forms ($= \{0\}$ if $k > n$).

- Exterior product: if $\omega \in \mathrm{Alt}^k(\mathbb{R}^n)$ and $\mu \in \mathrm{Alt}^\ell(\mathbb{R}^n)$, $\omega \wedge \mu \in \mathrm{Alt}^{k+\ell}(\mathbb{R}^n)$ defined by:

$$(\omega \wedge \mu)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+\ell}) := \sum_{\sigma \in \Sigma_{k,\ell}} \mathrm{sign}(\sigma)\, \omega(\boldsymbol{v}_{\sigma_1}, \ldots, \boldsymbol{v}_{\sigma_k})\, \mu(\boldsymbol{v}_{\sigma_{k+1}}, \ldots, \boldsymbol{v}_{\sigma_{k+\ell}}),$$

  with $\Sigma_{k,\ell} = \{\text{permutations } \sigma \text{ s.t. } \sigma_1 < \cdots < \sigma_k\,,\ \sigma_{k+1} < \cdots < \sigma_{k+\ell}\}$.

  Example: for 1-forms: $(\omega \wedge \mu)(\boldsymbol{v}_1, \boldsymbol{v}_2) = \omega(\boldsymbol{v}_1)\mu(\boldsymbol{v}_2) - \omega(\boldsymbol{v}_2)\mu(\boldsymbol{v}_1)$.

- Anti-symmetry: $\omega \wedge \mu = (-1)^{k\ell} \mu \wedge \omega$.

Crash course on alternating forms II

- Canonical basis of linear forms: $(\mathrm{d}x^i)_{i=1,\dots,n}$ s.t. $\mathrm{d}x^i(\boldsymbol{a}) = a_i$.
- Canonical basis of $\mathrm{Alt}^k(\mathbb{R}^n)$:

$$\{\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k} \ : \ \sigma \text{ s.t. } 1 \leq \sigma_1 < \cdots < \sigma_k \leq n\}.$$

  Example for $k = 2$: $\{\mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^1 \wedge \mathrm{d}x^3, \mathrm{d}x^2 \wedge \mathrm{d}x^3\}$.

- Hodge star operator: $\star : \mathrm{Alt}^k(\mathbb{R}^n) \to \mathrm{Alt}^{n-k}(\mathbb{R}^n)$ such that

$$\langle \star\omega, \mu \rangle \mathrm{vol} = \omega \wedge \mu \text{ for all } \mu \in \mathrm{Alt}^{n-k}(\mathbb{R}^n)$$

  where $\langle \cdot, \cdot \rangle$ the inner product on $\mathrm{Alt}^k(\mathbb{R}^n)$ for which the canonical basis is orthonormal.

- We have $\star(\star\omega) = (-1)^{k(n-k)}\omega$.

Crash course on alternating forms III

- Vector proxy of alternating form: if $n = 3$,

k	k-form	Scalar/vector proxy
0	$\omega = a$	$\omega_\sharp = a$
1	$\omega = a\mathrm{d}x^1 + b\mathrm{d}x^2 + c\mathrm{d}x^3$	$\omega_\sharp = (a, b, c)$
2	$\omega = a(\mathrm{d}x^2 \wedge \mathrm{d}x^3) - b(\mathrm{d}x^1 \wedge \mathrm{d}x^3) + c(\mathrm{d}x^1 \wedge \mathrm{d}x^2)$	$\omega_\sharp = (a, b, c)$
3	$\omega = a\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$	$\omega_\sharp = a$

Crash course on alternating forms IV

- Trace: if $\omega \in \mathrm{Alt}^k(\mathbb{R}^n)$ and V is a subspace of $\mathbb{R}^n$, $\mathrm{tr}_V\,\omega \in \mathrm{Alt}^k(V)$ such that

$$\mathrm{tr}_V\,\omega(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k) = \omega(i_V\boldsymbol{v}_1,\ldots,i_V\boldsymbol{v}_k) \text{ for all } \boldsymbol{v}_1,\ldots,\boldsymbol{v}_k \in V$$

  where $i_V : V \hookrightarrow \mathbb{R}^n$ embedding.

