Basics for polynomial interpolation on simplices

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Outline

Introduction
- $n = 1$ and $k = 0$
- $n > 1$ and $k = 0$
- $n = 1$ and $k = 1$
- Polynomial differential forms
- Degrees of freedom

Interpolation of differential forms
- Lebesgue constant
- From interpolation at nodes to interpolation at edges

Numerical results

Conclusions

Last achievements

Polynomial interpolation

of *scalar values* over an *interval* $I \subset \mathbb{R}$

The discrete representation of I depends on the type of scalar values

Definition of the polynomial interpolation problem

$I \subset \mathbb{R}$ interval and $\mathbb{P}_r(I)$ polynomial space
$N_r = \dim(\mathbb{P}_r(I)) = \binom{n+r}{r}$

We have $\{y_i\}$ values at points $\{x_i\}$ in I, $i = 1, ..., N_r$

** We wish to represent $\{y_i\}$ by a polynomial function $\Pi_r f$ and here, we construct $\Pi_r f$ that interpolates the $\{y_i\}$ at the $\{x_i\}$ **

$\Pi_r f$ is function such that

(1) $\quad \Pi_r f \in \mathbb{P}_r(I),$

(2) $\quad \Pi_r f(x_i) = y_i, \qquad \forall i = 1, ..., N_r \qquad (\, x_i \neq x_j$ for $i \neq j \,)$

Prop. $\exists\, !\ \Pi_r f \in \mathbb{P}_r(I)$ that interpolates $\{y_i\}_i$ at the $\{x_i\}_i$

$\rightarrow$! (Uniqueness) as if there were *two*, their difference would be a polynomial of degree $\leq r$ (here $N_r = r + 1$) with $r + 1$ zeros in I, so it would be identically zero on I.
$\rightarrow$ $\exists$ (Existence) by construction

$$\Pi_r f(x) = \sum_{k=1}^{N_r} y_k\, \varphi_k(x), \qquad \varphi_k(x) = \prod_{\substack{j=1 \\ j \neq k}}^{N_r} \frac{(x - x_j)}{(x_k - x_j)}$$

φ_k is the Lagrangian[1] polynomial in $\mathbb{P}_r(I)$ associated with x_k

$\{\varphi_k\}$ is the basis of $\mathbb{P}_r(I)$ in duality with the values at the $\{x_k\}$

$$\varphi_k(x_j) = \delta_{j,k} = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

[1] Giuseppe Ludovico De la Grange Tournier (1736-1813)

To compute the function φ_k

To compute φ_k with a general technique we can

- choose a basis $\{\psi_\ell\}$ in $\mathbb{P}_r(I)$ and set $(V)_{j,\ell} = \psi_\ell(x_j)$
- write $\varphi_k(x) = \sum_{\ell=1}^{N_r} c_\ell^k \, \psi_\ell(x)$
- find the vector $\mathbf{c}^k$ of coefficient c_ℓ^k by solving $V\,\mathbf{c}^k = \mathbf{e}_k$.

V is the generalised Vandermonde matrix[2] as if $\psi_\ell(x) = x^{\ell-1}$ then

$$\det(V) = \det \begin{pmatrix} 1 & x_1 & \dots & x_1^r \\ 1 & x_2 & \dots & x_2^r \\ \dots & \dots & \dots & \dots \\ 1 & x_{N_r} & \dots & x_{N_r}^r \end{pmatrix} = \prod_{1 \le j \le \ell \le N_r} (x_\ell - x_j)$$

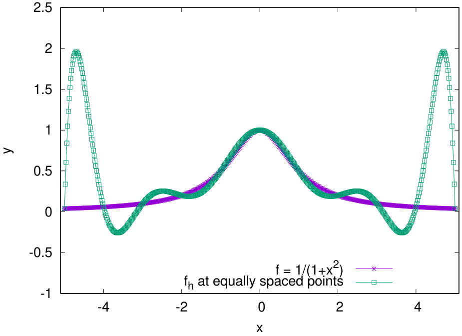
$\mathrm{cond}(V)$ matters (for high r) and it depends on the basis $\{\psi_\ell\}$

[2] Alexandre-Théophile Vandermonde (1735-1796)

Runge phenomenon[4]

The approximation of f by $\Pi_r f$ may give bad results[3]

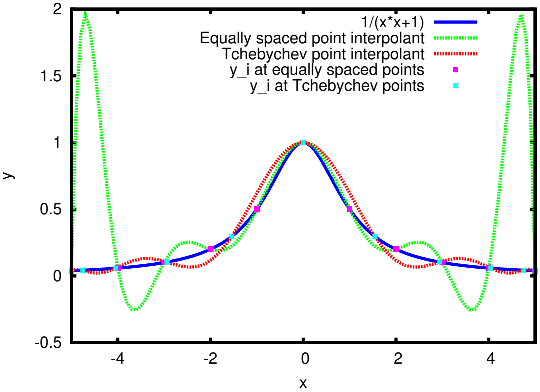
$$\lim_{r \to +\infty} ||f - \Pi_r f|| \neq 0 \qquad \text{if } f(x) = \frac{1}{(1+x^2)} \text{ on } I = [-5, 5]$$



[3] Maria Gaetana Agnesi (1718 - 1799), look for “Witch of Agnesi”
[4] Carl David Tolmé Runge (1856-1927) discovered it in 1901

Runge phenomenon

Taking other distributions of points, things improve.



