Modeling Multiphase Multicomponent Porous Flows

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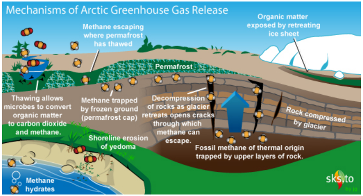
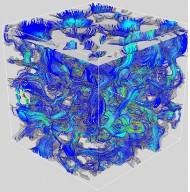




Application: Release of CH_4 and CO_2 as permafrost melts.

Computational Limitations:

- A scalar equation to represent the balance of mass for each conserved component (species or molecule).
- A scalar equation to represent the macroscopic balance of energy if the system is not isothermal.

Momentum & energy balances, chemistry, etc. are all “modeled”.

Macroscopic Model:

- There exist N_c Components which are conserved.
- The components combine to form N_p immiscible Phases

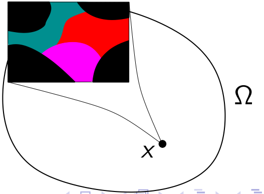
Component Densities: $\Omega \subset \mathbb{R}^d$

- m_c = mass of component c per unit volume of Ω
- $M_{c\pi}$ = mass of component c in phase π per unit volume of Ω

$$m_c = \sum_{\pi=1}^{N_p} M_{c\pi} \qquad \text{write} \qquad \mathbf{m} = M\mathbf{1}$$

Notation:

- $\mathbf{m} = (m_1, \ldots, m_c, \ldots, m_{N_c})$
- $\mathbf{1} = (1, 1, \ldots, 1)$



Phase Densities: $\boldsymbol{\rho} = M^\top \mathbf{1}$

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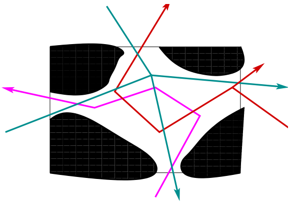
Balances of Mass: $\partial_t m_c + div\left(\sum_{\pi=1}^{N_p} M_{c\pi}\mathbf{v}_\pi + \mathbf{h}_c\right) = 0.$

Constraint: $\mathbf{m} = M\mathbf{1}$

Phase Velocities: $\{\mathbf{v}_1, \ldots, \mathbf{v}_{N_p}\}$

Mass Averages: $\mathbf{v}_\pi = \frac{\int_{V_\pi} \rho \mathbf{v}}{\int_V \rho}$

Mass Diffusion: $\{\mathbf{h}_1, \ldots, \mathbf{h}_{N_c}\}$



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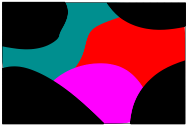
Postulate: At the pore scale, components form Immiscible, Homogeneous phases in Thermodynamic Equilibrium.

(transport time scale >> time to equiliberate)



- Ensemble Free Energy: (rigid, chemically inert medium)

$$\Psi(\mathbf{s}, M) = \sum_{\pi=1}^{N_p} \Psi_\pi(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}, \theta) + \Upsilon(\mathbf{s}, \theta)$$

  - Saturations: s_π volume of phase π per unit volume of Ω
  - Interfacial Energy: $\Upsilon(\mathbf{s})$ (surface tension/wetting energy)
  - Homogeneity:

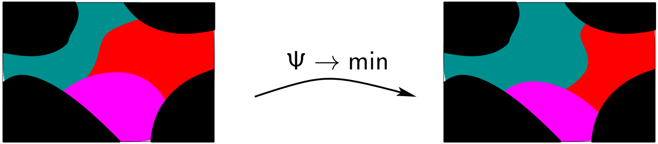
$$\lambda\,\Psi_\pi(\mathcal{V}_\pi, \{\mathcal{M}_{c\pi}\}_{c=1}^{N_c}) = \Psi_\pi(\lambda\mathcal{V}_\pi, \{\lambda\mathcal{M}_{c\pi}\}_{c=1}^{N_c}), \quad \lambda \geq 0$$

