A structure-preserving semi-implicit IMEX finite volume scheme for ideal MHD at all Mach and Alfvén numbers

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joint work with Walter Boscheri [†]

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Three scales in the model

- Transport via the local flow velocity $\mathbf{v} \rightarrow$ material wave u
- Influence of the magnetic field $\mathbf{B} \rightarrow$ Alfvén speed b
- Influence of the pressure $p \rightarrow$ sound speeds c

Parameters in the model

- Mach number: $M_c = |u|/c$
- Alfvén "Mach" number: $M_b = |u|/b$

Asymptotic process $M_c \rightarrow 0$:
Transition between compressible and incompressible Euler equations + magnetic field evolution

Whishlist of properties for the numerical scheme

Structure-preserving

- Discrete consistency of the numerical scheme with involution constraint $\nabla \cdot \mathbf{B} = 0$
- Discrete consistency with low acoustic Mach number limit, i.e. preserving asymptotics from compressible to incompressible flow (Boscarino Russo Scandurra 2016)
- Applicability of the scheme in all regimes with respect to Mach and Alfvén numbers (Ability to resolve shocks)

Efficiency due to

- stability under large time steps restricted by the fluid flow
- avoiding staggering of meshes
- avoiding non-linear implicit systems

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{v}\\ \rho E\\ \mathbf{B}\end{pmatrix}+\nabla\cdot\begin{pmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\otimes\mathbf{v}+\left(p+\frac{\|\mathbf{B}\|^2}{8\pi}\right)\mathbb{I}-\frac{1}{4\pi}\mathbf{B}\otimes\mathbf{B}\\ \left(\rho E+p+\frac{\|\mathbf{B}\|^2}{8\pi}\right)\mathbf{v}-\frac{1}{4\pi}\mathbf{B}(\mathbf{v}\cdot\mathbf{B})\\ \mathbf{B}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{B}\end{pmatrix}=0 \qquad \text{(MHD)}$$

Computational domain: $\Omega(\mathbf{x},t)\subset\mathbb{R}^3$ with $\mathbf{x}=(x,y,z)\in\Omega$ and $t\in\mathbb{R}_0^+$

$\rho>0$	$\rightarrow$	density
$\mathbf{v}=(u,v,w)$	$\rightarrow$	velocity field
ρE	$\rightarrow$	total energy
$p>0$	$\rightarrow$	hydrodynamic pressure
$\mathbf{B}=(B_x,B_y,B_z)$	$\rightarrow$	magnetic field
$\mathbb{I}$	$\rightarrow$	identity matrix

Involution: $\nabla\cdot\mathbf{B}=0$ (solenoidal property of the magnetic field)

Total energy

$$\rho E = \rho e + \rho k + m \qquad \text{with} \qquad k = \frac{1}{2}\|\mathbf{v}\|^2, \quad m = \frac{\|\mathbf{B}\|^2}{8\pi}.$$

Equation of state: ideal gas

$$\rho e = \frac{p}{\gamma - 1} \qquad \text{with} \qquad \gamma > 0.$$

Eigenvalues in normal direction n

$$\lambda_{1,8} = \mathbf{v}\cdot\mathbf{n} \mp c_f, \quad \lambda_{2,7} = \mathbf{v}\cdot\mathbf{n} \mp b_n, \quad \lambda_{3,6} = \mathbf{v}\cdot\mathbf{n} \mp c_s, \quad \lambda_{4,5} = \mathbf{v}\cdot\mathbf{n},$$

$$c_{s,f}^2 = \frac{1}{2}\left(c^2 + b^2 \mp \sqrt{(c^2+b^2)^2 - 4c^2 b_n^2}\right),$$
$$c^2 = \frac{\gamma p}{\rho}, \quad b^2 = \frac{\|\mathbf{B}\|^2}{4\pi\rho}, \quad b_n^2 = \frac{(\mathbf{B}\cdot\mathbf{n})^2}{4\pi\rho}.$$

Non-dimensional variables: $\tilde{q} = q/q_{\text{ref}}$

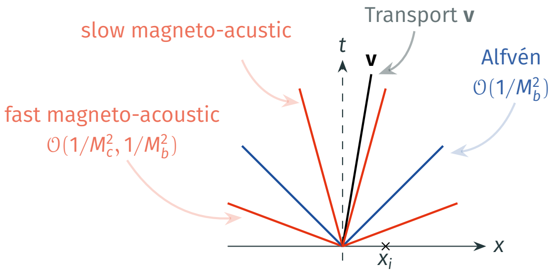
Stiffness parameters: $M_c = \|\mathbf{v}\|/c$ (acoustic Mach number)
$M_b = \|\mathbf{v}\|/b$ (Alfvén Mach number)

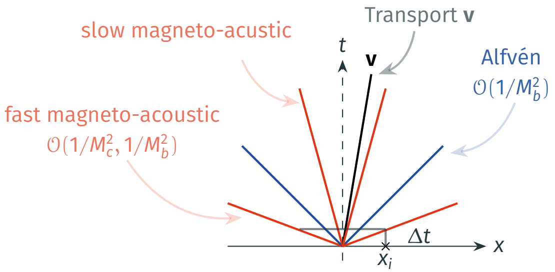
$$\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{\mathbf{v}} \\ \tilde{\rho}\tilde{E} \\ \tilde{\mathbf{B}} \end{pmatrix} + \nabla\cdot\begin{pmatrix} \tilde{\rho}\tilde{\mathbf{v}} \\ \tilde{\rho}\tilde{\mathbf{v}}\otimes\tilde{\mathbf{v}} + \left(\frac{\tilde{p}}{M_c^2} + \frac{1}{M_b^2}\frac{\|\tilde{\mathbf{B}}\|^2}{2}\right)\mathbb{I} - \frac{1}{M_b^2}\tilde{\mathbf{B}}\otimes\tilde{\mathbf{B}} \\ \left(\tilde{\rho}\tilde{E} + \frac{\tilde{p}}{M_c^2} + \frac{1}{M_b^2}\frac{\|\tilde{\mathbf{B}}\|^2}{2}\right)\tilde{\mathbf{v}} - \frac{1}{M_b^2}\tilde{\mathbf{B}}(\tilde{\mathbf{v}}\cdot\tilde{\mathbf{B}}) \\ \tilde{\mathbf{B}}\otimes\tilde{\mathbf{v}} - \tilde{\mathbf{v}}\otimes\tilde{\mathbf{B}} \end{pmatrix} = \mathbf{0}, \quad \nabla\cdot\tilde{\mathbf{B}} = 0$$

where the total energy is given by

$$\tilde{E} = \frac{1}{M_c^2}\tilde{e} + \frac{1}{M_b^2}\frac{\tilde{m}}{\tilde{\rho}} + \tilde{k}.$$

