Polytopal methods on Riemannian manifolds

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Based on a collaboration with Jérôme Droniou and Todd Oliynyk

https://arxiv.org/abs/2401.16130

Partial differential equations on Manifolds

Several approaches

- Regge calculus
- Lattice based

T. Regge *General relativity without coordinates* 1961

L. Brewin *Riemann normal coordinates, smooth lattices and numerical relativity* 1998

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- Lattice based
- Finite elements

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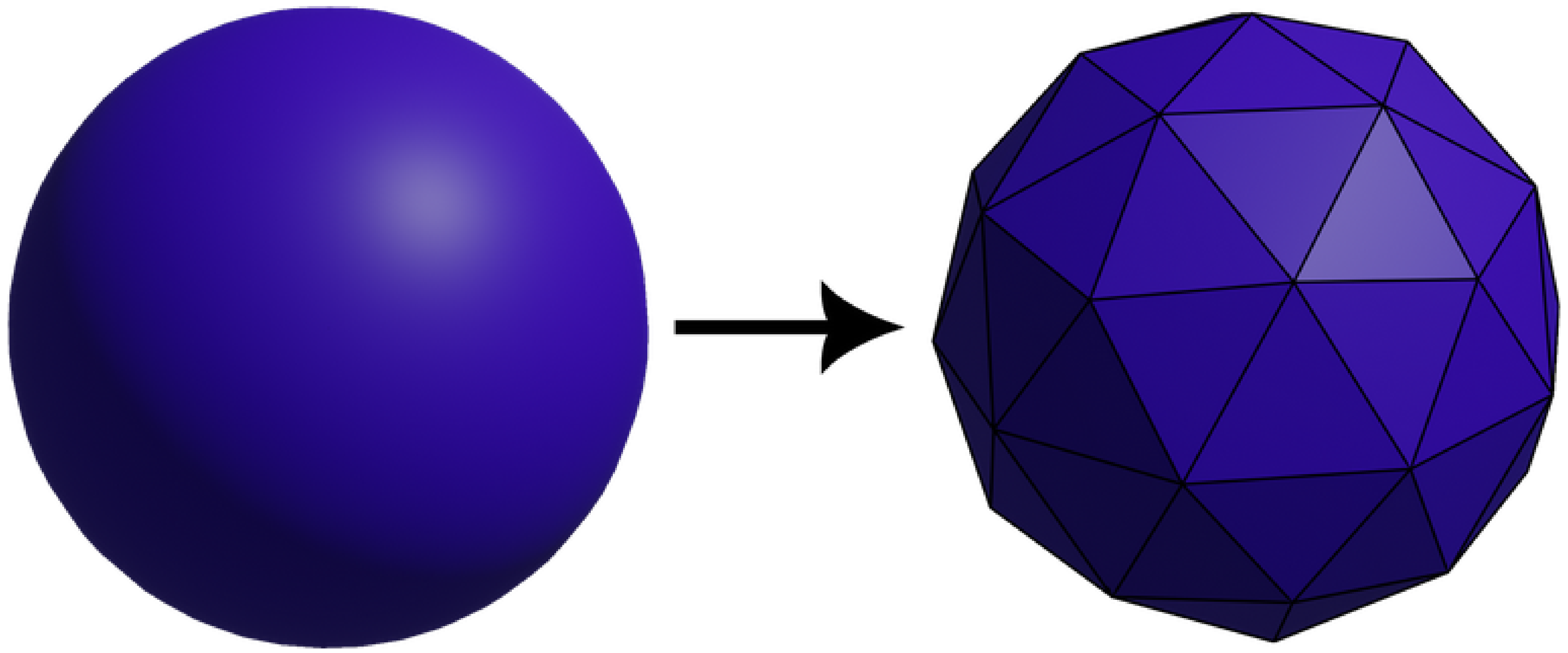
First idea: discretizing the Manifold

Consider the Manifold as embedded in $\mathbb{R}^n$ (usually a surface in $\mathbb{R}^3$), and approximate it with piecewise flat elements.