  - Ideal Gas:

$$\Psi(\mathcal{V}, \mathcal{M}; \theta) = k\theta\,\mathcal{M}\left(c + \ln\left(\frac{\mathcal{M}}{\mathcal{V}}\right) - \ln\left((const)\theta^c\right)\right).$$

Classical Thermodynamics

Canonical Ensemble (Isothermal System)



- Ensemble Free Energy: (rigid, chemically inert medium)

$$\Psi(\mathbf{s}, M) = \sum_{\pi=1}^{N_p} \Psi_\pi(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}, \theta) + \Upsilon(\mathbf{s}, \theta)$$

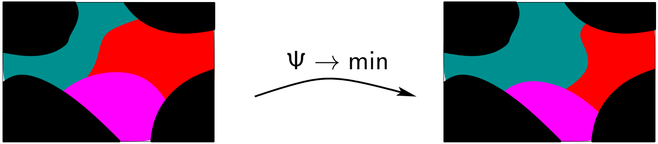
- Equilibrium: Fix $(\phi, \mathbf{m})$, then Ψ is *minimized* subject to:

$$\mathbf{s}.\mathbf{1} = \phi, \qquad \text{and} \qquad \mathbf{m} = M\mathbf{1}.$$

Porosity: $\phi = 1 - s_0$.

Macroscopic Free Energy:

$$\psi(\mathbf{m}) = \inf_{(\mathbf{s},M)} \left\{ \Psi(\mathbf{s}, M) \mid \mathbf{s}.\mathbf{1} = \phi \text{ and } M\mathbf{1} = \mathbf{m} \right\}$$



- Ensemble Free Energy: (rigid, chemically inert medium)

$$\Psi(\mathbf{s}, M) = \sum_{\pi=1}^{N_p} \Psi_\pi(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}, \theta) + \Upsilon(\mathbf{s}, \theta)$$

- Equilibrium: Fix $(\phi, \mathbf{m})$, then Ψ is *minimized* subject to:

$$\mathbf{s}.\mathbf{1} = \phi, \qquad \text{and} \qquad \mathbf{m} = M\mathbf{1}.$$

- Lagrange Multipliers:

$$-\frac{\partial \Psi_\pi}{\partial s_\pi} = p + \frac{\partial \Upsilon}{\partial s_\pi} \equiv p_\pi \quad \text{and} \quad \frac{\partial \Psi_\pi}{\partial M_{c\pi}} = \mu_c, \qquad 1 \le \pi \le N_p.$$

Mass Balances: Densities $\mathbf{m} = (m_1, \ldots, m_c, \ldots, m_{N_c}) \in [0, \infty)^{N_c}$

$$\partial_t m_c + div\left(\sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi + \mathbf{h}_c \right) = 0.$$

Constitutive Input: Given $\mathbf{m}$ and porosity $\phi \in [0, 1]$

- Phases: (flash calculation)

$$(\mathbf{s}, M) = \underset{(\mathbf{s}, M)}{\arg\min} \left\{ \Psi(\mathbf{s}, M) \;\middle|\; \mathbf{s}.\mathbf{1} = \phi \text{ and } M\mathbf{1} = \mathbf{m} \right\}$$

- Fluxes: Darcy and Fick Laws ($\rho_\pi \equiv \sum_{c=1}^{N_c} M_{c\pi}$)

$$\mathbf{v}_\pi = -K_\pi (s_\pi \nabla p_\pi - \rho_\pi \mathbf{g}) \quad \text{and} \quad \mathbf{h}_c = -K_c \nabla \mu_c.$$

- Energy Law: ...

Classical Darcy Law:

$$\mathbf{v}_\pi = -K_\pi(\mathbf{s})\left(s_\pi \nabla p_\pi - \rho_\pi \mathbf{g}\right) \quad \text{where} \quad \rho_\pi = \sum_{c=1}^{N_c} M_{c\pi}$$

Homogeneity: $\Psi_\pi(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c})$ is homogeneous

- Euler's Identity: $\Psi_\pi = -s_\pi p_\pi + \sum_{c=1}^{N_c} M_{c\pi}\mu_c$
- Gibbs–Duhem Identity: $-s_\pi \nabla p_\pi + \sum_{c=1}^{N_c} M_{c\pi} \nabla \mu_c = 0$