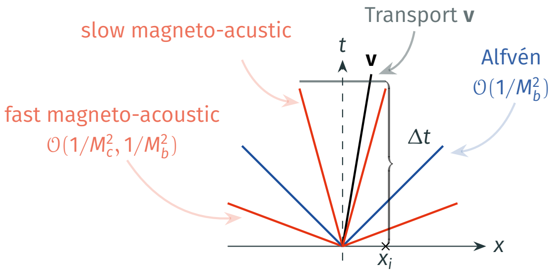
Ideal MHD equations: wave speeds and time step restrictions





Explicit scheme:

$$\Delta t \leqslant \frac{\Delta x}{\max|\lambda_k|} \to 0 \quad \text{for} \quad M_b \to 0^+ \text{ or } M_c \to 0^+$$



Explicit scheme:

$$\Delta t \leqslant \frac{\Delta x}{\max|\lambda_k|} \to 0 \quad \text{for} \quad M_b \to 0^+ \text{ or } M_c \to 0^+$$

Semi-implicit/implicit-explicit scheme:

$$\Delta t \leqslant \frac{\Delta x}{\max|\mathbf{v}|} \quad \text{for any} \quad M_b, M_c > 0$$

IMplicit - EXplicit [Ascher (1997), Pareschi (2005)]

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^{Ex}(\mathbf{q}) + \mathbf{f}^{Im}(\mathbf{q})$$

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^{Ex}(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{f}^{Im}(\mathbf{q})}{\partial x} = \mathbf{0}.$$

- Explicit hyperbolic sub-system with eigenvalues λ^{Ex}:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^{Ex}(\mathbf{q})}{\partial x} = \mathbf{0}.$$

- Implicit (hyperbolic) sub-system with eigenvalues λ^{Im}:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^{Im}(\mathbf{q})}{\partial x} = \mathbf{0}.$$

stability condition of the numerical scheme: $\Delta t \leqslant \min_{\Omega} \frac{\Delta x}{|\lambda^{Ex}|}$

1D MHD equations

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \mathbf{0}, \quad \frac{\partial B_x}{\partial x} = 0$$

$$\mathbf{q} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \\ B_x \\ B_y \\ B_z \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{pmatrix} \rho u \\ \rho u^2 + p + \frac{\|\mathbf{B}\|^2}{8\pi} - \frac{1}{4\pi} B_x^2 \\ \rho u v - \frac{1}{4\pi} B_x B_y \\ \rho u w - \frac{1}{4\pi} B_x B_z \\ \left(\rho E + p + \frac{\|\mathbf{B}\|^2}{8\pi}\right) u - \frac{1}{4\pi} B_x (\mathbf{v} \cdot \mathbf{B}) \\ 0 \\ u B_y - v B_x \\ u B_z - w B_x \end{pmatrix}.$$

1D MHD equations: splitting scheme of [Dumbser (2019)]

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^c(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{f}^p(\mathbf{q})}{\partial x} = \mathbf{0}$$

$$\mathbf{f}^c = \begin{pmatrix} \rho u \\ \rho u^2 + \frac{\|\mathbf{B}\|^2}{8\pi} - \frac{1}{4\pi} B_x^2 \\ \rho u v - \frac{1}{4\pi} B_x B_y \\ \rho u w - \frac{1}{4\pi} B_x B_z \\ (\rho k + 2m) u - \frac{1}{4\pi} B_x (\mathbf{v} \cdot \mathbf{B}) \\ 0 \\ u B_y - v B_x \\ u B_z - w B_x \end{pmatrix}, \quad \mathbf{f}^p = \begin{pmatrix} 0 \\ p \\ 0 \\ 0 \\ h \rho u \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad h = e + \frac{p}{\rho}.$$

⇒ stiffness of the acoustic Mach number (M_c)

⇒ dependence of time step on Alfvén Mach number (M_b)

1D MHD equations: splitting scheme of [Fambri (2021)] $\rightarrow$ talk of E. Sonnendrücker

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^c(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{f}^p(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{f}^b(\mathbf{q})}{\partial x} = \mathbf{0}$$

$$\mathbf{f}^c = \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho u v \\ \rho u w \\ \rho k u \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}^p = \begin{pmatrix} 0 \\ p \\ 0 \\ 0 \\ h\rho u \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}^b = \begin{pmatrix} 0 \\ \frac{\|\mathbf{B}\|^2}{8\pi} - \frac{1}{4\pi}B_x^2 \\ -\frac{1}{4\pi}B_x B_y \\ -\frac{1}{4\pi}B_x B_z \\ 2mu - \frac{1}{4\pi}B_x(\mathbf{v}\cdot\mathbf{B}) \\ 0 \\ uB_y - vB_x \\ uB_z - wB_x \end{pmatrix}.$$

$\Rightarrow$ stiffness of the acoustic **and** Alfvén Mach number (M_c, M_b)

$\Rightarrow$ dependence of time step only on convection!