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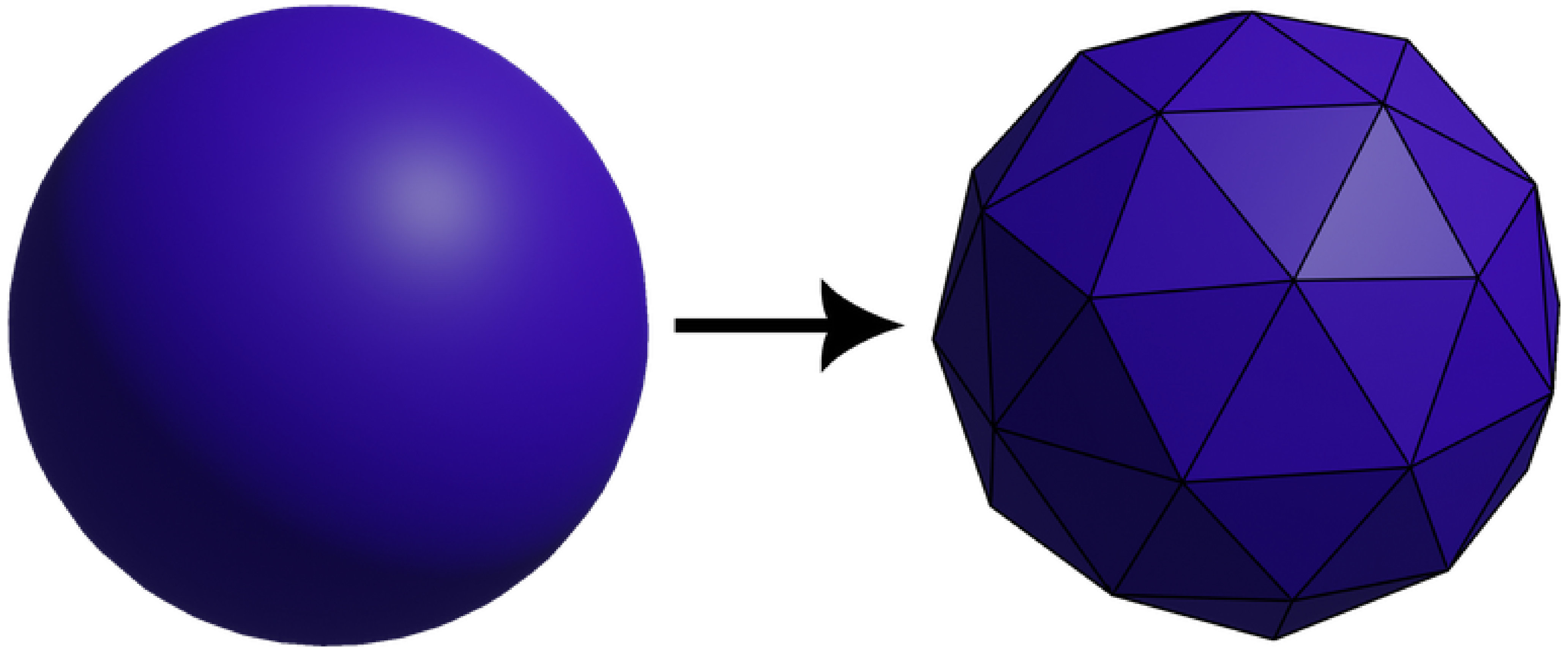
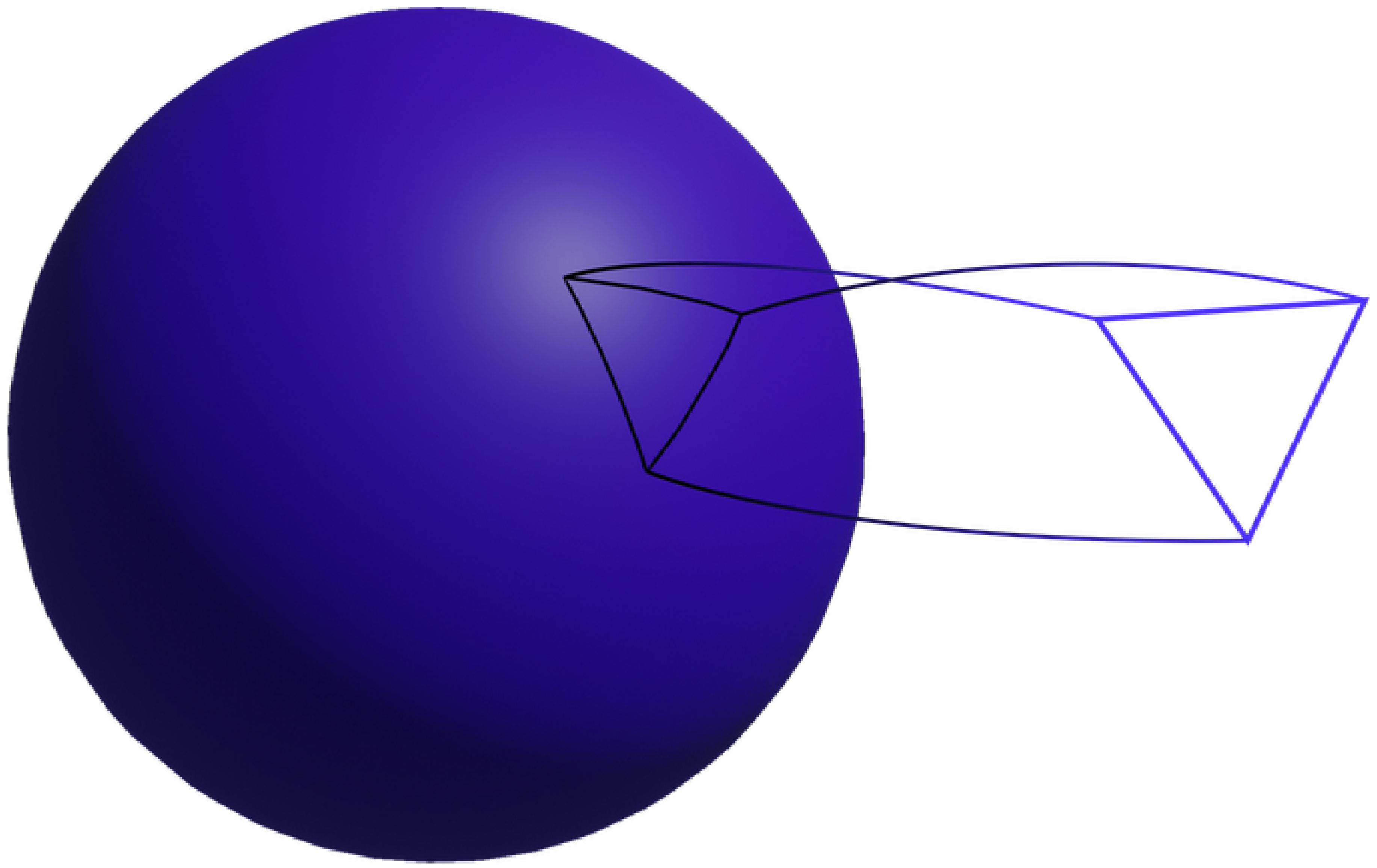