Darcy Law: $\mathbf{v}_\pi = -K_\pi(\mathbf{s}) \sum_{c=1}^{N_c} M_{c\pi}\left(\nabla \mu_c - \mathbf{g}\right)$

Weak Statement: (zero flux b.c)

$$\int_\Omega (\mathbf{m}_t, \hat{\mu}) + \sum_{\pi=1}^{N_p} \sum_{c,c'=1}^{N_c} \left(M_{c\pi} \nabla \mu_c, M_{c'\pi} \nabla \hat{\mu}_{c'}\right)_{K_\pi} + \sum_{c=1}^{N_c} \left(\nabla \mu_c, \nabla \hat{\mu}_{c'}\right)_{K_c}$$

$$= \int_\Omega \sum_{\pi=1}^{N_p} \sum_{c,c'=1}^{N_c} \left(M_{c\pi} \mathbf{g}, M_{c'\pi} \nabla \hat{\mu}_{c'}\right)_{K_\pi}$$

Energy Law: $\partial\psi/\partial m_c = \mu_c \ \ldots$

$$\frac{d}{dt} \int_\Omega \psi(\mathbf{m}) + \int_\Omega \sum_{\pi=1}^{N_p} \left| \sum_{c=1}^{N_c} M_{c\pi} \nabla \mu_c \right|_{K_\pi}^2 + \sum_{c=1}^{N_c} \left| \nabla \mu_c \right|_{K_c}^2$$

$$= \int_\Omega \sum_{\pi=1}^{N_p} \sum_{c,c'=1}^{N_c} \left(M_{c\pi} \mathbf{g}, M_{c'\pi} \nabla \mu_{c'}\right)_{K_\pi}$$

Balance Law: $\partial_t m_c - \mathrm{div}(\sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi) = 0$

Euler Approximation: $m_c^n - m_c^{n-1} - \tau\, \mathrm{div}(\sum_{\pi=1}^{N_p} M_{c\pi}^{n-1} \mathbf{v}_\pi) = 0$

Variational Problem:

$$I(\boldsymbol{\mu}) = \int_\Omega \left(\psi^*(\boldsymbol{\mu}) - \boldsymbol{\mu}.\mathbf{m}^{n-1} + \frac{\tau}{2} \sum_{\pi=1}^{N_p} \Big| \sum_{c=1}^{N_c} M_{c\pi}^{n-1} (\nabla \mu_c - \mathbf{g}) \Big|_{K^{n-1}}^2 \right)$$

Constitutive Relation: $\boldsymbol{\mu} \in \partial\psi(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial\psi^*(\boldsymbol{\mu})$.

Problem: Only $\psi(\mathbf{m})$ is computable, ψ^* *is not available*.

Saddle Point Problem:

$$L(\mathbf{m}, \boldsymbol{\mu}) = \int_\Omega \left(-\psi(\mathbf{m}) + \boldsymbol{\mu}.(\mathbf{m} - \mathbf{m}^{n-1}) + \frac{\tau}{2} \sum_{\pi=1}^{N_p} \Big| \sum_{c=1}^{N_c} M_{c\pi}^{n-1} (\nabla \mu_c - \mathbf{g}) \Big|_{K^{n-1}}^2 \right)$$

Balance of Energy[1]:

$$\partial_t e + div\Big(\sum_{\pi=1}^{N_p}(e_\pi + s_\pi p_\pi)\mathbf{v}_\pi + \sum_{c=1}^{N_c}\mu_c \mathbf{h}_c + \mathbf{q}\Big) = r + \sum_{\pi=1}^{N_p}\rho_\pi \mathbf{g}\cdot\mathbf{v}_\pi$$

- e Internal energy per unit volume of Ω.
- $\{e_\pi\}_{\pi=1}^{N_p}$ internal energy of phase π per unit volume.
- $\mathbf{q}$ heat flux.
- r heat supply (radiation) per unit volume.