The distribution of $\{x_i\}$ has to be optimized ! Yes, but how ?

The Lebesgue[5] constant Λ

Prop. There exists a constant Λ such that

$$||f - \Pi_r f|| \leq (1 + \Lambda)\, ||f - f^*||$$

where $||g|| = \sup_{x \in I} |g(x)|$ and $||f - f^*|| = \inf_{g \in \mathbb{P}_r(I)} ||f - g||$

Proof.

$$\begin{aligned} ||f - \Pi_r f|| &= ||f - f^* + f^* - \Pi_r f|| \\ &= ||f - f^* + \Pi_r f^* - \Pi_r f|| \\ &\leq ||f - f^*|| + ||\Pi_r (f - f^*)|| \\ &\leq (1 + ||\Pi_r||)\, ||f - f^*|| \qquad \leq (1 + \Lambda)\, ||f - f^*||. \end{aligned}$$

since $||\Pi_r|| = \sup_{g,\, ||g||=1} ||\Pi_r g||$ and

$$||\Pi_r|| = \sup_{g, ||g||=1} \max_{x \in I} |\sum_i g(x_i)\, \varphi_i(x)| \leq \max_{x \in I} \sum_i |\varphi_i(x)| = \Lambda$$

[5] Henri-Léon Lebesgue (1875-1941)

Λ is the condition number for the interpolation problem

Prop. If $\{\tilde{y}_i\}$ are perturbations of $\{y_i\}$ with $\max_i |y_i - \tilde{y}_i| \leq \epsilon$, then

$$||\Pi_r f - \Pi_r \tilde{f}|| \leq \epsilon \Lambda$$

where $\Pi_r \tilde{f}$ *interpolates* $\{\tilde{y}_i\}$

Proof.

$$\begin{aligned} ||\Pi_r f - \Pi_r \tilde{f}|| &= \max_{x \in I} \left|\sum_i (y_i - \tilde{y}_i)\, \varphi_i(x)\right| \\ &\leq (\max_i |y_i - \tilde{y}_i|)\, (\max_{x \in I} \sum_i |\varphi_i(x)|) \leq \epsilon \Lambda. \end{aligned}$$

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* Small changes on y_i yield small changes on $\Pi_r f$ only if Λ is small

Remarks

We have $\lim_{r \to +\infty} (1 + \Lambda) \, ||f - f^*|| = \infty \, . \, 0$

* If Λ grows faster in r than the best-fit error dies away, convergence in r may be impossible to attain (cf. Runge)

* If Λ grows slowly with r, then $\Pi_r f$ is as good as the f^* ($\Pi_r f$ is easier than f^* to compute !)

* Λ does not depend on the basis $\{\psi_\ell\}$ used to have small $\mathrm{cond}(V)$

* Λ depends heavily on the distribution of points x_i in I

$\mathbf{x}_1 \bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet \, \mathbf{x}_{N_r}$ (uniform) $\Lambda \sim c \, exp(r)$

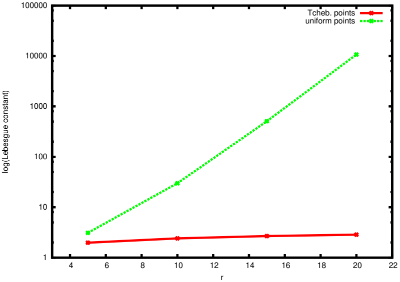
$\mathbf{x}_1 \bullet$ — $\bullet$ ———— $\bullet$ ———— $\bullet$ — $\bullet \, \mathbf{x}_{N_r}$ (Tcheb., Fekete) $\Lambda \sim \tilde{c} \, r$

How to compute $\Lambda = \max_{x \in I} \sum_{i=1}^{N_r} |\varphi_i(x)|$?

We replace the interval I by a discrete repres. of same type as $\{x_i\}$

- $S = \{z_q\}$ is a finite set of points $z_q \in I$
- $\mathrm{card}(S) \gg N_r$

and compute[6] $\Lambda \approx \Lambda_h = \max_{z_q \in S} \sum_{i=1}^{N_r} |\varphi_i(z_q)|$

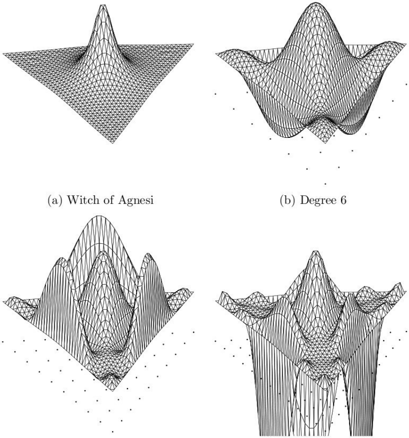


[6] If $S \equiv \{x_i\}$, then $\Lambda_h = 1$.