Semi-implicit flux splitting for MHD [Boscheri and T. (in revision, ArXiv)]

$$\mathbf{f}^c = \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho u v \\ \rho u w \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}^p = \begin{pmatrix} 0 \\ p \\ 0 \\ 0 \\ (\rho E + p) u \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}^b = \begin{pmatrix} 0 \\ \frac{\|\mathbf{B}\|^2}{8\pi} - \frac{1}{4\pi} B_x^2 \\ -\frac{1}{4\pi} B_x B_y \\ -\frac{1}{4\pi} B_x B_z \\ \frac{\|\mathbf{B}\|^2}{8\pi} u - \frac{1}{4\pi} B_x (\mathbf{v} \cdot \mathbf{B}) \\ 0 \\ u B_y - v B_x \\ u B_z - w B_x \end{pmatrix}.$$

Eigenvalues of the three sub-systems

Convective sub-system: $\partial_t q + \partial_x \mathbf{f}^c = \mathbf{0}$

Real eigenvalues (weakly hyperbolic):

$$\lambda^c_{1,2,3,4} = 0, \quad \lambda^c_{5,6,7,8} = u.$$

Pressure sub-system: $\partial_t q + \partial_x \mathbf{f}^p = \mathbf{0}$

Real/Complex eigenvalues (full set of eigenvectors):

$$\lambda^p_{1,2,3,4,5,6} = 0, \quad \lambda^p_{7,8} = \frac{1}{2}\left(u \mp \sqrt{u^2 + 4(c^2 - (\gamma - 1)(m + k + u^2))}\right).$$

Magnetic sub-system: $\partial_t q + \partial_x \mathbf{f}^b = \mathbf{0}$

Real eigenvalues (weakly hyperbolic):

$$\lambda^b_{1,2,3,4} = 0, \quad \lambda^b_{5,6} = \frac{1}{2}\left(u \mp \sqrt{u^2 + 4\left(\frac{B_x}{\sqrt{4\pi\rho}}\right)^2}\right), \quad \lambda^b_{7,8} = \frac{1}{2}\left(u \mp \sqrt{u^2 + 4\left(\frac{\|\mathbf{B}\|^2}{\sqrt{4\pi\rho}}\right)^2}\right).$$

Numerical scheme

Semi-discrete first order scheme 1/3

1. **Explicit sub-system** ($\rho^\star = \rho^{n+1}$):

$$\mathbf{q}^\star = \mathbf{q}^n - \Delta t \, \frac{\partial \mathbf{f}^c(\mathbf{q}^n)}{\partial x}.$$

1. **Explicit sub-system** ($\rho^\star = \rho^{n+1}$):

$$\mathbf{q}^\star = \mathbf{q}^n - \Delta t \frac{\partial \mathbf{f}^c(\mathbf{q}^n)}{\partial x}.$$

2. **Implicit magnetic sub-system** ($B_x = const$):

$$(\rho u)^{n+1} = \rho u^\star - \Delta t \frac{\partial}{\partial x}\left(p^n + \frac{\mathbf{B}^n \cdot \mathbf{B}^{n+1}}{8\pi} - \frac{1}{4\pi} B_x^2 \right),$$

$$(\rho v)^{n+1} = \rho v^\star - \Delta t \frac{\partial}{\partial x}\left(-\frac{1}{4\pi} B_x B_y^{n+1} \right),$$

$$(\rho w)^{n+1} = \rho w^\star - \Delta t \frac{\partial}{\partial x}\left(-\frac{1}{4\pi} B_x B_z^{n+1} \right),$$

$$B_y^{n+1} = B_y^n - \Delta t \frac{\partial}{\partial x}\left(\frac{(\rho u)^{n+1}}{\rho^{n+1}} B_y^n - \frac{(\rho v)^{n+1}}{\rho^{n+1}} B_x \right),$$

$$B_z^{n+1} = B_z^n - \Delta t \frac{\partial}{\partial x}\left(\frac{(\rho u)^{n+1}}{\rho^{n+1}} B_z^n - \frac{(\rho w)^{n+1}}{\rho^{n+1}} B_x \right).$$

Linear coupling of momentum and magnetic field equations

$$B_y^{n+1} = B_y^\star + \Delta t^2 \frac{\partial}{\partial x}\left(\frac{B_y^n}{\rho^{n+1}}\frac{\partial}{\partial x}\left(\frac{B_y^n B_y^{n+1} + B_z^n B_z^{n+1}}{8\pi}\right) + \frac{B_x}{\rho^{n+1}}\frac{\partial}{\partial x}\left(\frac{B_x B_y^{n+1}}{4\pi}\right)\right)$$

$$B_z^{n+1} = B_z^\star + \Delta t^2 \frac{\partial}{\partial x}\left(\frac{B_z^n}{\rho^{n+1}}\frac{\partial}{\partial x}\left(\frac{B_y^n B_y^{n+1} + B_z^n B_z^{n+1}}{8\pi}\right) + \frac{B_x}{\rho^{n+1}}\frac{\partial}{\partial x}\left(\frac{B_x B_z^{n+1}}{4\pi}\right)\right)$$

where

$$B_y^\star = B_y^n - \Delta t \frac{\partial}{\partial x}\left(\frac{B_y^n}{\rho^{n+1}}\left(\rho u^\star - \Delta t \frac{\partial}{\partial x}\left(p^n - \frac{B_x^2}{8\pi}\right)\right) + \frac{B_x}{\rho^{n+1}}\rho v^\star\right),$$

$$B_z^\star = B_z^n - \Delta t \frac{\partial}{\partial x}\left(\frac{B_z^n}{\rho^{n+1}}\left(\rho u^\star - \Delta t \frac{\partial}{\partial x}\left(p^n - \frac{B_x^2}{8\pi}\right)\right) + \frac{B_x}{\rho^{n+1}}\rho w^\star\right).$$

Remark. At this stage we seek only an update for $\mathbf{B}$, $\rho\mathbf{v}$ is not updated
Implicit diffusion can be added to this step to ensure stability throughout all flow regimes

3. Implicit energy (pressure) sub-system: [Boscarino, Russo, Scandurra (2018)] (AP for $M_c \to 0$)

$$(\rho u)^{n+1} = (\rho u)^{\star} - \Delta t \frac{\partial}{\partial x}\left(p^{n+1} + \frac{\|\mathbf{B}^{n+1}\|^2}{8\pi} - \frac{1}{4\pi}B_x^2\right),$$

$$(\rho E)^{n+1} = (\rho E)^{n} - \Delta t \frac{\partial}{\partial x}\left((\rho E^n + p^n)\frac{(\rho u)^{n+1}}{\rho^{n+1}} + \frac{\|\mathbf{B}^{n+1}\|^2}{8\pi}u^n - \frac{B_x}{4\pi}(\mathbf{v}^n \cdot \mathbf{B}^{n+1})\right).$$