  In vector proxy: if $n = 3$ and V is a hyperplane of $\mathbb{R}^3$ with normal $\mathbf{n}_V$,

  - $k = 0$: $\mathrm{tr}_V\,\omega \leftrightarrow \omega_\sharp$.
  - $k = 1$: $\mathrm{tr}_V\,\omega \leftrightarrow \mathbf{n}_V \times (\omega_\sharp \times \mathbf{n}_V)$= **tangential projection** of $\omega_\sharp$ on V.
  - $k = 2$: $\mathrm{tr}_V\,\omega \leftrightarrow \omega_\sharp \cdot \mathbf{n}_V$= **normal component** of $\omega_\sharp$ to V (if $\mathbf{n}_V$ positively oriented).

Crash course on differential forms I

Ω domain of $\mathbb{R}^n$.

- Differential form: $\omega \in \Lambda^k(\Omega)$ if $\omega : \Omega \to \mathrm{Alt}^k(\mathbb{R}^n)$.

- Decomposition on basis: $\omega_x = \sum_\sigma a_\sigma(x)\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$.

  Various regularities:

  - $\omega \in L^2\Lambda^k(\Omega)$ if $a_\sigma \in L^2(\Omega)$.
  - $\omega \in \mathcal{P}_r\Lambda^k(\Omega)$ if $a_\sigma \in \mathcal{P}_r(\Omega)$.

- Exterior derivative: if $\omega \in C^1\Lambda^k(\Omega)$, $\mathrm{d}^k\omega$ is the $(k+1)$-form such that

$$\mathrm{d}^k\omega_x = \sum_\sigma \sum_{i=1}^n \frac{\partial a_\sigma}{\partial x_i}(x)\mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}.$$

  In vector proxy:

  - $k = 0$: $d^0\omega \leftrightarrow \mathbf{grad}\,\omega_\sharp$.
  - $k = 1$: $d^1\omega \leftrightarrow \mathbf{curl}\,\omega_\sharp$.
  - $k = 2$: $d^2\omega \leftrightarrow \mathrm{div}\,\omega_\sharp$.

Crash course on differential forms II

- We have $\mathrm{d}^k \circ \mathrm{d}^{k+1} = 0$.

- De Rham complex: with $H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) \,:\, \mathrm{d}^k\omega \in L^2\Lambda^{k+1}(\Omega)\}$.

$$\begin{array}{ccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega).
\end{array}$$

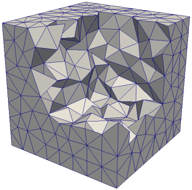
- Stokes formula: embeds all formulas for gradient, curl, divergence: if $\omega \in C^1\Lambda^k(\Omega)$ and $\nu \in C^1\Lambda^{n-k-1}(\Omega)$,

$$\int_\Omega \mathrm{d}^k\omega \wedge \mu = (-1)^{k+1} \int_\Omega \omega \wedge \mathrm{d}^{n-k-1}\mu + \int_{\partial\Omega} \mathrm{tr}_{\partial\Omega}\, \omega \wedge \mathrm{tr}_{\partial\Omega}\, \mu.$$

Outline

Outline

Global complex



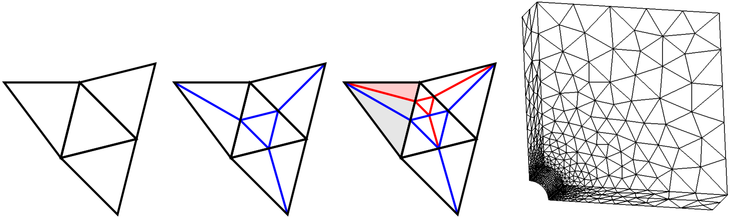
$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\quad \mathrm{d}^0 \quad} & V_h^1 & \xrightarrow{\quad \mathrm{d}^1 \quad} & V_h^2 & \xrightarrow{\quad \mathrm{d}^2 \quad} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) \end{array}$$