Polynomial interpolation of a scalar field over a n-simplex $T \subset \mathbb{R}^n$, with $n > 1$

T is a triangle (2-simplex) or a tetra (3-simplex)

Runge phenomenon in a triangle with equally spaced points



(a) Witch of Agnesi
(b) Degree 6
(c) Degree 9
(d) Degree 12

Figure: From the PhD of Michael James Roth, Univ. of Victoria, 2005

Which distribution of points in a n-simplex ?

Straightforward extension to higher dimension on tensorial domains (products of 1D intervals)

What can we do on n-simplices ?

Lebesgue points minimizing Λ are not known in 2D and 3D

Fekete points[7] are among the best for $r > 10$ and $\Lambda \leq N_r$

Warp&blend points $\approx$ Fekete points and have explicit formula

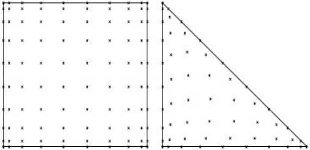
$$\Lambda = \max_{(x,y)\in T} \sum_i |\varphi_i(x,y)|, \qquad (n=2)$$

[7] Michael Fekete (1886-1957) Hungarian mathematician

Fekete points ($N = r$ and $n = N_r$)

Let $\mathcal{P}_N(T)$ the space of polynomials over T of degree $\leq N$ and dim$\mathcal{P}_N = n$

Given the basis $\{\psi_j\}_{j=1}^n$ of $\mathcal{P}_N(T)$, Fekete's points $\{\mathbf{x}_i\}$ maximize over T the determinant of the Vandermonde matrix V, with $V_{ij} = \psi_j(\mathbf{x}_i)$, $i, j = 1, \cdots, n$.



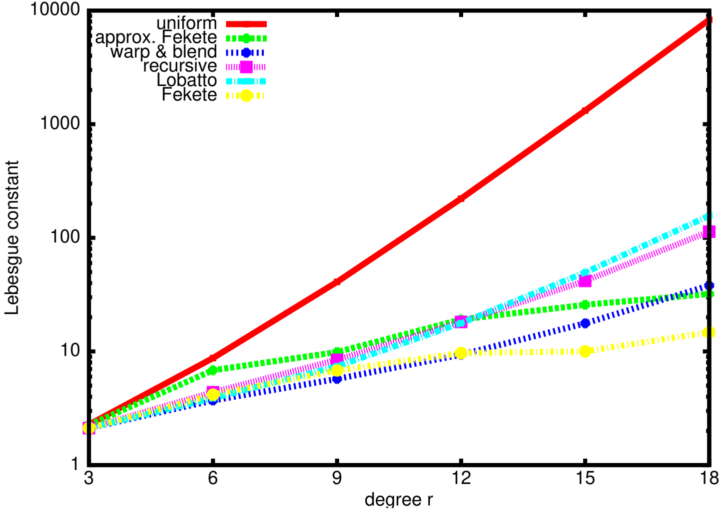
The cardinal function associated to the node (ξ_i, η_i) is

$$\phi_i(x,y) = \frac{\det V(x_1, y_1, ..., x_{i-1}, y_{i-1}, x, y, x_{i+1}, y_{i+1}, ..., x_n, y_n)}{\det V(x_1, y_1, ..., x_n, y_n)}, \qquad i = 1, n$$

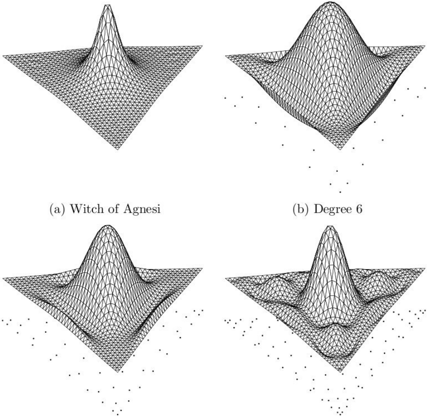
We remark that $|\phi_i(x,y)| \leq 1$.

Fekete points $\{\mathbf{x}_i\}_{i=1}^n$ do not depend on the choice of the basis $\{\psi_j\}_{j=1}^n$.

The Lebesgue constant $k = 0$, $n = 2$



Runge phenomenon in a triangle with Fekete points



(a) Witch of Agnesi
(b) Degree 6
(c) Degree 9
(d) Degree 12

Figure: From the PhD of Michael James Roth, Univ. of Victoria, 2005

If instead of values at points, we used averages ...

$$I = \bigcup_{i=1}^{N_r - 1} [x_i, x_{i+1}] = \bigcup_{i=1}^{N_r - 1} \sigma_i, \qquad a_i = \int_{x_i}^{x_{i+1}} f\, dx = \int_{\sigma_i} f\, dx$$

We have $\{a_i\}$ averages on sub-intervals $\{\sigma_i\}$ in I, $i = 1, ..., N_r - 1$

** We wish to represent $\{a_i\}$ by a polynomial function $\Pi_{r-1} f$ and here, we construct $\Pi_{r-1} f$ that interpolates the $\{a_i\}$ on the $\{\sigma_i\}$ **