Linear elliptic equation with $p^{n+1} = (\rho E^{n+1} - \rho k^n - \rho m^{n+1})(\gamma - 1)$

$$(\rho E)^{n+1} = (\rho E)^{\star} + (\gamma - 1)\Delta t^2 \frac{\partial}{\partial x}\left(\frac{\rho E^n + p^n}{\rho^{n+1}}\frac{\partial}{\partial x}((\rho E)^{n+1})\right),$$

where

$$\rho E^{\star} = \rho E^{n} - \Delta t \frac{\partial}{\partial x}\left(\frac{\rho E^n + p^n}{\rho^{n+1}}(\rho u)^{\star\star} + \frac{\|\mathbf{B}^{n+1}\|^2}{8\pi}u^n - \frac{1}{4\pi}B_x(\mathbf{v}^n \cdot \mathbf{B}^{n+1})\right)$$

$$(\rho u)^{\star\star} = (\rho u)^{\star} - \Delta t \frac{\partial}{\partial x}\left(-(\gamma - 1)\left(\rho^n k^n + \rho^{n+1}\frac{\|\mathbf{B}^{n+1}\|^2}{8\pi}\right) + \frac{\|\mathbf{B}^{n+1}\|^2}{8\pi} - \frac{B_x^2}{4\pi}\right)$$

Autonomous system:

$$\frac{\partial \mathbf{q}(t)}{\partial t} + \mathcal{H}(\mathbf{q}_{Ex}(t), \mathbf{q}_{Im}(t)) = \mathbf{0}, \qquad \forall t > t_0, \qquad \text{with} \qquad \mathbf{q}(t_0) = \mathbf{q}_0.$$

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}^c(\mathbf{q}_{Ex})}{\partial x} + \frac{\partial \mathbf{f}^p(\mathbf{q}_{Ex}, \mathbf{q}_{Im})}{\partial x} + \frac{\partial \mathbf{f}^b(\mathbf{q}_{Ex}, \mathbf{q}_{Im})}{\partial x} = \mathbf{0}$$

Autonomous system

$$\frac{\partial \mathbf{q}}{\partial t} = \mathcal{H}(\mathbf{q}_{Ex}(t), \mathbf{q}_{Im}(t)), \qquad \forall t > t_0, \qquad \text{with} \qquad \mathbf{q}(t_0) = \mathbf{q}_0.$$

Stage fluxes for $i = 1, \dots, s$ $\quad (\mathbf{q}^n_{Ex} = \mathbf{q}^n_{Im} = \mathbf{q}^n)$

$$\mathbf{q}^i_{Ex} = \mathbf{q}^n_{Ex} + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} k_j, \qquad 2 \leqslant i \leqslant s,$$

$$\tilde{\mathbf{q}}^i_{Im} = \mathbf{q}^n_{Ex} + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j, \qquad 2 \leqslant i \leqslant s,$$

$$k_i = \mathcal{H}\left(\mathbf{q}^i_{Ex}, \tilde{\mathbf{q}}^i_{Im} + \Delta t\, a_{ii}\, k_i\right), \qquad 1 \leqslant i \leqslant s.$$

Numerical solution $\mathbf{q}^{n+1} = \mathbf{q}^n + \Delta t \sum_{i=1}^{s} b_i k_i.$

$\tilde{a}_{ij}, a_{ij}, b_i \Rightarrow$ Butcher tableaux

Finite volume data in each cell ω_{ijk} with 2^nd order TVD reconstruction

$$q_{ijk} = \frac{1}{|\omega_{ijk}|} \int_{\omega_{ijk}} q(\mathbf{x})\, d\mathbf{x}, \qquad r_{ijk}(x) = c_0\, q_{ijk} + c_1\,(x - x_i).$$

Flux operators for the convective sub-system

- Numerical flux operator $\mathbb{F}(f(q))$

$$\mathbb{F}_x(f(q)) = \frac{\mathcal{F}_{i+\frac{1}{2}jk}(f(q)) - \mathcal{F}_{i-\frac{1}{2}jk}(f(q))}{\Delta x},$$

  with a Rusanov–type numerical flux function

$$\mathcal{F}_{i+\frac{1}{2}jk}(q) = \frac{1}{2}\Big(f(q^{+}_{i+\frac{1}{2}jk}) + f(q^{-}_{i+\frac{1}{2}jk})\Big) - \frac{1}{2}\alpha_{i+\frac{1}{2}jk}\Big(q^{+}_{i+\frac{1}{2}jk} - q^{-}_{i+\frac{1}{2}jk}\Big),$$

  The numerical dissipation is only proportional to λ^c:

$$\alpha_{i+\frac{1}{2}jk} = \max\Big(|u_{i+1jk}|, |u_{ijk}|\Big).$$

- Central flux operator $\mathbb{B}(f(q))$. The same as $\mathbb{F}(f(q))$ with $\alpha_{i+\frac{1}{2}jk} = 0$.

Magnetic vector potential A: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ [Helzel (2011)]

$$\begin{aligned} \mathbf{B} - \nabla \times \mathbf{A} &= \mathbf{0} \\ \frac{\partial \mathbf{A}}{\partial t} + (\nabla \times \mathbf{A}) \times \mathbf{v} &= \mathbf{0} \end{aligned}$$

Mimetic FD [Hyman & Shashkov (1997)] and DG [Boscheri (2023)]

- Gradient and curl operators:

$$\mathbb{G}(q) = \begin{pmatrix} \mathbb{G}_x(q) \\ \mathbb{G}_y(q) \\ \mathbb{G}_z(q) \end{pmatrix} = \begin{pmatrix} \frac{q_{i+1jk} - q_{i-1jk}}{2\Delta x} \\ \frac{q_{ij+1k} - q_{ij-1k}}{2\Delta y} \\ \frac{q_{ijk+1} - q_{ijk-1}}{2\Delta z} \end{pmatrix}, \quad \mathbb{C}(\mathbf{q}) = \begin{pmatrix} \mathbb{C}_x(\mathbf{q}) \\ \mathbb{C}_y(\mathbf{q}) \\ \mathbb{C}_z(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} \mathbb{G}_x(q_z) - \mathbb{G}_z(q_y) \\ \mathbb{G}_z(q_x) - \mathbb{G}_x(q_z) \\ \mathbb{G}_x(q_y) - \mathbb{G}_y(q_x) \end{pmatrix}$$