- Example: conforming $\mathcal{P}_k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].
- **Gluing only works on special meshes!**

Shortcomings



- Approach limited to conforming meshes with standard elements

$\Longrightarrow$ local refinement requires to trade mesh size for mesh quality

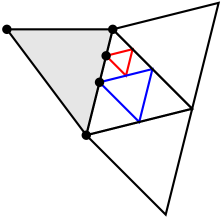
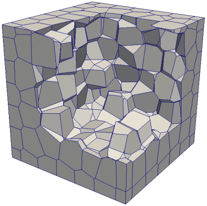
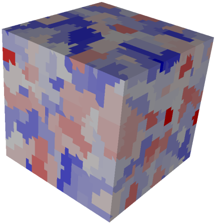
$\Longrightarrow$ complex geometries may require a large number of elements

$\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions

$\Longrightarrow$ significant increase of DOFs on hexahedral elements

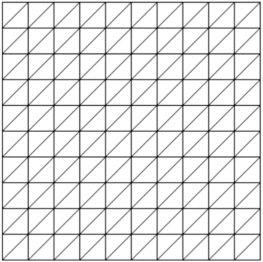
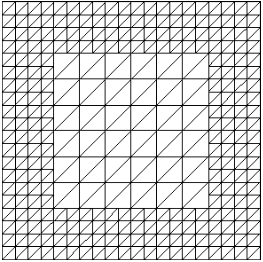
Polytopal meshes I

- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

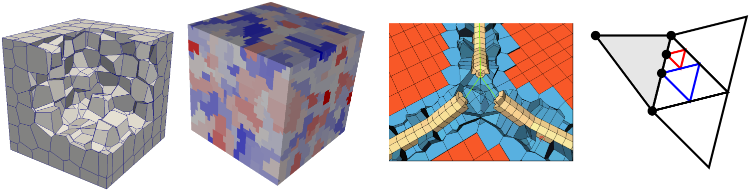
Polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathcal{P}_2$-($\mathcal{P}_1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

Outline

Domain and polytopal mesh



- Assume $\Omega \subset \mathbb{R}^n$ polytopal (polygon if $n = 2$, polyhedron if $n = 3, \ldots$)
- We consider a polytopal mesh $\mathcal{M}_h$ with flat d-cells, $0 \leq d \leq n$
- d-cells in $\mathcal{M}_h$ are collected in $\Delta_d(\mathcal{M}_h)$.

  When $n = 3$:

  - $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$: set of vertices
  - $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$: set of edges
  - $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$: set of faces
  - $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$: set of elements

We remove k from d^k

- Stokes formula: if $f \in \Delta_d(\mathcal{M}_h)$ and $(\omega, \mu) \in C^1\Lambda^k(f) \times C^1\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge \mathrm{d}\mu + \int_{\partial f} \mathrm{tr}_{\partial f}\, \omega \wedge \mathrm{tr}_{\partial f}\, \mu$$

- Inner product on $L^2\Lambda^k(f)$:

$$(\omega, \beta)_f = \int_f \omega \wedge \star\beta.$$

- Stokes formula with inner products: for $\omega \in H\Lambda^k(f)$ and $\beta \in H\Lambda^{k+1}(f)$,

$$(\mathrm{d}\omega, \beta)_f = (-1)^{k+1}(\omega, \delta\beta)_f + (\mathrm{tr}_{\partial f}\, \omega, \star^{-1}\, \mathrm{tr}_{\partial f}(\star\beta))_{\partial f}$$

$\delta = \star^{-1}\mathrm{d}\star$ (co-derivative).