We assume $\sigma_i \cap \sigma_j = \emptyset$, for $i \neq j$, thus $\Pi_{r-1} f$ is function such that

$$\begin{array}{ll} (1) & \Pi_{r-1} f \in \mathbb{P}_{r-1}(I), \\ (2) & \int_{\sigma_i} \Pi_{r-1} f\, dx = \int_{\sigma_i} f\, dx, \quad \forall i = 1, ..., N_r - 1 \end{array}$$

$$\Pi_{r-1} f(x) = \sum_{i=1}^{N_r - 1} \Big(\int_{\sigma_i} f \, dx \Big) \, \varphi_i(x), \qquad \int_{\sigma_j} \varphi_i \, dx = \delta_{i,j}$$

To compute φ_i use general technique (as before)

- choose a basis $\{\psi_\ell\}$ in $\mathbb{P}_{r-1}(I)$ and set $(V)_{j,\ell} = \int_{\sigma_j} \psi_\ell \, dx$
- write $\varphi_i(x) = \sum_{\ell=1}^{N_r-1} c_\ell^i \, \psi_\ell(x)$
- find the vector $\mathbf{c}^i$ of coefficient c_ℓ^k by solving $V \, \mathbf{c}^i = \mathbf{e}_i$.

Runge phenomenon if $\{\sigma_i\}$ is a uniform distribution in I

Similar estimates on the interpolation error ... the norm changes

Generalized Lebesgue constant Λ^1 (Alonso & R., JCP'21)

The mass[8] of a segment s (1-simplex) is $|s|_0 = \mathrm{diam}(s)$
If $s = \sum_{j \in J} c_j\, s_j$ then $|s|_0 = \sum_{j \in J} |c_j|\, |s_j|_0$

$$\Lambda^1 = \max_{s \subset I} \frac{1}{|s|_0} \sum_i |\sigma_i|_0\, |\int_{\sigma_i} \varphi_i\, dx| \qquad (\varphi_i\, dx \text{ is a } 1-\mathrm{form})$$

* the mass of any point x (0-simplex) is $|x|_0 = 1$
* $\int_x \varphi_i dx = \varphi_i(x)$
* If $\sigma_i \rightsquigarrow x_i$ and $s \rightsquigarrow x$, then $\Lambda^1 \rightsquigarrow \Lambda^0 = \Lambda \qquad (||g||_0 \rightsquigarrow ||g||)$

We can still prove that

$$||f - \Pi_{r-1} f||_0 \leq (1 + \Lambda^1)\, ||f - \tilde{f}^*||_0, \qquad ||g||_0 = \sup_{s \neq 0,\, s \subset I} \frac{|\int_s g\, dx|}{|s|_0}$$

[8] The mass $|\sigma|_0$ of a k-simplex σ is its k-dimensional Hausdorff measure.

Estimated Λ_h^0 and Λ_h^1 for $n = 1$

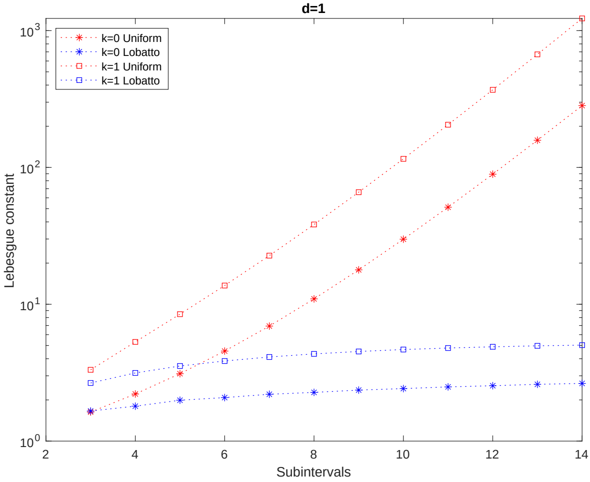
Estimated generalised Lebesgue constants in an *interval* associated with the uniform and the GLLobatto distribution of nodes.

$k = 0$	Λ_{Un}	Λ_{Lb}
3	1.63	1.66
4	2.21	1.80
5	3.11	1.99
6	4.55	2.08
7	6.93	2.20
8	10.95	2.27
9	17.85	2.36
10	29.90	2.42
11	51.21	2.49
12	89.32	2.54
13	158.09	2.60
14	283.18	2.64

$k = 1$	Λ_{Un}	Λ_{Lb}
3	3.32	2.66
4	5.31	3.15
5	8.47	3.54
6	13.71	3.85
7	22.68	4.12
8	38.30	4.34
9	65.97	4.52
10	115.57	4.67
11	205.40	4.79
12	369.40	4.89
13	670.91	4.97
14	1228.48	5.03

In the first column the number of subintervals. On the left it is the degree of the polynomial differential form, on the right it is the degree plus one.

Λ_h^0 and Λ_h^1 for $n = 1$



Polynomial interpolation of any field over

a n-simplex $T \subset \mathbb{R}^n$

Can we still talk about Lebesgue constant, etc. ?

Fields of any type

Let $T \subset \mathbb{R}^3$ be a tetrahedron.

$$\begin{array}{ccccccc}
H^1(T) & \xrightarrow{\text{grad}} & H(\text{curl}; T) & \xrightarrow{\text{curl}} & H(\text{div}; T) & \xrightarrow{\text{div}} & L^2(T) \\
L_r(T) & \longrightarrow & N_r(T) & \longrightarrow & RT_r(T) & \longrightarrow & DP_{r-1}(T)
\end{array}$$

$L_r(T)$ is $\mathbb{P}_r(T) = W_r^0(T)$

$N_r(T)$ is $W_r^1(T)$

$RT_r(T)$ is $W_r^2(T)$

$DP_{r-1}(T)$ is discontinuous-$\mathbb{P}_{r-1}(T) = W_{r-1}^3(T)$.