- Divergence operator: $\mathbb{D}(\mathbf{q}) = \mathbb{G}_x(q_x) + \mathbb{G}_y(q_y) + \mathbb{G}_z(q_z)$

div-curl discrete property:

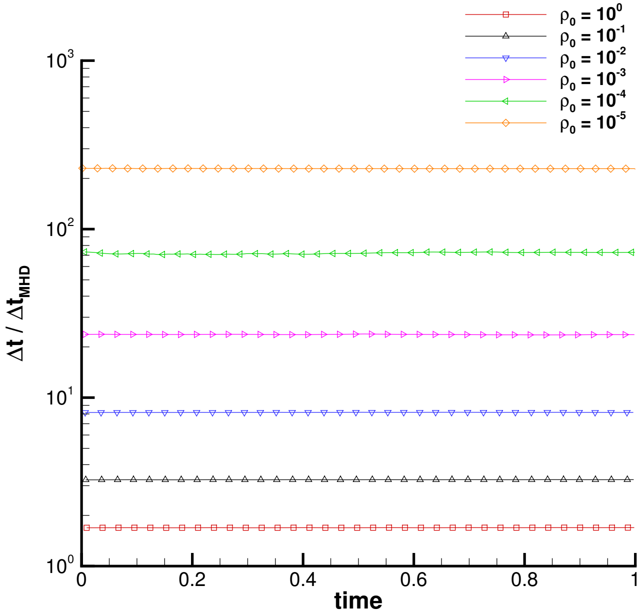
$$\mathbb{D}(\mathbb{C}(\mathbf{q})) = \mathbb{G}_x(\mathbb{C}_x(\mathbf{q})) + \mathbb{G}_y(\mathbb{C}_y(\mathbf{q})) + \mathbb{G}_z(\mathbb{C}_z(\mathbf{q})) = 0$$

Numerical results

$$
\begin{aligned}
\rho &= \rho_0 = 10^{-k}, \qquad k = 0, \ldots, 5 \\
(u, v) &= \mathbf{v}_0 + \frac{\tilde{v}}{2\pi} \exp\left(\frac{1 - r^2}{2}\right) \cdot (-r \sin(\theta), r \cos(\theta)) \\
(B_x, B_y) &= \mathbf{0} + \frac{\tilde{B}}{2\pi} \exp\left(\frac{1 - r^2}{2}\right) \cdot (-r \sin(\theta), r \cos(\theta)) \\
p &= p_0 + \frac{1}{2} e^{1-r^2} \left(\frac{1}{8\pi} \frac{\tilde{B}^2}{(2\pi)^2} (1 - r^2) - \frac{1}{2} \rho \left(\frac{\tilde{v}}{2\pi}\right)^2\right)
\end{aligned}
$$

	Background density $\rho_0 = 10^{-k}$					
	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
M_c	1.606E+01	4.489E-01	1.529E-01	4.832E-02	1.529E-02	4.832E-03
b	1.549E-01	4.900E-01	1.549E+00	4.900E+00	1.549E+01	4.900E+01

ρ_0	$N_x = N_y$	ρ		u		p		B_x		A_z	
10^0	32	7.13e-02		4.40e-02		5.86e-02		8.26e-02		4.94e-02	
	64	1.63e-02	2.13	9.16e-03	2.26	1.65e-02	1.83	2.20e-02	1.91	1.25e-02	1.99
	128	3.79e-03	2.10	2.18e-03	2.07	4.19e-03	1.98	5.66e-03	1.96	3.14e-03	1.99
	256	9.26e-04	2.03	6.79e-04	1.69	1.05e-03	1.99	1.54e-03	1.88	8.12e-04	1.95
10^{-1}	32	2.33e-03		3.50e-02		6.84e-03		8.21e-02		4.84e-02	
	64	6.39e-04	1.87	8.55e-03	2.04	1.91e-03	1.84	2.17e-02	1.92	1.23e-02	1.98
	128	1.62e-04	1.98	2.14e-03	2.00	4.90e-04	1.96	5.53e-03	1.97	3.09e-03	1.99
	256	4.08e-05	2.00	6.71e-04	1.67	1.24e-04	1.99	1.51e-03	1.87	7.99e-04	1.95
10^{-2}	32	4.12e-04		9.14e-02		3.90e-03		9.05e-02		5.02e-02	
	64	7.64e-05	2.43	2.45e-02	1.90	9.82e-04	1.99	2.32e-02	1.96	1.24e-02	2.02
	128	1.63e-05	2.23	6.27e-03	1.97	2.69e-04	1.87	5.85e-03	1.99	3.08e-03	2.00
	256	3.92e-06	2.06	1.63e-03	1.95	6.83e-05	1.98	1.58e-03	1.89	7.95e-04	1.95
10^{-3}	32	2.80e-04		2.07e-01		3.45e-03		9.44e-02		4.80e-02	
	64	4.50e-05	2.64	5.88e-02	1.81	9.63e-04	1.84	2.06e-02	2.20	8.76e-03	2.46
	128	5.95e-06	2.92	1.72e-02	1.77	1.98e-04	2.28	4.45e-03	2.21	1.79e-03	2.29
	256	8.10e-07	2.88	4.69e-03	1.88	4.74e-05	2.06	1.19e-03	1.90	4.61e-04	1.96
10^{-4}	32	1.55e-04		1.67e-00		1.28e-02		2.60e-01		2.15e-01	
	64	4.98e-05	1.64	2.25e-01	2.89	4.49e-03	1.51	8.01e-02	1.70	4.92e-02	2.12
	128	5.87e-06	3.09	3.44e-02	2.71	8.21e-04	2.45	1.55e-02	2.37	7.77e-03	2.66
	256	7.22e-07	3.02	9.02e-03	1.93	1.30e-04	2.66	2.45e-03	2.66	1.14e-03	2.76
10^{-5}	32	1.85e-05		8.14e-00		1.32e-02		2.85e-01		2.35e-01	
	64	9.85e-06	0.91	8.62e-01	3.24	6.75e-03	0.96	1.24e-01	1.20	8.47e-02	1.47
	128	1.41e-06	2.81	8.42e-02	3.36	1.64e-03	2.04	3.04e-02	2.03	1.70e-02	2.32
	256	1.61e-07	3.12	2.05e-02	2.04	2.71e-04	2.60	4.85e-03	2.65	2.54e-03	2.74