A) Compute the projection of $\mathrm{d}\omega$ on $\mathcal{P}_r\Lambda^{k+1}(f)$?
Take $\beta \in \mathcal{P}_r\Lambda^{k+1}(f)$

$$(\mathrm{d}\omega, \beta)_f = (-1)^{k+1}(\omega, \delta\beta)_f + (\mathrm{tr}_{\partial f}\,\omega, \underbrace{\star^{-1}\,\mathrm{tr}_{\partial f}(\star\beta)}_{\in \mathcal{P}_r\Lambda^k(\partial f)})_{\partial f}$$

Requires:

- the projection of ω on $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$,
- the projection of $\mathrm{tr}_{\partial f}\,\omega$ on $\mathcal{P}_r\Lambda^k(\partial f)$.

A) Compute the projection of $\mathrm{d}\omega$ on $\mathcal{P}_r\Lambda^{k+1}(f)$?
Take $\beta \in \mathcal{P}_r\Lambda^{k+1}(f)$

$$(\mathrm{d}\omega, \beta)_f = (-1)^{k+1}(\omega, \delta\beta)_f + (\mathrm{tr}_{\partial f}\,\omega, \underbrace{\star^{-1}\,\mathrm{tr}_{\partial f}(\star\beta)}_{\in \mathcal{P}_r\Lambda^k(\partial f)})_{\partial f}$$

Requires:

- the projection of ω on $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$,
- the projection of $\mathrm{tr}_{\partial f}\,\omega$ on $\mathcal{P}_r\Lambda^k(\partial f)$.

B) Compute the projection of ω on $\mathcal{P}_r\Lambda^k(f)$?
Reverse the Stokes formula and take $\beta \in \mathcal{P}_{r+1}\Lambda^{k+1}(f)$:

$$(-1)^{k+1}(\omega, \delta\beta)_f = (\mathrm{d}\omega, \beta)_f - (\mathrm{tr}_{\partial f}\,\omega, \star^{-1}\,\mathrm{tr}_{\partial f}(\star\beta))_{\partial f}$$

$\Rightarrow$ $\mathrm{d}\omega$ and $\mathrm{tr}_{\partial f}\,\omega$ give the projection of ω on $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f) \subset \mathcal{P}_r\Lambda^k(f)$.

$\rightsquigarrow$ whole projection of ω on $\mathcal{P}_r\Lambda^k(f)$ additionally requires the projection of ω on a complement of $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$ in $\mathcal{P}_r\Lambda^k(f)$.

- *Conclusion*: to get the projection of $\mathrm{d}\omega$ and ω on $\mathcal{P}_r$, we need:
  - ω on $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$,
  - ω on a complement of $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$ in $\mathcal{P}_r\Lambda^k(f)$,
  - $\mathrm{tr}_{\partial f}\,\omega$ on $\mathcal{P}_r\Lambda^k(\partial f)$ *(can be reconstructed...)*

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- **Koszul complement**:

$$\mathcal{P}_r\Lambda^{d-k}(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{d-k-1}(f) \oplus \mathcal{K}_r^{d-k}(f).$$

Since $\delta = \star^{-1}\mathrm{d}\star$ and $\star : \Lambda^{d-k}(f) \to \Lambda^k(f)$ is an isomorphism,

$$\mathcal{P}_r\Lambda^k(f) = \delta\mathcal{P}_{r+1}\Lambda^{k+1}(f) \oplus \star^{-1}\mathcal{K}_r^{d-k}(f).$$

- **Trimmed space**:

$$\mathcal{P}_r^-\Lambda^{d-k}(f) := \mathrm{d}\mathcal{P}_r\Lambda^{d-k-1}(f) \oplus \mathcal{K}_r^{d-k}(f)$$

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  - ω on a complement of $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$ in $\mathcal{P}_r\Lambda^k(f)$,
  - $\mathrm{tr}_{\partial f}\,\omega$ on $\mathcal{P}_r\Lambda^k(\partial f)$ *(can be reconstructed...)*

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- *Conclusion, revisited*: we need ω on $\star^{-1}\mathcal{P}_r^-\Lambda^{d-k}(f)$.