They can be identified with the spaces of trimmed polynomial differential k-forms

$$\mathcal{P}_r^- \Lambda^k(T) \quad k = 0, 1, 2, 3 \text{ respectively.}$$

Notation

Given a 3-simplex $T = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and $j \in \{0, 1, 2, 3\}$

- $\Delta_j(T)$ denotes the set of j-subsimplices of T;
- $\lambda_j(\mathbf{x})$ denote the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices of t.

The principal lattice of order r of t is the set of points

$$\Sigma_r(t) = \left\{ \mathbf{x} \in t \, : \, \lambda_j(\mathbf{x}) \in \left\{ 0, \frac{1}{r}, \ldots \frac{r-1}{r}, 1 \right\} \quad \forall j \in \{0, 1, 2, 3\} \right\}$$

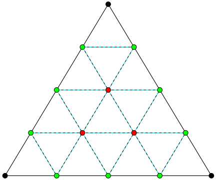


Figure: The principal lattice of a triangle for $r = 4$.

$L_r(T)$: degrees of freedom

Classical degrees of freedom for $f_h \in L_r(T)$ are the values (weights) of f_h at the points of the principal lattices of T

$$f_h(\mathbf{x}_i) \text{ for each } \mathbf{x}_i \in \Sigma_r(T)$$

Alternative degrees of freedom are the moments:

Nodal $\varphi(\mathbf{v}_i)$ for $i = 0, 1, 2, 3$;

Edge $\int_e \varphi\, q$ for each $e \in \Delta_1(\mathcal{T}_h)$ and $q \in \mathbb{P}_{r-2}(e)$;

Face $\int_f \varphi\, q$ for each $f \in \Delta_2(\mathcal{T}_h)$ and $q \in \mathbb{P}_{r-3}(f)$;

Element $\int_t \varphi\, q$ for each $t \in \mathcal{T}_h$ and $q \in \mathbb{P}_{r-4}(t)$.

There is a clear correspondence with "nodal", "edge", "face" and "element" points of the principal lattice of T.

How is it possible to define weights for other types of fields ?

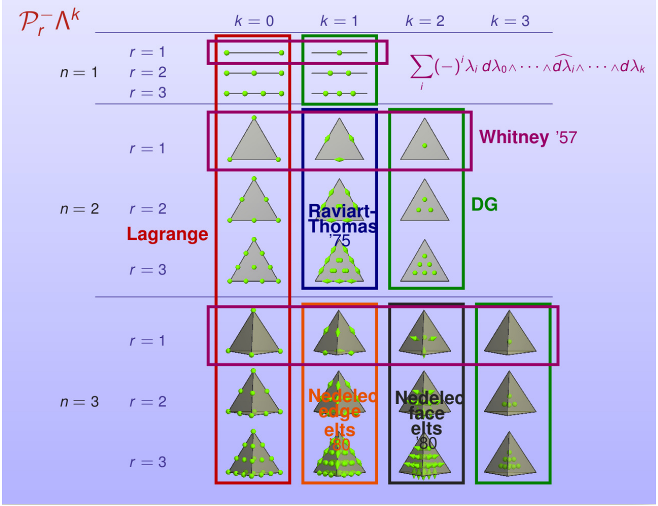


Figure: D.N.Arnold, Periodic table of FEs

Weights for fields in $\mathcal{P}_r^- \Lambda^k(T)$

The weights were introduced by R. and Bossavit (2009).

The degrees of freedom for a k-form $\omega \in \mathcal{P}_r^- \Lambda^k$ are integrals [9] on k-chains $\sigma \in C_k(T)$:

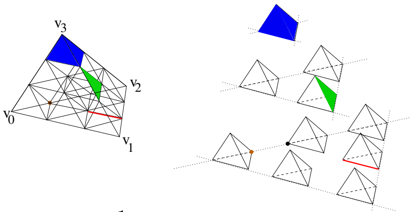
$$\int_\sigma \omega$$

Consider in particular the integrals on the so-called small k-simplices associated with the principal lattice of order r of T

$$\sigma = \sum_{\alpha,\, s} c_{\{\alpha,s\}} \{\alpha, s\}, \qquad \int_\sigma \omega = \sum_{\alpha,\ s} c_{\{\alpha,s\}} \int_{\{\alpha,s\}} \omega$$

[9] If $k = 0$, σ is a point and $\int_\sigma \omega = \omega(\sigma)$. If $k = 1, 2$ then $\int_\sigma \omega$ is the circulation or the flux on σ respectively.

Small k-simplices in a simplex T (R. & Bossavit, 2009)



- The small volumes are $\frac{1}{r}$ homothetic to T and their vertices are points of the principal lattice $\Sigma_r(T)$
- Small edges and small faces are edges and faces of the small volumes. Small nodes are the points of $\Sigma_r(T)$.
- A small k-simplex is $\{\alpha, s\}$, $\alpha \in \mathcal{I}(r-1, n)$, $s \in \Delta_k(T)$.