Problem:

Limited to low order discretization, as the geometrical error dominates.

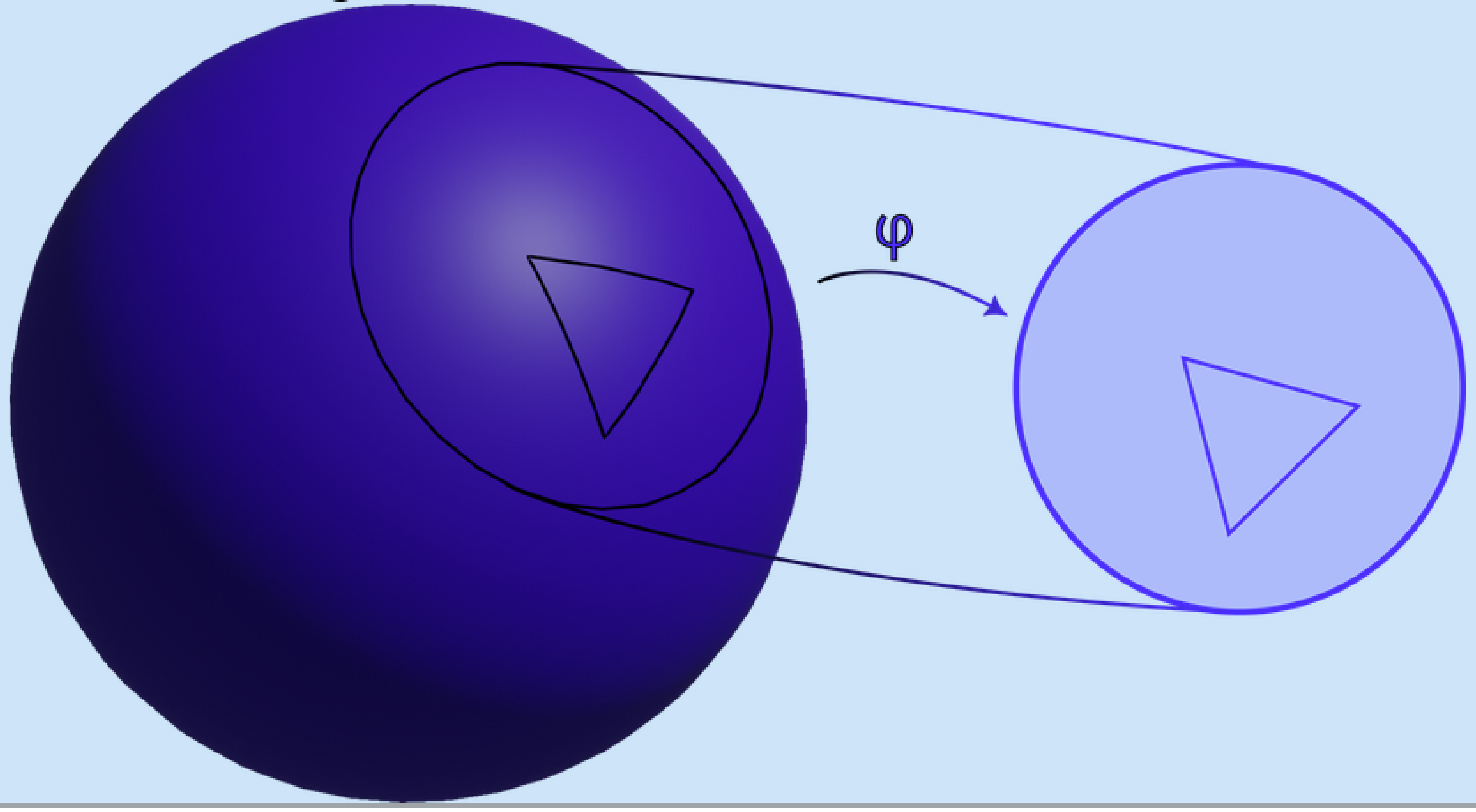
Intrinsic approach



- Define elements on a flat space
- Maps to the manifold using pullbacks

Remark

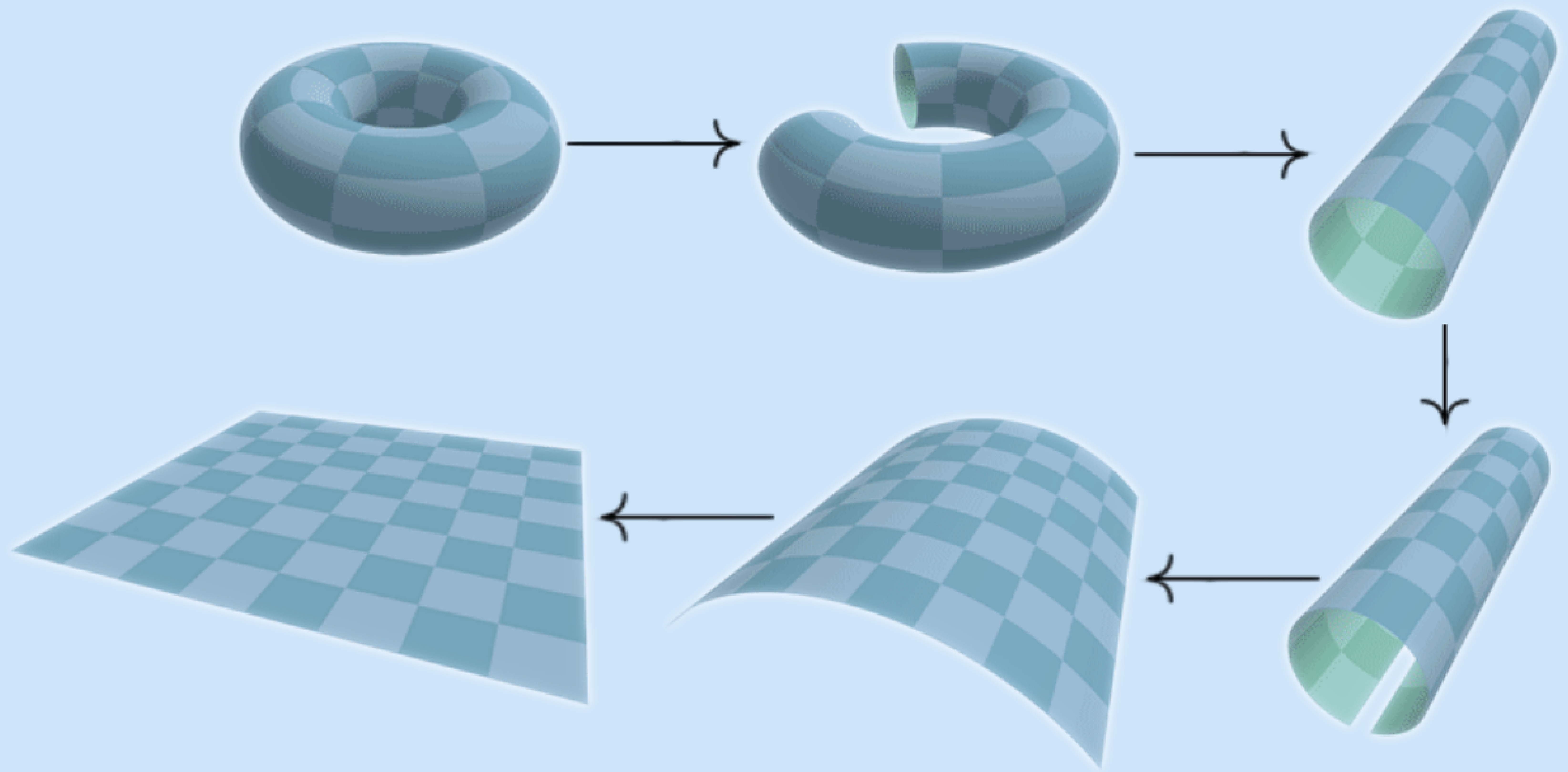
Equivalent to discretizing in a chart:



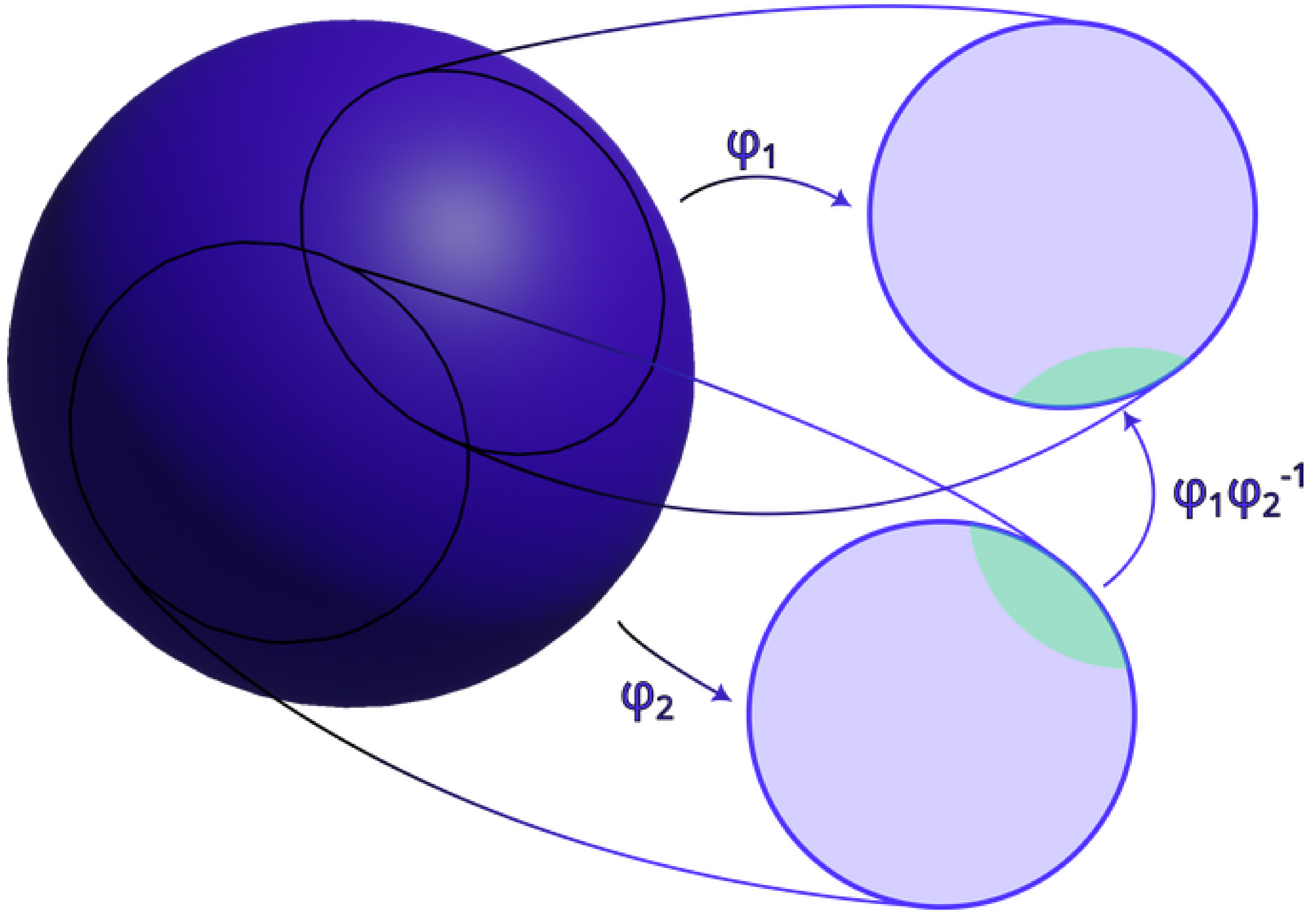
It works (sometimes™) !

Example

Torus viewed as a square with periodic boundary conditions



In general, we need several charts in an atlas



$\phi_1\phi_2^{-1}$ is a diffeomorphism, and do not need to preserve polynomials or flatness.

General difficulties

Lack of a global objects

- No coordinates systems $\Rightarrow$ hard to define polynomials
- No global chart $\Rightarrow$ hard to define a mesh

We need to define the method locally

Fully discrete methods

Flexibility for the interfaces

- Localized
- Mild compatibility conditions on interfaces

D. A. Di Pietro, J. Droniou *An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré Inequalities, and Consistency* 2021