DDR: Discrete $H\Lambda^k$ space

$$\underline{X}^k_{r,h} = \bigtimes_{d=k}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \star^{-1}\mathcal{P}_r^-\Lambda^{d-k}(f).$$

- Generic vector: $\underline{\omega}_h = (\omega_f)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k,n]}$.
- $\underline{X}^k_{r,f}$ and $\underline{\omega}_f$: restrictions to f and all $f'\in\Delta(f)$.

Space	$f_0\equiv V$	$f_1\equiv E$	$f_2\equiv F$	$f_3\equiv T$
$\underline{X}^0_{r,h}$	$\mathbb{R}=\mathcal{P}_r\Lambda^0(f_0)$	$\star^{-1}\mathcal{P}_r^-\Lambda^1(f_1)$	$\star^{-1}\mathcal{P}_r^-\Lambda^2(f_2)$	$\star^{-1}\mathcal{P}_r^-\Lambda^3(f_3)$
$\underline{X}^1_{r,h}$		$\mathcal{P}_r\Lambda^1(f_1)$	$\star^{-1}\mathcal{P}_r^-\Lambda^1(f_2)$	$\star^{-1}\mathcal{P}_r^-\Lambda^2(f_3)$
$\underline{X}^2_{r,h}$			$\mathcal{P}_r\Lambda^2(f_2)$	$\star^{-1}\mathcal{P}_r^-\Lambda^1(f_3)$
$\underline{X}^3_{r,h}$				$\mathcal{P}_r\Lambda^3(f_3)$
$(\underline{X}^0_{r,h})_\sharp$	$\mathbb{R}=\mathcal{P}_r(V)$	$\mathcal{P}_{r-1}(E)$	$\mathcal{P}_{r-1}(F)$	$\mathcal{P}_{r-1}(T)$
$(\underline{X}^1_{r,h})_\sharp$		$\mathcal{P}_r(E)$	$\mathcal{RT}_r(F)^\perp$	$\mathcal{RT}_r(T)$
$(\underline{X}^2_{r,h})_\sharp$			$\mathcal{P}_r(F)$	$\mathcal{N}_r(T)$
$(\underline{X}^3_{r,h})_\sharp$				$\mathcal{P}_r(T)$

DDR: Local discrete operators I

- Local spaces: if $f \in \Delta_d(\mathcal{M}_h)$ and $k \leq d$

$$\underline{X}^k_{r,f} = \Big(\bigtimes_{f' \in \Delta_k(f)} \mathcal{P}_r\Lambda^k(f') \Big) \times \Big(\bigtimes_{f' \in \Delta_{k+1}(f)} \star^{-1}\mathcal{P}^-_r\Lambda^1(f') \Big)$$
$$\times \cdots \times \Big(\bigtimes_{f' \in \Delta_{d-1}(f)} \star^{-1}\mathcal{P}^-_r\Lambda^{d-k-1}(f') \Big) \times \Big(\star^{-1}\,\mathcal{P}^-_r\Lambda^{d-k}(f) \Big).$$

- Local discrete exterior derivative and potential reconstruction

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f) \quad \text{and} \quad P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^k(f)$$

built using a hierarchical and recursive process (from lowest-dimensional f to highest-dimensional f).

DDR: Local discrete operators II

$$\underline{X}^k_{r,f} = \Big(\bigtimes_{f' \in \Delta_k(f)} \mathcal{P}_r \Lambda^k(f') \Big) \times \Big(\bigtimes_{f' \in \Delta_{k+1}(f)} \star^{-1} \mathcal{P}^-_r \Lambda^1(f') \Big)$$
$$\times \cdots \times \Big(\bigtimes_{f' \in \Delta_{d-1}(f)} \star^{-1} \mathcal{P}^-_r \Lambda^{d-k-1}(f') \Big) \times \Big(\star^{-1} \mathcal{P}^-_r \Lambda^{d-k}(f) \Big).$$

- $d = k$: then

$$\underline{X}^k_{r,f} = \mathcal{P}_r \Lambda^k(f)$$

and we set $P^k_{r,f} \underline{\omega}_f = \omega_f \in \mathcal{P}_r \Lambda^k(f)$.