For $r = 3$ (left): small edge $\{(1,1,0,0), [v_0, v_1]\}$,
small face $\{(0,1,0,1), [v_1, v_2, v_3]\}$, small tetra $\{(0,0,0,2), t\}$.

Unisolvence

The integrals of a k-form $\omega \in \mathcal{P}_r^- \Lambda^k$ on the small k simplices of T are determinant, namely, if $X_r^k(T)$ denotes the set of small k simplices of order r in T then

$$\text{if } \omega \in \mathcal{P}_r^- \Lambda^k, \quad \int_\sigma \omega = 0 \quad \forall\, \sigma \in X_r^k \Rightarrow \omega = 0\,.$$

For the proof see Christiansen and R. (2016).

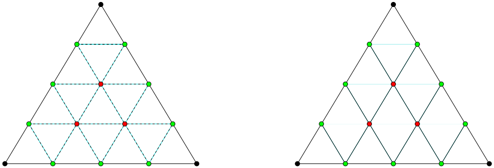
However, for $k = 1$ and $k = 2$ in $\mathbb{R}^3$, the number of elements of X_r^k is greater than the dimension of $\mathcal{P}_r^- \Lambda^k$.

Unisolvence for $k = 1, 2$: find $S_r^k \subset X_r^k$ s.t. $\# S_r^k = \dim P_r^- \Lambda^k$.
(See Alonso, Bruni Bruno and R. (2019).)

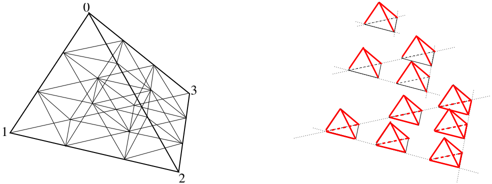
Example

Sets of unisolvent small edges ($k = 1$).

- 2D ($r = 4$) (interior of a face f, small edges $\parallel$ to 2 sides of f)



- 3D ($r = 3$) (interior of T, small edges $\parallel$ to 3 sides of T)



Interpolation of differential k-forms

- A set S_r^k of k-simplices $\sigma = \{\boldsymbol{\alpha}, s\}$ $\boldsymbol{\alpha} \in \mathcal{I}(r-1,n)$, $s \in \Delta_k(T)$, is unisolvent in $\mathcal{P}_r^- \Lambda^k$ then the weight matrix V is invertible

$$V_{i,j} = \int_{\sigma_i} \psi^{\sigma_j}, \qquad i,j = 1,\dots,\#S_r^k, \qquad \psi^\sigma = B_{\boldsymbol{\alpha}}^n \omega^s$$

$B_{\boldsymbol{\alpha}}^n = \binom{n}{\boldsymbol{\alpha}} \lambda^{\boldsymbol{\alpha}}$ Bernstein polyn., ω^s Whitney k-form of deg. 1.

- Given a set S_r^k of k-simplices that are unisolvent in $\mathcal{P}_r^- \Lambda^k$ the associated canonical basis $\{\varphi_\sigma\}_{\sigma \in S_r^k}$ is such that

$$\int_{\sigma'} \varphi_\sigma = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ 0 & \text{otherwise} \end{cases} \qquad \psi^{\sigma_j} = \sum_\ell V_{\ell,j}\, \varphi_{\sigma_\ell}.$$

Interpolation of differential k-forms

- If ω is a differential k-form we denote $\Pi_r^k \omega$ the unique element of $\mathcal{P}_r^- \Lambda^k$ such that

$$\int_\sigma \omega = \int_\sigma \Pi_r^k \omega \quad \forall\, \sigma \in S_r^k.$$

- If $\{\varphi_\sigma\}_{\sigma \in S_r^k}$ is the canonical basis associated with S_r^k then

$$\Pi_r^k \omega = \sum_{\sigma \in S_r^k} \left(\int_\sigma \omega \right) \varphi_\sigma$$

Interpolation of differential k-forms: Lebesgue constant

Let ω and $\widetilde{\omega}$ be two differential k-forms such that for any k-simplex σ of measure $|\sigma|$

$$\frac{1}{|\sigma|}\left|\int_\sigma (\omega - \widetilde{\omega})\right| \leq \epsilon.$$ [10]

Then

$$\frac{1}{|c|}\left|\int_c \left(\Pi_r^k \omega - \Pi_r^k \widetilde{\omega}\right)\right| \leq \epsilon \sum_{\sigma \in S_r^k} \frac{1}{|c|}|\sigma| \left|\int_c \varphi_\sigma\right|. \tag{1}$$

The generalised Lebesgue constant for differential k-forms is defined as

$$\Lambda(S_r^k) := \sup_c \sum_{\sigma \in S_r^k} \frac{1}{|c|}|\sigma| \left|\int_c \varphi_\sigma\right|.$$

being $\{\varphi_\sigma\}_{\sigma \in S_r^k}$ the canonical basis associated with S_r^k.

(See Alonso and Rapetti (2021)).

[10] $|\omega|_0 := \sup_c \frac{1}{|c|}\left|\int_c \omega\right|$ is a norm for regular k- forms.