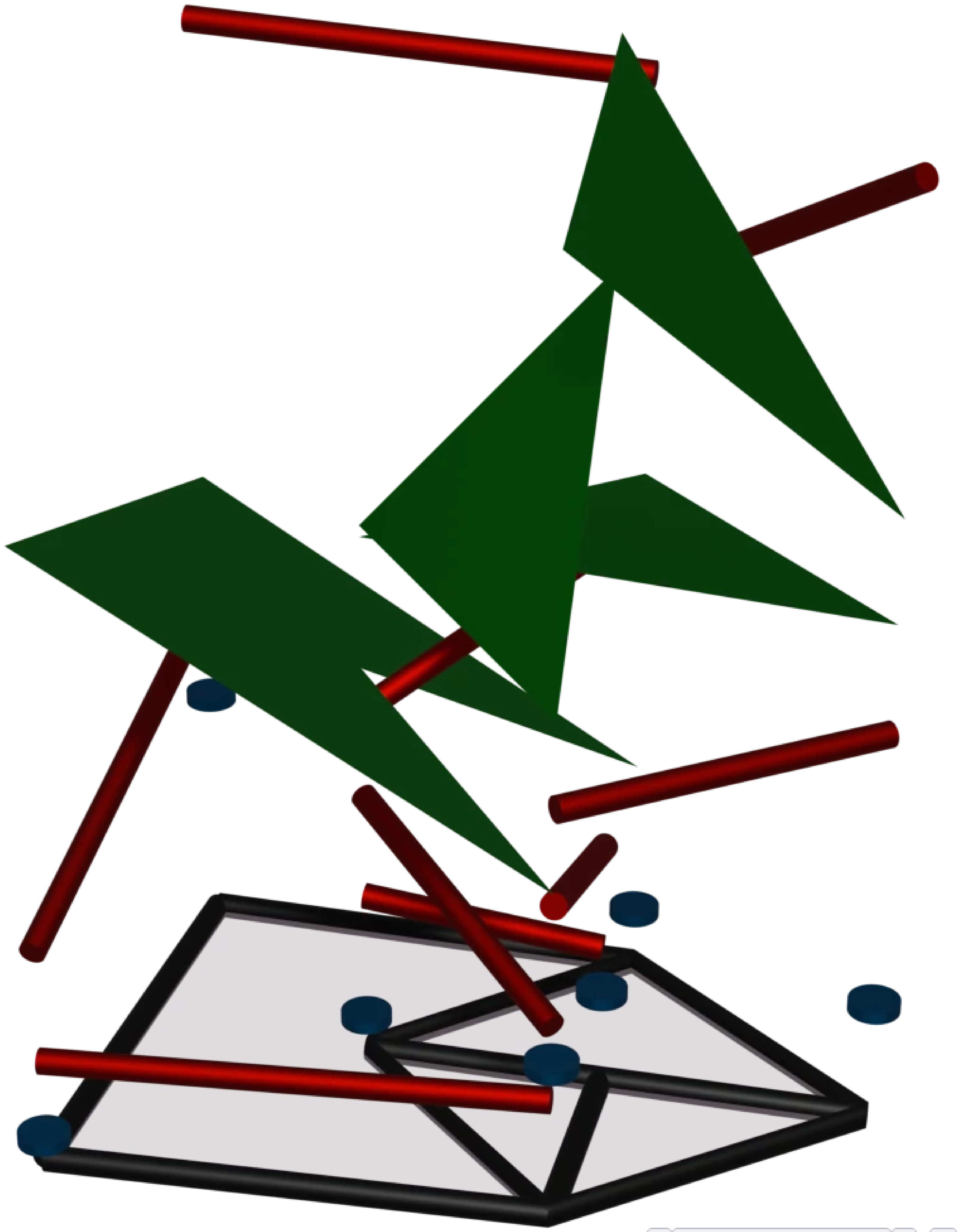
Discrete objects

An element $\underline{\boldsymbol{u}}$ of the discrete space $\underline{X}_h$ is a collection of local functions over the cells, edges and vertices:

$$\underline{\boldsymbol{u}} := \{((u_V)_{V\in\mathcal{V}_h}, (u_E)_{E\in\mathcal{E}_h}, (u_F)_{F\in\mathcal{F}_h})\}.$$

In general:

$$\underline{\boldsymbol{u}} := \{((u_f)_{f\in 0\text{-cells}}, (u_f)_{f\in 1\text{-cells}}, \ldots (u_f)_{f\in d\text{-cells}})\}.$$