Fourier/Darcy/Fick Laws:

$$\mathbf{q} = -K_\theta \nabla\theta, \qquad K_\pi \mathbf{v}_\pi = -(s_\pi \nabla p_\pi - \rho_\pi \mathbf{g}), \qquad \mathbf{h}_c = -K_c \nabla \mu_c .$$

[1] Farid F Smaï, A thermodynamic formulation for multiphase compositional flows in porous media.

- Ensemble Entropy:

$$S(\mathbf{e}, \mathbf{s}, M) = S_0(e_0) + \sum_{\pi=1}^{N_p} S_\pi\left(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}\right)$$

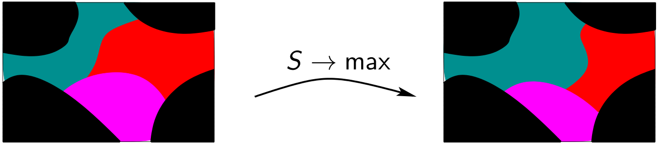
  - Medium: e_0 internal energy of medium per unit volume of Ω.
  - Concavity: $S_\pi(.,.,.)$ is concave.
  - Homogeneity:

$$\lambda S_\pi\left(\mathcal{E}_\pi, \mathcal{V}_\pi, \{\mathcal{M}_{c\pi}\}_{c=1}^{N_c}\right) = S_\pi\left(\lambda\mathcal{E}_\pi, \lambda\mathcal{V}_\pi, \{\lambda\mathcal{M}_{c\pi}\}_{c=1}^{N_c}\right)$$

  - Ideal Gas: $S(\mathcal{E}, \mathcal{V}, \mathcal{M}) = k\,\mathcal{M} \ln\left[(const)\frac{\mathcal{V}}{\mathcal{M}}\left(\frac{\mathcal{E}}{ck\,\mathcal{M}}\right)^c\right]$

Classical Thermodynamics

Microcanonical Ensemble (Non–Isothermal Problem)



- **Ensemble Entropy:**

$$S(\mathbf{e}, \mathbf{s}, M) = S_0(e_0) + \sum_{\pi=1}^{N_p} S_\pi \left(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c} \right)$$

- **Equilibrium:** Fix $(e, \phi, \mathbf{m})$, then S is maximized subject to:

$$e = e_0 + \mathbf{e}.\mathbf{1} + \Upsilon(\mathbf{s}), \qquad \phi = \mathbf{s}.\mathbf{1}, \qquad \mathbf{m} = M\mathbf{1}.$$

- **Lagrange Multipliers:**

$$\theta \frac{\partial S_\pi}{\partial e_\pi} = 1 \qquad \theta \frac{\partial S_\pi}{\partial s_\pi} = p + \frac{\partial \Upsilon}{\partial s_\pi} \equiv p_\pi, \qquad \theta \frac{\partial S_\pi}{\partial M_{c\pi}} = \mu_c.$$

Dissipation Relation: $\eta_t = \frac{1}{\theta}(e_t - \mathbf{m}_t.\boldsymbol{\mu})$ where

$$\eta(e, \mathbf{m}) = \max \left\{ S(\mathbf{e}, \mathbf{s}, M) \mid e = e_0 + \mathbf{e}.\mathbf{1} + \Upsilon(\mathbf{s}),\, \phi = \mathbf{s}.\mathbf{1},\, \mathbf{m} = M\mathbf{1} \right\}$$

$$S(\mathbf{e}, \mathbf{s}, M) = S_0(e_0) + \sum_{\pi=1}^{N_p} S_\pi \left(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c} \right)$$

Homogeneity: $S_\pi(e_\pi, s_\pi, \{M_{c\pi}\}_{c=1}^{N_c})$ is homogeneous.