Proof of (1)

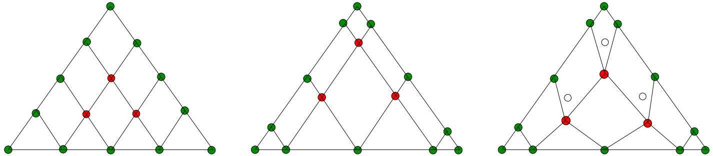
$$\frac{1}{|c|}\left|\int_c \left(\Pi_r^1\omega - \Pi_r^1\widetilde{\omega}\right)\right| = \frac{1}{|c|}\left|\int_c \sum_{\sigma\in S_r^k}\left(\int_\sigma(\omega-\widetilde{\omega})\right)\varphi_\sigma\right|$$

$$= \frac{1}{|c|}\left|\sum_{\sigma\in S_r^k}\int_\sigma(\omega-\widetilde{\omega})\int_c\varphi_\sigma\right| \le \frac{1}{|c|}\sum_{\sigma\in S_r^k}\left|\int_\sigma(\omega-\widetilde{\omega})\right|\left|\int_c\varphi_\sigma\right|$$

$$\le \frac{1}{|c|}\sum_{\sigma\in S_r^k}\epsilon|\sigma|\left|\int_c\varphi_\sigma\right| = \epsilon\sum_{\sigma\in S_r^k}\frac{1}{|c|}|\sigma|\left|\int_c\varphi_\sigma\right|$$

Interpolation nodes and edges on the simplex

We investigate if spatial distributions of nodes that are suitable for high-order Lagrange interpolation on the triangle and tetrahedron[11] induce (by a simplicial map) small k-simplices suitable for the interpolation in $\mathcal{P}_r^- \Lambda^k(T)$.



Interpolation nodes: Uniform, Lobatto, and symmetrised Lobatto.

[11] See Blyth, Luo, and Pozrikidis (2006), Warburton (2006).

Estimated Lebesgue constant

We estimate the generalised Lebesgue constant for different configurations of nodes.

- We consider a "reference" mesh $\mathcal{T}_R$ of t and compute

$$\max_{c \in \Delta_k(\mathcal{T}_R)} \sum_{\sigma \in S_r^k} \frac{1}{|c|} |\sigma| \left| \int_c \varphi_\sigma \right| \approx \Lambda(S_r^k).$$

We compare the classical results for $k = 0$ with those obtained for $k = 1$ in dimension 1, 2 and 3 when increasing the polynomial degree.

Estimated Lebesgue constant: $d = 2$, $k = 0$

$k = 0$	uniform in 2D	nonuniform in 2D	
r	Λ_{Un}	$\Lambda_{Lb\,sym}$	Λ_{WB}
3	2.27	2.11	2.11
4	3.47	2.66	2.66
5	5.45	3.14	3.12
6	8.75	3.87	3.70
7	14.35	4.66	4.27
8	24.01	5.93	4.96
9	40.92	7.39	5.74
10	70.89	9.83	6.67
11	124.53	12.92	7.90
12	221.41	17.78	9.36

Lebesgue constants in a *triangle* T associated with a uniform and nonuniform (symmetrised Lobatto and "warp and blend") distribution of nodes for different polynomial degrees $r \geq 3$, as computed in Warburton (2006).

Estimated Lebesgue constant: $n = 2$, $k = 1$

$k = 1$	uniform in 2D	nonuniform in 2D		
r	Λ_{Un}	Λ_{Lb}	$\Lambda_{Lb\,sym}$	Λ_{WB}
3	7.92	6.67	6.71	6.71
4	12.17	9.17	8.16	8.16
5	18.92	14.51	9.61	9.60
6	29.95	23.49	11.80	11.62
7	48.31	41.55	14.71	14.51
8	79.45	77.15	18.13	17.65
9	133.03	154.18	20.99	20.32
10	226.20	327.36	28.74	24.44
11	389.59	827.80	38.15	29.19
12	678.10	2142.45	52.97	35.85

Lebesgue constants in a *triangle* T, associated with uniform and nonuniform distributions of small edges for different polynomial degrees.

The ending points of the small edges are either in the uniform or in the nonuniform (Lobatto, symmetrised Lobatto and "warp and blend") sets.

Estimated Lebesgue constant: $n = 3$, $k = 0$

$k = 0$	uniform in 3D	nonuniform in 3D	
r	Λ_{Un}	$\Lambda_{Lb\,sym}$	Λ_{WB}
3	2.94	2.93	3.11
4	4.88	4.07	4.07
5	8.09	5.38	5.32
6	13.66	7.53	7.01
7	23.38	10.17	9.21
8	40.55	14.63	12.54
9	71.15	20.46	17.02

Lebesgue constants in a *tetrahedron* T associated with a uniform and nonuniform (symmetrised Lobatto and "warp and blend") distributions of nodes for different polynomial degrees $r \geq 3$, as computed in Warburton (2006).