Mesh on Manifolds

Intrinsic definition, not relying on charts

- "Flat" is not meaningfull
- Since we do not work on a vector space, the convex hull is not meaningfull
- The topological information remains: faces are in the boundary of cells...

Let Ω be an n dimensional manifold.

Definition

A d-cell is a submanifold of Ω of dimension d. We denote by $\Delta_d(\mathcal{M}_h)$ the set of d-cells.

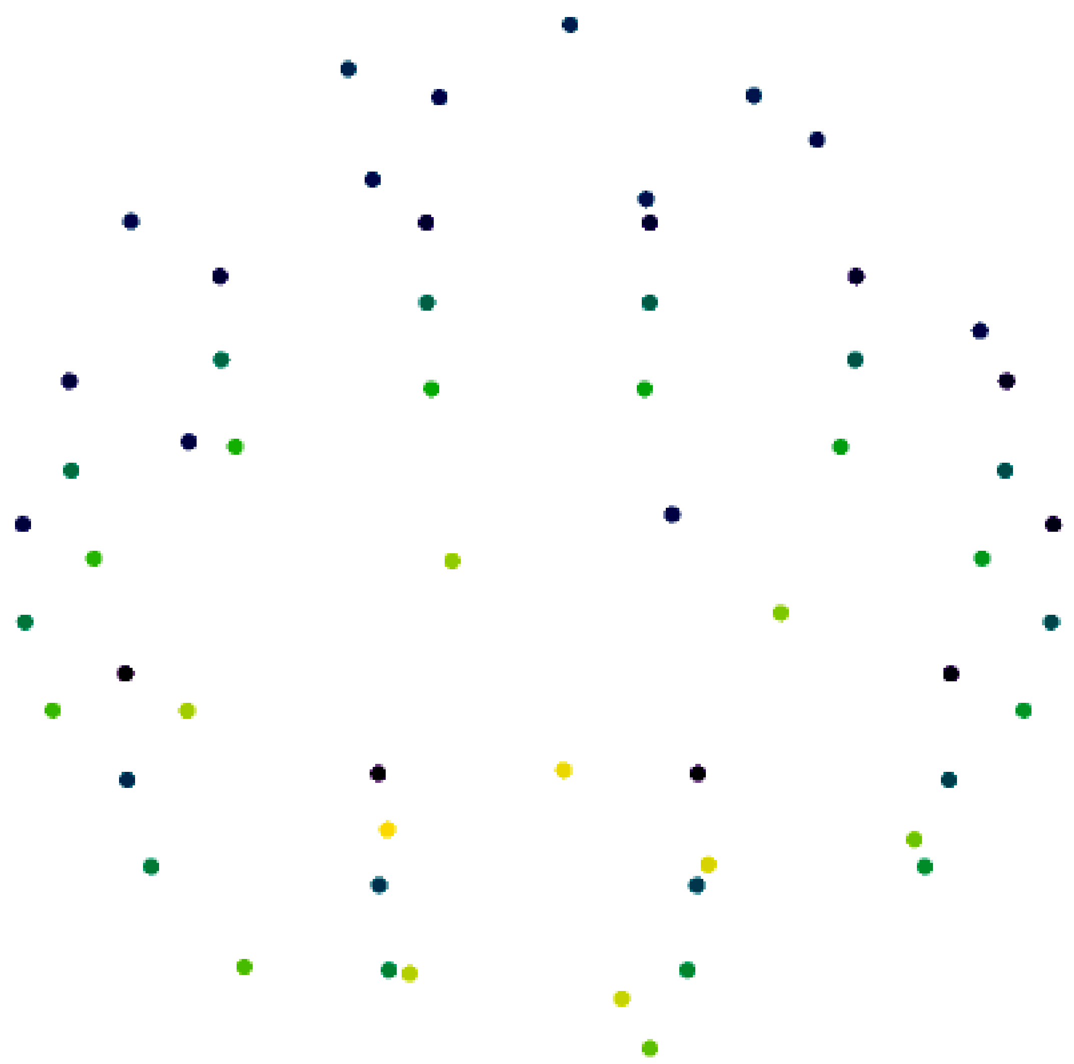
- The submanifolds cover Ω:

$$\bigcup_{d\in[0,n]} \bigcup_{f\in\Delta_d(\mathcal{M}_h)} f = \Omega.$$

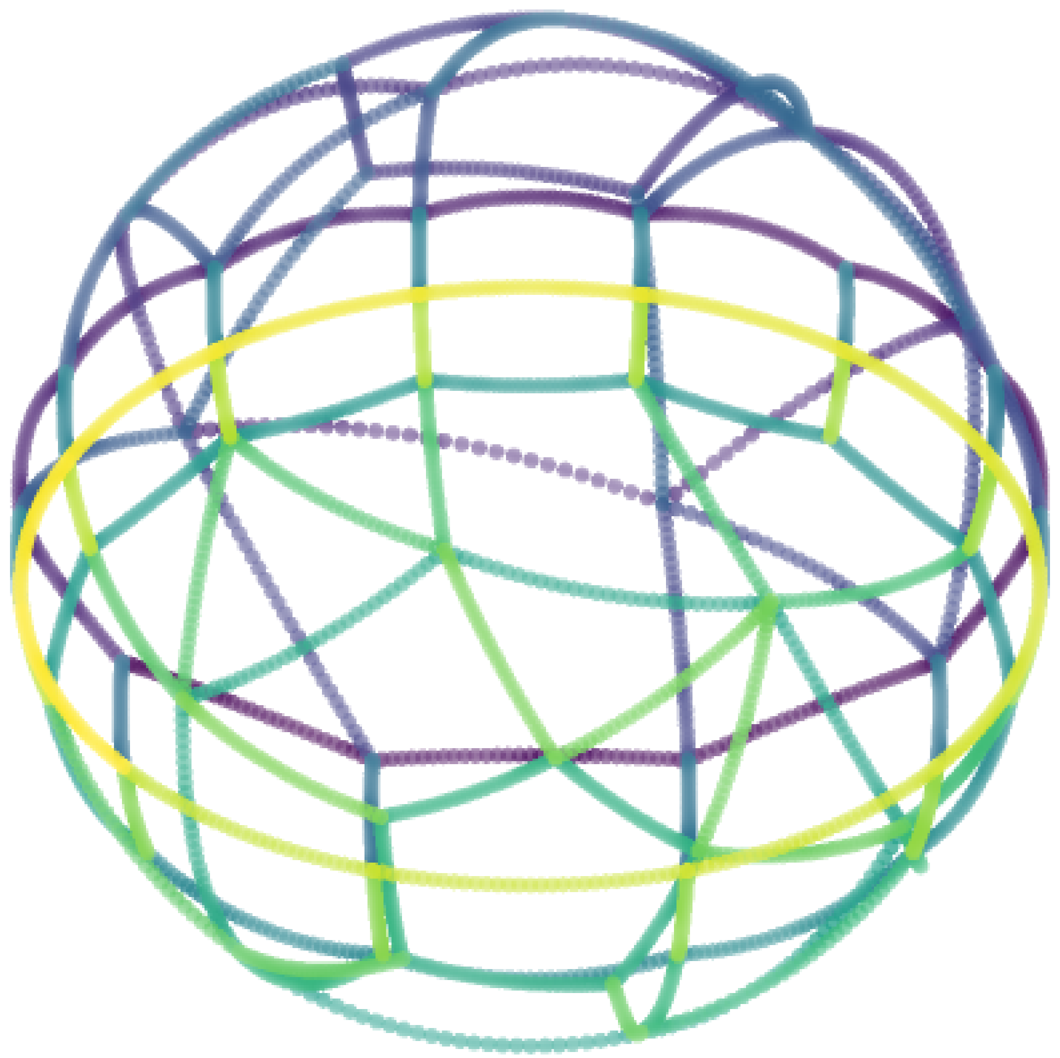
- The submanifolds are pairwise disjoint: $\forall f, f' \in \mathcal{M}_h, f \neq f' \Rightarrow f \cap f' = \emptyset$.
- Boundaries of the submanifolds are submanifolds:

$$\forall d \in [1, n], f \in \Delta_d(\mathcal{M}_h), d' \in [0, d-1], f' \in \Delta_{d'}(\mathcal{M}_h), f' \cap \overline{f} \neq \emptyset \Rightarrow f' \subset \partial f.$$