- Euler's Representation: $\theta\eta_\pi = e_\pi + s_\pi p_\pi - \sum_{c=1}^{N_c} M_{c\pi}\mu_c$
- Gibbs–Duhem Relation: $\eta_\pi \nabla\theta - s_\pi \nabla p_\pi + \sum_{c=1}^{N_c} M_{c\pi} \nabla \mu_c = 0.$

Dissipation Relation: $\eta_t = \dfrac{1}{\theta}(e_t - \mathbf{m}_t.\boldsymbol{\mu})$ where

$$\eta(e,\mathbf{m}) = \max\left\{ S(\mathbf{e},\mathbf{s},M) \mid e = e_0 + \mathbf{e}.\mathbf{1} + \Upsilon(\mathbf{s}),\ \phi = \mathbf{s}.\mathbf{1},\ \mathbf{m} = M\mathbf{1} \right\}$$

Weak Statement: Set $\beta = 1/\theta \ldots$

$$(\mathbf{m}_t, \beta\hat{\boldsymbol{\mu}}) - \sum_{\pi=1}^{N_p}\sum_{c=1}^{N_c}\left(M_{c\pi}\mathbf{v}_\pi, \beta\nabla\hat{\mu}_c + \hat{\mu}_c\nabla\beta\right) - \sum_{c=1}^{N_c}\left(\mathbf{h}_c, \beta\nabla\hat{\mu}_c + \hat{\mu}_c\nabla\beta\right) = 0.$$

$$(e_t, \hat{\beta}) - \left(\sum_{\pi=1}^{N_p}\left(\eta_\pi/\beta + \sum_{c=1}^{N_c} M_{c\pi}\mu_c\right)\mathbf{v}_\pi + \sum_{c=1}^{N_c}\mu_c\mathbf{h}_c + \mathbf{q}, \nabla\hat{\beta}\right)$$
$$+ \sum_{\pi=1}^{N_p}\left((\eta_\pi/\beta^2)\nabla\beta - \sum_{c=1}^{N_c} M_{c\pi}\nabla\mu_c, \mathbf{v}_\pi\hat{\beta}\right) = \left(r + \sum_{\pi=1}^{N_p}|\mathbf{v}_\pi|^2_{K_\pi}, \hat{\beta}\right).$$

Dissipation Relation: $\eta_t = \frac{1}{\theta}(e_t - \mathbf{m}_t.\boldsymbol{\mu})$ where

$$\eta(e, \mathbf{m}) = \max\left\{ S(\mathbf{e}, \mathbf{s}, M) \;\middle|\; e = e_0 + \mathbf{e}.\mathbf{1} + \Upsilon(\mathbf{s}),\, \phi = \mathbf{s}.\mathbf{1},\, \mathbf{m} = M\mathbf{1} \right\}$$

Dissipation Relation: Set $(\hat{\boldsymbol{\mu}}, \hat{\beta}) = (\boldsymbol{\mu}, \beta)$ ...

$$(\mathbf{m}_t, \beta\boldsymbol{\mu}) - \sum_{\pi=1}^{N_p}\sum_{c=1}^{N_c}\left(M_{c\pi}\mathbf{v}_\pi, \beta\nabla\mu_c + \mu_c\nabla\beta\right) - \sum_{c=1}^{N_c}\left(\mathbf{h}_c, \beta\nabla\mu_c + \mu_c\nabla\beta\right) = 0.$$

$$(e_t, \beta) - \left(\sum_{\pi=1}^{N_p}\left(\eta_\pi/\beta + \sum_{c=1}^{N_c} M_{c\pi}\mu_c\right)\mathbf{v}_\pi + \sum_{c=1}^{N_c}\mu_c\mathbf{h}_c + \mathbf{q}, \nabla\beta\right)$$
$$+ \sum_{\pi=1}^{N_p}\left((\eta_\pi/\beta^2)\nabla\beta - \sum_{c=1}^{N_c} M_{c\pi}\nabla\mu_c, \mathbf{v}_\pi\beta\right) = \left(r + \sum_{\pi=1}^{N_p}|\mathbf{v}_\pi|^2_{K_\pi}, \beta\right).$$

Dissipation Relation: $\eta_t = \frac{1}{\theta}(e_t - \mathbf{m}_t.\boldsymbol{\mu})$ where

$$\eta(e, \mathbf{m}) = \max\left\{S(\mathbf{e}, \mathbf{s}, M) \mid e = e_0 + \mathbf{e}.\mathbf{1} + \Upsilon(\mathbf{s}),\, \phi = \mathbf{s}.\mathbf{1},\, \mathbf{m} = M\mathbf{1}\right\}$$