Note: no d^k *exterior derivative on* k*-cells.*

DDR: Local discrete operators III

$$\underline{X}^k_{r,f} = \Big(\bigtimes_{f' \in \Delta_k(f)} \mathcal{P}_r \Lambda^k(f') \Big) \times \Big(\bigtimes_{f' \in \Delta_{k+1}(f)} \star^{-1} \mathcal{P}^-_r \Lambda^1(f') \Big) \\ \times \cdots \times \Big(\bigtimes_{f' \in \Delta_{d-1}(f)} \star^{-1} \mathcal{P}^-_r \Lambda^{d-k-1}(f') \Big) \times \Big(\star^{-1} \mathcal{P}^-_r \Lambda^{d-k}(f) \Big).$$

- $d = k + 1$: then

$$\underline{X}^k_{r,f} = \Big(\bigtimes_{f' \in \Delta_k(f)} \mathcal{P}_r \Lambda^k(f') \Big) \times \Big(\star^{-1} \mathcal{P}^-_r \Lambda^1(f) \Big).$$

  with $\mathcal{P}^-_r \Lambda^1(f) = \mathrm{d}\mathcal{P}_r \Lambda^0(f) \oplus \mathcal{K}^1_r(f)$.

  - Build $\mathrm{d}^k_{r,f} \underline{\omega}_f$ from $P^k_{r,\partial f} \underline{\omega}_{\partial f}$ and the component of ω_f on $\star^{-1} \mathrm{d}\mathcal{P}_r \Lambda^0(f)$.
  - Build $P^k_{r,f} \underline{\omega}_f$ from $\mathrm{d}^k_{r,f} \underline{\omega}_f$ and ω_f on $\star^{-1} \mathcal{K}^1_r(f)$.

- etc.

DDR: Local discrete operators IV

Formulas:

- Define $\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^{k+1}(f)$ such that, for all $\mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu.$$

- Define $P^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^k(f)$ using $\mathcal{P}_r\Lambda^{d-k}(f) = \mathrm{d}\mathcal{K}^{d-k-1}_{r+1}(f) \oplus \mathcal{K}^{d-k}_r(f)$:
  - For all $\mu \in \mathcal{K}^{d-k-1}_{r+1}(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

  - For all $\nu \in \mathcal{K}^{d-k}_r(f)$:

$$\int_f P^k_{r,f}\underline{\omega}_f \wedge \nu = \int_f \omega_f \wedge \nu.$$

Consistency

Interpolator: $\underline{I}_{r,h}^k : C^0\Lambda^k(\overline{\Omega}) \to \underline{X}_{r,h}^k(\mathcal{M}_h)$ such that

$$\underline{I}_{r,h}^k\omega = (\star^{-1}\pi_{r,f}^{-,d-k}(\star\,\mathrm{tr}_f\,\omega))_{f\in\Delta_d(\mathcal{M}_h),\,d\in[k,n]}.$$

Theorem

For all integers $0 \le k \le d \le n$ and all $f \in \Delta_d(\mathcal{M}_h)$, it holds

- *Polynomial consistency:*

$$P_{r,f}^k\underline{I}_{r,f}^k\omega = \omega \qquad \forall\omega \in \mathcal{P}_r\Lambda^k(f),$$
$$\mathrm{d}_{r,f}^k\underline{I}_{r,f}^k\omega = \mathrm{d}\omega \qquad \forall\omega \in \mathcal{P}_{r+1}^-\Lambda^k(f) \quad \text{if } d \ge k+1.$$