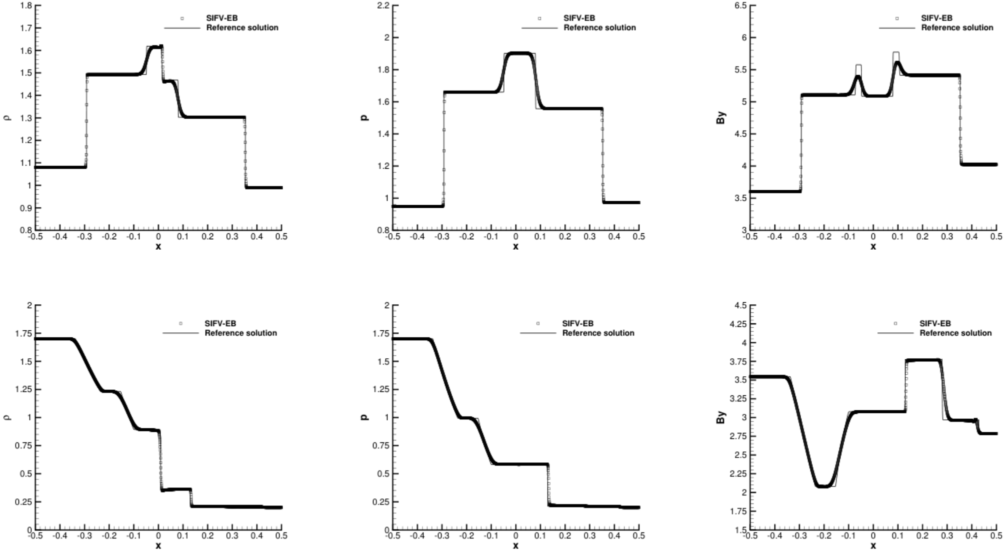


$\rho_0 = 10^0$
$\rho_0 = 10^{-1}$
$\rho_0 = 10^{-2}$
$\rho_0 = 10^{-3}$
$\rho_0 = 10^{-4}$
$\rho_0 = 10^{-5}$
10^3
10^2
10^1
10^0
$\Delta t / \Delta t_{MHD}$
0
0.2
0.4
0.6
0.8
1
time

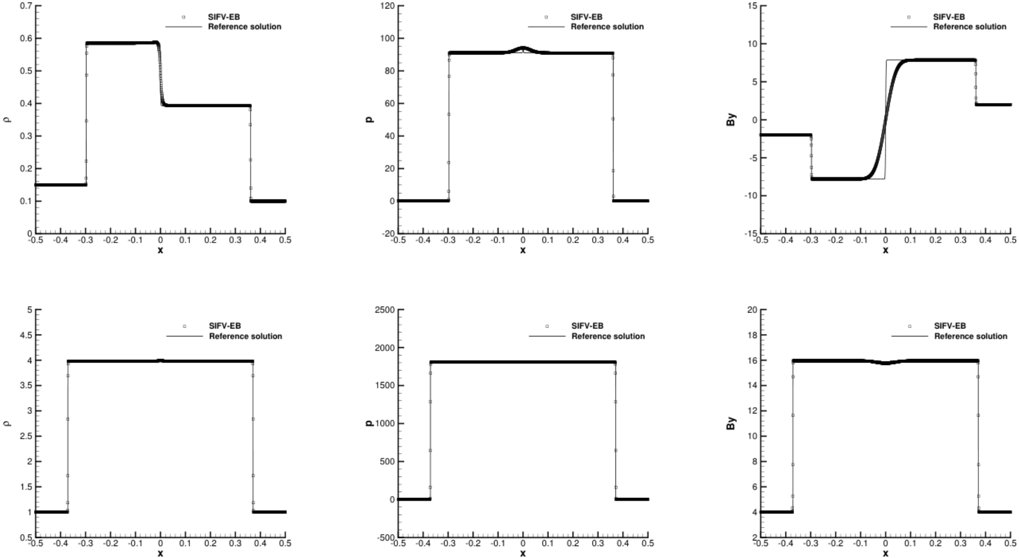
Riemann problems [Balsara (2016),Dumbser (2019)]

Case		ρ	u	v	w	p	B_x	B_y	B_z	t_f
RP1	L:	1.0	0.0	0.0	0.0	1.0	$0.75\sqrt{4\pi}$	$+\sqrt{4\pi}$	0.0	0.10
	R:	0.125	0.0	0.0	0.0	0.1	$0.75\sqrt{4\pi}$	$-\sqrt{4\pi}$	0.0	
RP2	L:	1.08	1.2	0.01	0.5	0.95	2.0	3.6	2.0	0.2
	R:	0.9891	−0.0131	0.0269	0.010037	0.97159	2.0	4.0244	2.0026	
RP3	L:	1.7	0.0	0.0	0.0	1.7	3.899398	3.544908	0.0	0.15
	R:	0.2	0.0	0.0	−1.496891	0.2	3.899398	2.785898	2.192064	
RP4	L:	1.0	0.0	0.0	0.0	1.0	$1.3\sqrt{4\pi}$	$+\sqrt{4\pi}$	0.0	0.16
	R:	0.4	0.0	0.0	0.0	0.4	$1.3\sqrt{4\pi}$	$-\sqrt{4\pi}$	0.0	
RP5	L:	0.15	21.55	1.0	1.0	0.28	0.05	−2.0	−1.0	0.04
	R:	0.10	−26.45	0.0	0.0	0.10	0.05	+2.0	+1.0	
RP6	L:	1.0	36.87	−0.115	−0.0386	1.0	4.0	4.0	1.0	0.03
	R:	1.0	−36.87	0.0	0.0	1.0	4.0	4.0	1.0	
RP7	L:	$1/\mu_0$	−1.0	+1.0	−1.0	1.0	1.0	−1.0	1.0	0.25
	R:	$1/\mu_0$	−1.0	−1.0	−1.0	1.0	1.0	+1.0	1.0	