Dissipation Relation: Set $(\hat{\boldsymbol{\mu}}, \hat{\beta}) = (\boldsymbol{\mu}, \beta)$ ...

$$-\frac{d}{dt}\int_\Omega \eta + \int_\Omega \left\{|\nabla \ln(\beta)|^2_{K_\theta} + \beta\left(\sum_{\pi=1}^{N_p} |\mathbf{v}_\pi|^2_{K_\pi} + \sum_{c=1}^{N_c} |\nabla \mu_c|^2_{K_c}\right)\right\} = \int_\Omega -\beta r.$$

Fourier/Darcy/Fick Laws:

$$\mathbf{q} = -K_\theta \nabla\theta = (1/\beta)^2 K_\theta \nabla\beta, \quad K_\pi \mathbf{v}_\pi = -(s_\pi \nabla p_\pi - \rho_\pi \mathbf{g}), \quad \mathbf{h}_c = -K_c \nabla \mu_c.$$

Classical Single Phase Poroelasticity:

$$\frac{\partial}{\partial t} \rho_{ref} \left(\alpha p + a \, \mathrm{div}(\mathbf{u}) \right) - \mathrm{div} \left(K(\nabla p - \rho_{ref} \mathbf{g}) \right) = 0,$$

$$-\mathrm{div} \left(\mathbb{C}(\nabla \mathbf{u}) - a \, p \, I \right) = \mathbf{f}.$$

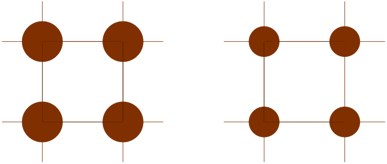
- Fluid pressure p
- Macroscopic displacement $\mathbf{u}$.
- Elasticity tensor, e.g. $\mathbb{C}(\nabla \mathbf{u}) = 2\mu D(\mathbf{u}) + \lambda D(\mathbf{u})$.
- Fluid compressibility α, $\rho/\rho_{ref} = \alpha(p - p_{ref})$
- Biot constant a ($\sim$ bulk modulus of medium).

Classical Single Phase Poroelasticity:

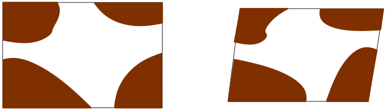
$$\frac{\partial}{\partial t}\rho_{ref}\left(\alpha p + a\,\mathrm{div}(\mathbf{u})\right) - \mathrm{div}\left(K(\nabla p - \rho_{ref}\mathbf{g})\right) = 0,$$

$$-\mathrm{div}\left(\mathbb{C}(\nabla \mathbf{u}) - a\,p\,I\right) = \mathbf{f}.$$

Medium Kinematics[2]: p pore pressure, s_0 solid volume



$\delta p > 0,\ \nabla \mathbf{u} = 0,\ \delta s_0 < 0$



$\delta p > 0,\ \delta s_0 = \nabla \mathbf{u} : A < 0$

[2] Sanchez Palencia, Non-Homogeneous Media and Vibration Theory (Springer)

Balances of Mass: $\partial_t m_c + div\left(\sum_{\pi=1}^{N_p} M_{c\pi}(\mathbf{u}_t + \mathbf{v}_\pi) + \mathbf{h}_c\right) = 0,$

Elastic Medium: $-\mathrm{div}(T) = \mathbf{f}$ with $T = \dfrac{\partial \psi}{\partial \nabla \mathbf{u}}$

Ensemble Free Energy:

$$\Psi(\mathbf{s}, M, \nabla \mathbf{u}) = \Psi_0(s_0, \nabla \mathbf{u}) + \sum_{\pi=1}^{N_p} \Psi_\pi\left(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}\right) + \Upsilon(\mathbf{s}).$$

Equilibrium:

$$\psi(\mathbf{m}, \nabla \mathbf{u}) = \inf\left\{\Psi(\mathbf{s}, M, \nabla \mathbf{u}) \mid s_0 + \mathbf{s}.\mathbf{1} = 1 + A : \nabla \mathbf{u},\ M\mathbf{1} = \mathbf{m}\right\}$$