Estimated Lebesgue constant: $n = 3$, $k = 1$

$k = 1$	uniform in 3D	nonuniform in 3D		
r	Λ_{Un}	Λ_{Lb}	$\Lambda_{Lb\,sym}$	Λ_{WB}
3	11.23	11.40	10.80	10.80
4	18.04	22.38	15.25	15.25
5	29.37	69.45	20.09	20.79
6	46.76	274.58	26.73	28.32
7	74.19	1168.36	36.57	36.03
8	127.53	5443.19	48.66	45.82
9	218.19	26323.67	61.90	57.24

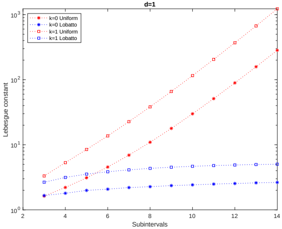
Lebesgue constants in a *tetrahedron* T, associated with uniform and nonuniform distributions of small edges for different polynomial degrees $r \geq 3$. The ending points of the small edges are either in the uniform or in the nonuniform (Lobatto, symmetrised Lobatto or "warp and blend") sets.

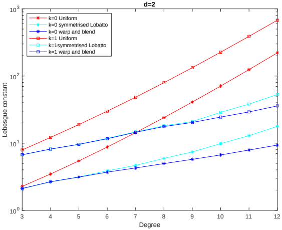
Legend

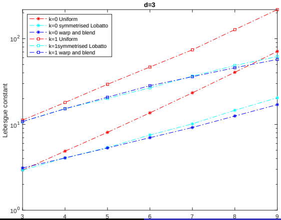
- $k = 0 \rightsquigarrow *$ $k = 1 \rightsquigarrow \square$
- Uniform nodes $\rightsquigarrow$ red Nonuniform nodes $\rightsquigarrow$ blue (or cyan)
- $d = 1 \rightsquigarrow \cdots$ $d = 2 \rightsquigarrow$ — $d = 3 \rightsquigarrow$ -·-

The $*$ lines and the $\square$ lines are almost parallel.

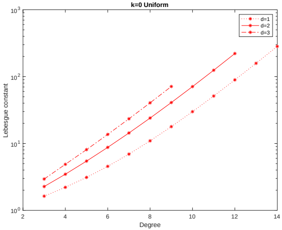
Estimated Lebesgue constant: $k = 0$ and $k = 1$

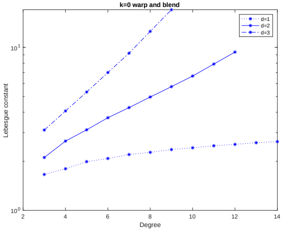


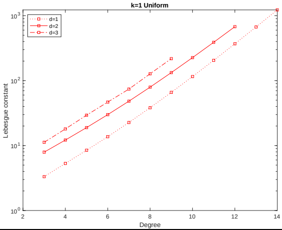


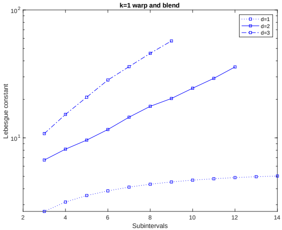


Estimated Lebesgue constant for n=1, 2, and 3









Conclusions

The $*$ lines and the $\square$ lines are essentially parallel.

So, the well-known results for $k = 0$ hold also true for $k = 1$:

- the interpolation on uniform distribution of the support of the degrees of freedom is not stable on the polynomial degree;
- the problem increases with the dimension of the space;
- the Lebesgue constant "measures" the stability on the polynomial degree of the polynomial interpolation problem;
- the distribution of the supports that minimises the Lebesgue constant is not uniform.

For $k = 1$ (and $k = 2$), the generalized Lebesgue constant depends on the shape of the element.

Last achievements

Weights

allow to extend naturally the matrix of the gradient operator: if $\{\phi_i^0\}$ is the cardinal basis in L_r and $\{z_j^1\}$ is the one for N_r, verifying

$$\phi_i^0(n_k) = \delta_{i,k}, \qquad \int_{e_\ell} z_j^1 = \delta_{j,\ell}$$

then, $\forall \varphi_h \in L_r$ and $\forall w_h \in N_r$ we have, respectively,

$$\varphi_h = \sum_{n_i} \varphi_h(n_i)\, \phi_i^0, \qquad w_h = \sum_{e_\ell} (\int_{e_j} w_h)\, z_j^1.$$

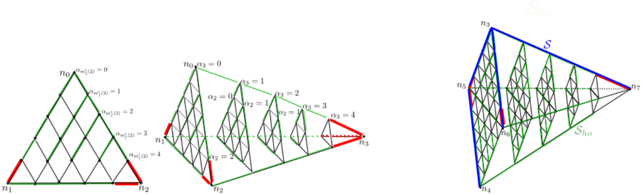
In particular:

$$\mathrm{grad}\varphi_h = \sum_{e_j} (\int_{e_j} \mathrm{grad}\varphi_h)\, z_j^1 = \sum_{e_j} (\varphi_h(n_{jend}) - \varphi_h(n_{jini}))\, z_j^1$$

Graph techniques

Related with this topic recently we have

- proposed an algorithm for the construction of the high order tree and, in the case of not simply connected domains, a high order belted tree;



- proved that there exists a particular set of moments and an isomorphism between weights and this set of moments that preserves the matrix of the grad operator;
- used the tree-cotree decomposition for the construction of a basis of divergence free Raviart-Thomas finite element space of arbitrary polynomial degree.

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