- *Smooth functions: if $\omega \in C^\infty\Lambda^k(f)$,*

$$\|P_{r,f}^k\underline{I}_{r,f}^k\omega - \omega\|_{L^2\Lambda^k(f)} \le C_\omega h_f^{r+1},$$
$$\|\mathrm{d}_{r,f}^k\underline{I}_{r,f}^k\omega - \mathrm{d}\omega\|_{L^2\Lambda^{k+1}(f)} \le C_\omega h_f^{r+1} \quad \text{if } d \ge k+1.$$

Global discrete exterior derivative and DDR complex

- Global discrete exterior derivative $\underline{\mathrm{d}}_{r,h}^{k} : \underline{X}_{r,h}^{k} \to \underline{X}_{r,h}^{k+1}$ s.t.

$$\underline{\mathrm{d}}_{r,h}^{k}\underline{\omega}_{h} \coloneqq \left(\star^{-1} \pi_{r,f}^{-,d-k-1}(\star \mathrm{d}_{r,f}^{k}\underline{\omega}_{f}) \right)_{f \in \Delta_{[k+1\dots n]}(\mathcal{M}_{h})}$$

- The DDR sequence then reads

$$\underline{X}_{r,h}^{0} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{0}} \underline{X}_{r,h}^{1} \longrightarrow \cdots \longrightarrow \underline{X}_{r,h}^{n-1} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{n-1}} \underline{X}_{r,h}^{n} \longrightarrow \{0\}$$

Cohomology

Theorem (Cohomology of the Discrete de Rham complex)

The DDR sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all k,

$$\mathrm{Ker}\,\underline{\mathrm{d}}_{r,h}^{k}/\mathrm{Im}\,\underline{\mathrm{d}}_{r,h}^{k-1} \cong \mathrm{Ker}\,\mathrm{d}^{k}/\mathrm{Im}\,\mathrm{d}^{k-1}.$$

Discrete L^2-products

- Discrete L^2-product $(\cdot,\cdot)_{k,h} : \underline{X}^k_{r,h} \times \underline{X}^k_{r,h} \to \mathbb{R}$:

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} \left(\int_f P^k_{r,f} \underline{\omega}_f \wedge \star P^k_{r,f} \underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \right)$$

  with $s_{k,f} : \underline{X}^k_{r,f} \times \underline{X}^k_{r,f} \to \mathbb{R}$ a stabilisation that satisfies

$$s_{k,f}(\underline{I}^k_{r,f}\omega, \underline{\mu}_f) = 0 \qquad \forall \omega \in \mathcal{P}_r\Lambda^k(f).$$

- Numerical schemes are obtained replacing spaces, differential operators, and L^2-products with their discrete counterparts. Yield stable schemes, with $\mathcal{O}(h^{k+1})$ rates of convergence in energy norm. [Di Pietro and Droniou, 2021, Beirão da Veiga et al., 2022, Droniou and Qian, 2023, Di Pietro and Droniou, 2023, Di Pietro and Droniou, 2022]

Conclusion

- Polytopal exterior calculus: framework for discrete polytopal complexes of arbitrary order, in the langage of differential forms.

  $\rightsquigarrow$ Unifies the analysis of all operators.

  $\rightsquigarrow$ Also gives discretisation method for PDEs, cf. Marien Hanot's talk.

- Consistency and same cohomology as the continuous de Rham complex.

  Ensures accuracy and robustness of schemes.

- Ongoing work: Poincaré inequalities, analysis tools (adjoint consistency, etc.).

- Notes and series of introductory lectures to DDR (vector proxy form):

  https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

Thank you!

The ERC Synergy NEMESIS project is hiring PhDs and post-docs.
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A little bit more...

The Virtual Element construction

General ideas

- Inspired by the VEM complex in vector proxy of [Beirão da Veiga et al., 2017, Beirão da Veiga et al., 2018, Beirão da Veiga et al., 2022].

  The exterior calculus construction in [Bonaldi et al., 2023] is done without virtual functions, in a fully discrete fashion.