Lagrange Multipliers: $\dfrac{\partial \psi}{\partial m_c} = \mu_c$ and $\dfrac{\partial \psi}{\partial \nabla \mathbf{u}} = \dfrac{\partial \Psi_0}{\partial \nabla \mathbf{u}} - pA$

$$-p = \frac{\partial \Psi_0}{\partial s_0} = \frac{\partial \Psi_\pi}{\partial s_\pi} + \frac{\partial \Upsilon}{\partial s_\pi} \quad \text{and} \quad \mu_c = \frac{\partial \Psi_\pi}{\partial M_{c\pi}}, \qquad 1 \le \pi \le N_p.$$

Balances of Mass: $\partial_t m_c + div\left(\sum_{\pi=1}^{N_p} M_{c\pi}(\mathbf{u}_t + \mathbf{v}_\pi) + \mathbf{h}_c\right) = 0,$

Elastic Medium: $-\mathrm{div}(T) = \mathbf{f}$ with $T = \frac{\partial \psi}{\partial \nabla \mathbf{u}}$

Ensemble Free Energy:

$$\Psi(\mathbf{s}, M, \nabla\mathbf{u}) = \Psi_0(s_0, \nabla\mathbf{u}) + \sum_{\pi=1}^{N_p} \Psi_\pi\left(s_\pi, \{M_{c\pi}\}_{c=1}^{N_c}\right) + \Upsilon(\mathbf{s}).$$

Linear Poroelasticity: Biot constant and tensor (a, A),

$$\Psi_0(s_0, \nabla\mathbf{u}) = \frac{1}{2}\mathbb{C}(\nabla\mathbf{u}) : (\nabla\mathbf{u}) + \frac{1}{2a}\left(A : \nabla\mathbf{u} + (s_0 - \phi)\right)^2$$

then

- $\frac{\partial \Psi_0}{\partial \nabla s_0} = \frac{1}{a}\left(A : \nabla\mathbf{u} + (s_0 - \phi)\right) = -p$
- $\frac{\partial \Psi_0}{\partial \nabla \mathbf{u}} = \mathbb{C}(\nabla\mathbf{u}) + \frac{1}{a}\left(A : \nabla\mathbf{u} + (s_0 - \phi)\right) A = \mathbb{C}(\nabla\mathbf{u}) - p\,A.$

Balances of Mass: $\partial_t m_c + div\left(\sum_{\pi=1}^{N_p} M_{c\pi} \mathbf{v}_\pi + \mathbf{h}_c \right) = 0,$

Elastic Medium: $-\mathrm{div}(T) = \mathbf{f}$ with $T = \dfrac{\partial \psi}{\partial \nabla \mathbf{u}} = \dfrac{\partial \Psi_0}{\partial \nabla \mathbf{u}} - p\,A$

Equilibrium:

$$\psi(\mathbf{m}, \nabla \mathbf{u}) = \inf \left\{ \Psi(\mathbf{s}, M, \nabla \mathbf{u}) \;\middle|\; s_0 + \mathbf{s}.\mathbf{1} = 1 + A : \nabla \mathbf{u},\;\; M\mathbf{1} = \mathbf{m} \right\}$$

Dissipation Relation: $\partial_t \psi = \mathbf{m}_t . \boldsymbol{\mu} + T : \nabla \mathbf{u}_t, \ldots$

$$\frac{d}{dt} \int_\Omega \psi + \int_\Omega \sum_\pi \Big| \sum_c M_{c\pi} (\nabla \mu_c - \mathbf{g}) \Big|^2_{K_\pi} + \sum_{c=1}^{N_c} |\nabla \mu_c|^2_{K_c} = \int_\Omega \mathbf{f}.\mathbf{u}_t.$$

Multiphase Linear Poroelasticity

Isothermal & Inertialess

Balances of Mass: $\partial_t m_c + div\Big(\sum_{\pi=1}^{N_p} M_{c\pi}\mathbf{v}_\pi + \mathbf{h}_c\Big) = 0,$