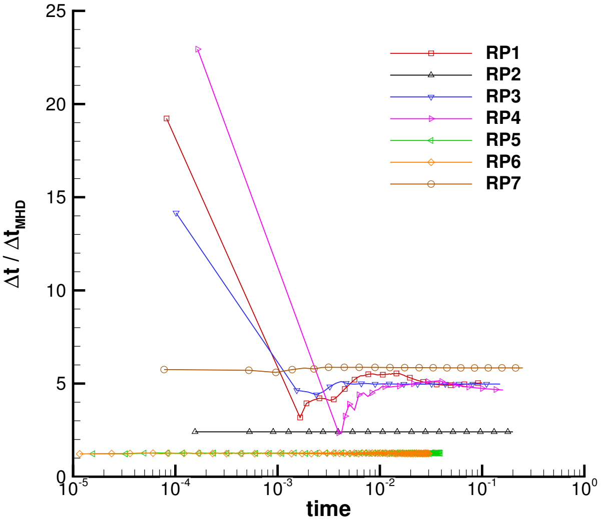
RP2 - RP3



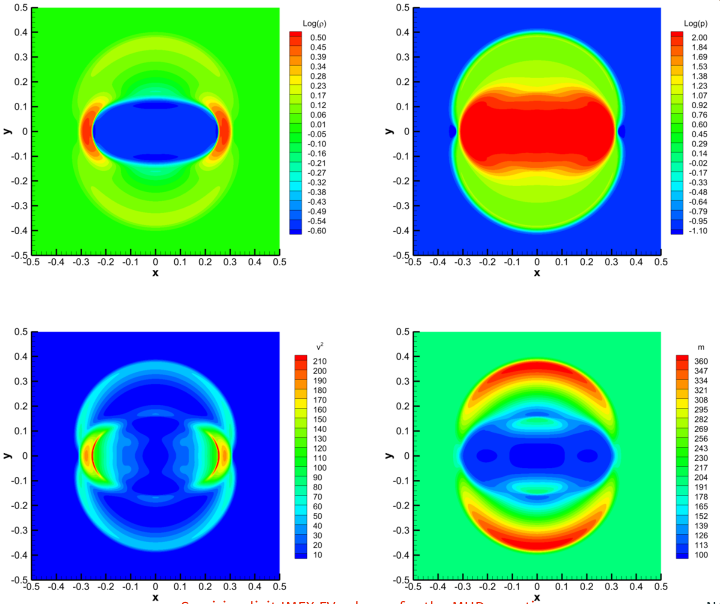
RP5 - RP6



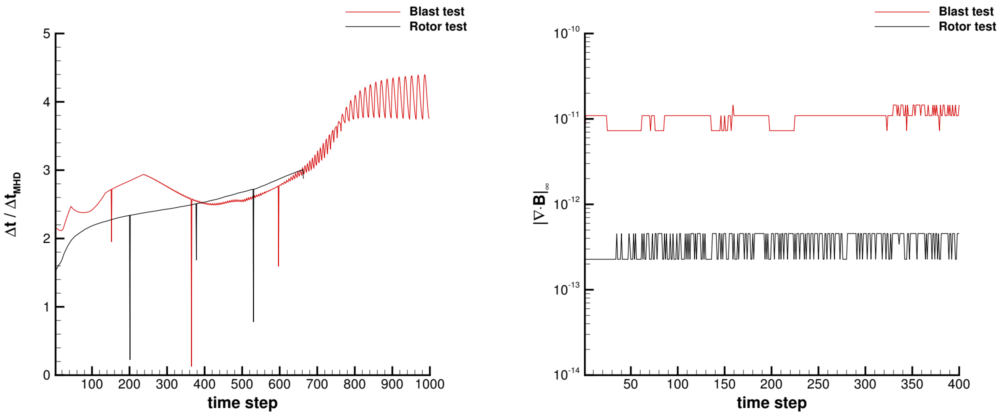
Time step comparison: Δt with material CFL, Δt_{MHD} explicit time step



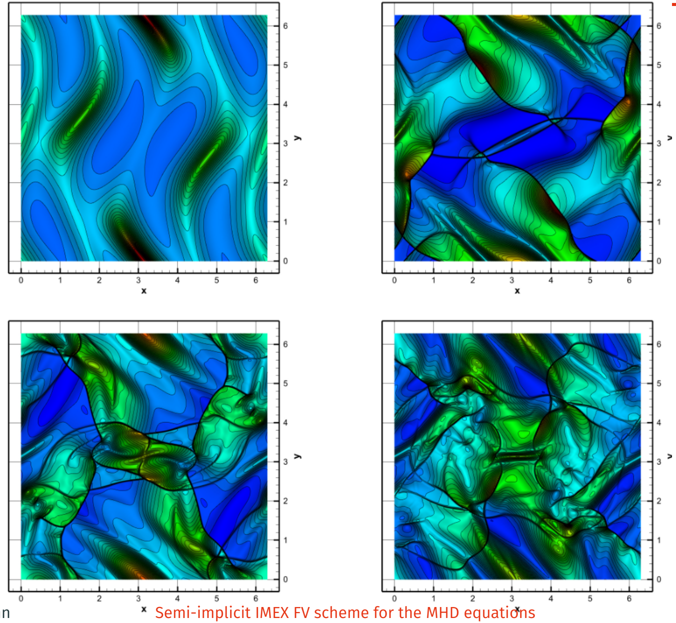
Blast problem [Balsara (1999)]



Time step comparison and divergence free property



Orszag-Tang vortex



Low acoustic Mach number regime

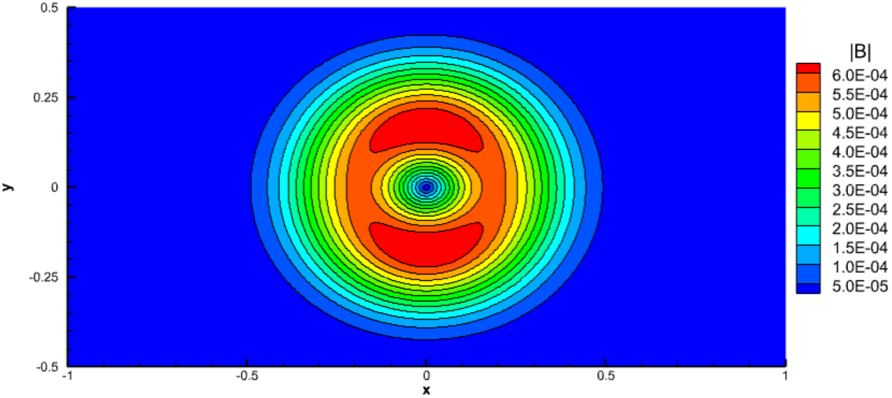
$$\rho = 1, \quad (u, v, w) = (2, 1, 0), \quad p = 10^5$$