- Some of the polynomial components in the discrete spaces represent projections of exterior derivatives (not just of the forms).
- Components on k- and $(k+1)$-cells play a different role to the other ones.
- Construction not hierarchical, construction of $\mathrm{d}^k_{r,h}$ and $P^k_{r,f}$ not intertwined. $\rightsquigarrow$ *larger spaces*.
- Stokes formula only used on the lowest-dimensional mesh entities.
- Similar consistency and cohomology properties.

Comparison DDR–VEM–RTN

k	Vector proxy	$r = 0$	$r = 1$	$r = 2$
0	$H^1(T)$	$4 \diamond 9 \diamond 4$	$15 \diamond 26 \diamond 10$	$32 \diamond 50 \diamond 20$
1	$\boldsymbol{H}(\mathbf{curl}; T)$	$6 \diamond 14 \diamond 6$	$28 \diamond 47 \diamond 20$	$65 \diamond 98 \diamond 45$
2	$\boldsymbol{H}(\mathrm{div}; T)$	$4 \diamond 7 \diamond 4$	$18 \diamond 26 \diamond 15$	$44 \diamond 59 \diamond 36$
3	$L^2(T)$	$1 \diamond 1 \diamond 1$	$4 \diamond 4 \diamond 4$	$10 \diamond 10 \diamond 10$

Table: Tetrahedron: dimensions of the local spaces in the DDR $\diamond$ VEM $\diamond$ RTN.

k	Vector proxy	$r = 0$	$r = 1$	$r = 2$
0	$H^1(T)$	$8 \diamond 15 \diamond 8$	$27 \diamond 42 \diamond 27$	$54 \diamond 78 \diamond 64$
1	$\boldsymbol{H}(\mathbf{curl}; T)$	$12 \diamond 22 \diamond 12$	$46 \diamond 69 \diamond 54$	$99 \diamond 138 \diamond 144$
2	$\boldsymbol{H}(\mathrm{div}; T)$	$6 \diamond 9 \diamond 6$	$24 \diamond 32 \diamond 36$	$56 \diamond 71 \diamond 108$
3	$L^2(T)$	$1 \diamond 1 \diamond 1$	$4 \diamond 4 \diamond 8$	$10 \diamond 10 \diamond 27$

Table: Hexahedron: dimensions of the local spaces in the DDR $\diamond$ VEM $\diamond$ RTN.

Comparison of *serendipity* DDR–VEM vs. RTN

k	Vector proxy	$r=0$	$r=1$	$r=2$
0	$H^1(T)$	4 ◊ 4 ◊ 4	10 ◊ 10 ◊ 10	20 ◊ 20 ◊ 20
1	$\boldsymbol{H}(\mathbf{curl};T)$	6 ◊ 9 ◊ 6	23 ◊ 31 ◊ 20	53 ◊ 68 ◊ 45
2	$\boldsymbol{H}(\mathrm{div};T)$	4 ◊ 7 ◊ 4	18 ◊ 26 ◊ 15	44 ◊ 59 ◊ 36
3	$L^2(T)$	1 ◊ 1 ◊ 1	4 ◊ 4 ◊ 4	10 ◊ 10 ◊ 10

Table: Tetrahedron: dimensions of the local spaces in the sDDR ◊ sVEM ◊ RTN.

k	Discrete space	$r=0$	$r=1$	$r=2$
0	$H^1(T)$	8 ◊ 8 ◊ 8	20 ◊ 20 ◊ 27	32 ◊ 32 ◊ 64
1	$\boldsymbol{H}(\mathbf{curl};T)$	12 ◊ 15 ◊ 12	39 ◊ 47 ◊ 54	77 ◊ 92 ◊ 144
2	$\boldsymbol{H}(\mathrm{div};T)$	6 ◊ 9 ◊ 6	24 ◊ 32 ◊ 36	56 ◊ 71 ◊ 108
3	$L^2(T)$	1 ◊ 1 ◊ 1	4 ◊ 4 ◊ 8	10 ◊ 10 ◊ 27

Table: Hexahedron: dimensions of the local spaces in the sDDR ◊ sVEM ◊ RTN.