Elastic Medium: $-\mathrm{div}(T) = \mathbf{f}$ with $T = \dfrac{\partial\psi}{\partial\nabla\mathbf{u}} = \dfrac{\partial\Psi_0}{\partial\nabla\mathbf{u}} - p\,A$

Equilibrium:

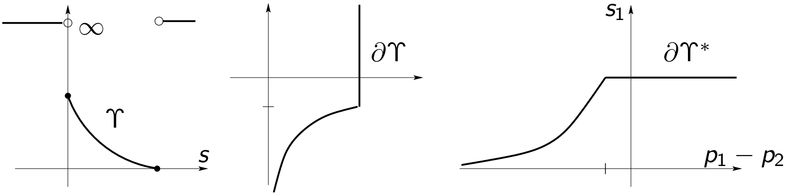
$$\psi(\mathbf{m}, \nabla\mathbf{u}) = \inf\big\{\Psi(\mathbf{s}, M, \nabla\mathbf{u}) \;\big|\; s_0 + \mathbf{s}.\mathbf{1} = 1 + A : \nabla\mathbf{u},\;\; M\mathbf{1} = \mathbf{m}\big\}$$

Dissipation Relation: $\partial_t\psi = \mathbf{m}_t.\boldsymbol{\mu} + T : \nabla\mathbf{u}_t, \;\ldots$

Weak Statement: $p = \ldots$

$$\int_\Omega(\mathbf{m}_t, \hat{\boldsymbol{\mu}}) + \sum_{\pi=1}^{N_p}\sum_{c,c'=1}^{N_c}\left(M_{c\pi}\nabla\mu_c, M_{c'\pi}(\nabla\hat{\mu}_{c'} - \mathbf{g})\right)_{K_\pi} + \sum_{c=1}^{N_c}\left(\nabla\mu_c, \nabla\hat{\mu}_{c'}\right)_{K_c} = 0$$

$$\left(\frac{\partial\Psi_0}{\partial\nabla\mathbf{u}} - p\,A, \nabla\mathbf{v}\right) = (\mathbf{f}, \mathbf{v})$$



Balances of Mass: $s_\pi^n - \tau \,\mathrm{div}\big(k_\pi^{n-1}(\nabla p_\pi^n - \mathbf{b}_\pi^{n-1})\big) = s_\pi^{n-1}$

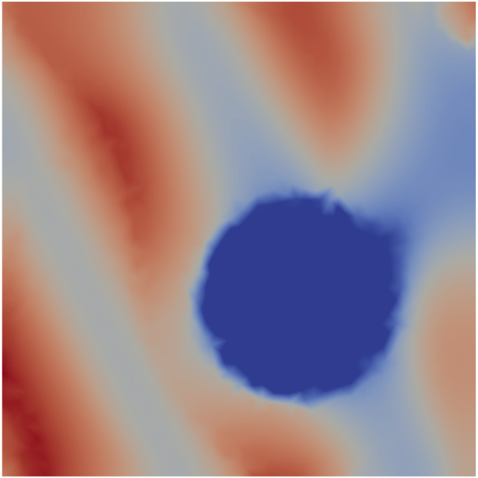
Interfacial Energy: $\Upsilon(s_1 - s_2) = \Upsilon(2s_1 - \phi)$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in p \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1/2)\partial\Upsilon(s_1 - s_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \partial\Upsilon^*(p_1 - p_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Energy Estimate:

$$\frac{d}{dt}\int_\Omega \Upsilon(s_1 - s_2) + \sum_{\pi=1}^{2} \int_\Omega |\nabla p_\pi|_{k_\pi}^2 = \sum_{\pi=1}^{2} \int_\Omega (\mathbf{b}_\pi, \nabla p_\pi)_{k_\pi}$$



Delaunay Mesh
Domain $[-1, 1]^2$
Time $t \in [0, 5]$
1592 Vertices
3061 Triangles
512 Time Steps
3184 Variables

25600 Nesterov
1854 Newton
iterations

Thank You