The velocity field is such that advection does not occure along a diagonal, so the fluxes in x- and y-directions are different

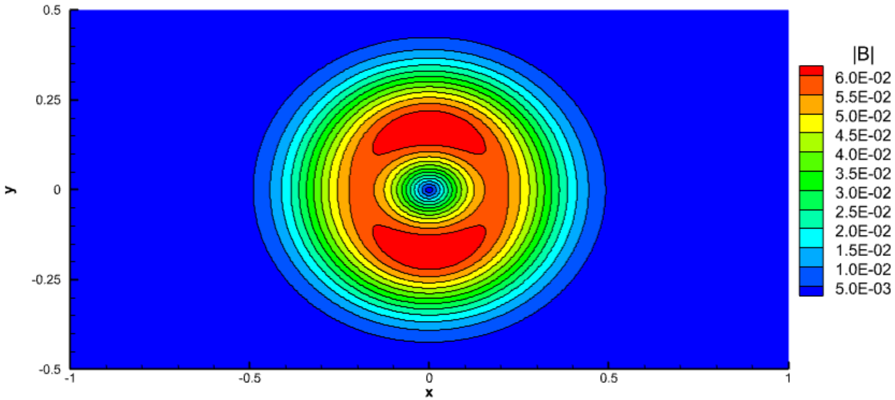
Magnetic vector potential scalable with $A_0 > 0$: cylindrical current distribution

$$A_z = \begin{cases} A_0(R - r) & \text{if} \quad r \leqslant R, \\ 0 & \text{else}, \end{cases}$$

$A_0 = 10^{-3}$ original test set up from [Dumbser (2019)]

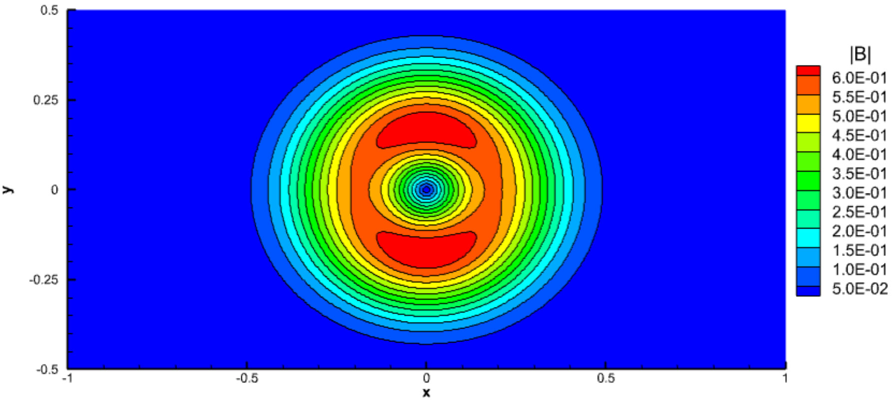


$A_0 = 10^{-1}$



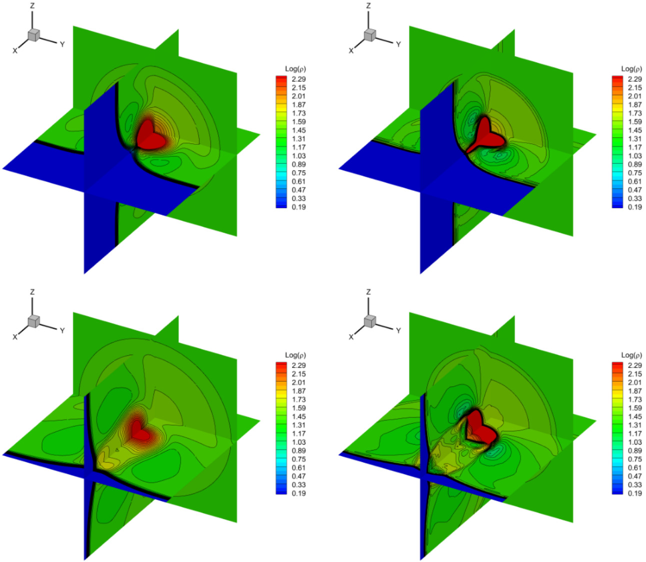
0.5
0.25
0
-0.25
-0.5
y
-1
-0.5
0
0.5
1
x
|B|
6.0E-02
5.5E-02
5.0E-02
4.5E-02
4.0E-02
3.5E-02
3.0E-02
2.5E-02
2.0E-02
1.5E-02
1.0E-02
5.0E-03

$A_0 = 10^0$



0.5
0.25
0
-0.25
-0.5
y
-1
-0.5
0
0.5
1
x
|B|
6.0E-01
5.5E-01
5.0E-01
4.5E-01
4.0E-01
3.5E-01
3.0E-01
2.5E-01
2.0E-01
1.5E-01
1.0E-01
5.0E-02

3D cloud-shock interaction problem [Helzel (2013)]



Conclusions

Conclusions and Outlook

Conclusions

- novel flux splitting for the ideal MHD equations;
- linear implicit-explicit time marching scheme;
- time step restriction dictated by transport velocity;
- cell-centered 2^{nd} order in space and time;
- compatible discrete div-curl operator on collocated grids;
- by construction asymptotic preserving for the low acoustic Mach number limit.

Outlook

- Extension to Resistive MHD with implicit viscosity;
- Well-balancing?

Thank you for your attention!

W. Boscheri and A. Thomann. *A structure-preserving semi-implicit IMEX finite volume scheme for ideal magnetohydrodynamics at all Mach and Alfvén numbers*. Under revision in J. Sci. Comput. 2024

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Rotor problem