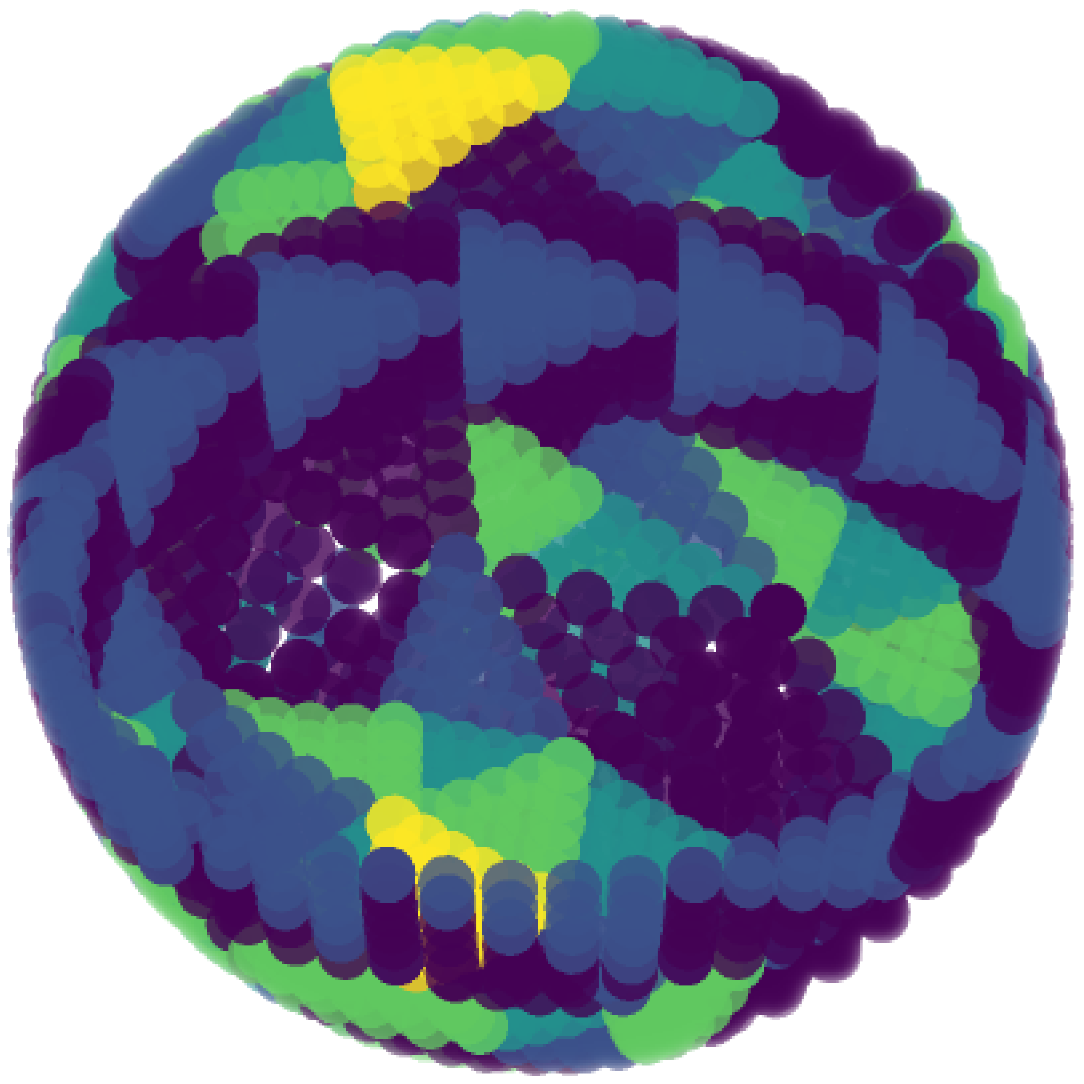
Example on the sphere



0-skeleton



1-skeleton



2-skeleton

Remarks:

- The manifold do not need to be embedded in $\mathbb{R}^n$.
- Cells do not need to be flat in any particular chart. In particular, we can apply the method to a domain of $\mathbb{R}^n$ discretized with curved elements.
- The only constraint on cells is that they must be C^2-diffeomorphic to contractible polytopes.

Basis functions

- Each cell is characterized by a reference polytope.
- The manifold is given by an atlas.
- The basis functions live on the (exterior algebra of) references polytopes.
- The charts are only used to specify the metric, and to compute the trace operators.
- The basis functions are mapped to the charts from the references elements by pullbacks.
- A single cell may live in any number of charts, the only requirement being that all elements on its boundary are also parametrized in these charts.

In practice, charts are also used to simplify mesh generation, keeping elements flat where possible.

Compatibility condition

Assumption

For every $0 \leq d \leq n$ and $f \in \Delta_d(\mathcal{M}_h)$, there is a C^2-diffeomorphism I_f from a subset of $\mathbb{R}^d$ into $\overline{f}$. Let $J_f := (I_f)^{-1}$. We assume that, for all $f' \in \Delta_{d-1}(\mathcal{M}_h)$, the transformation

$$\mathcal{T}_{f,f'} := J_f \circ \mathcal{I}_{f,f'} \circ I_{f'}$$

from (a subset of) $\mathbb{R}^{d-1}$ to $\mathbb{R}^d$ is affine.

This is similar to the regularity requirement of finite element, but much milder:

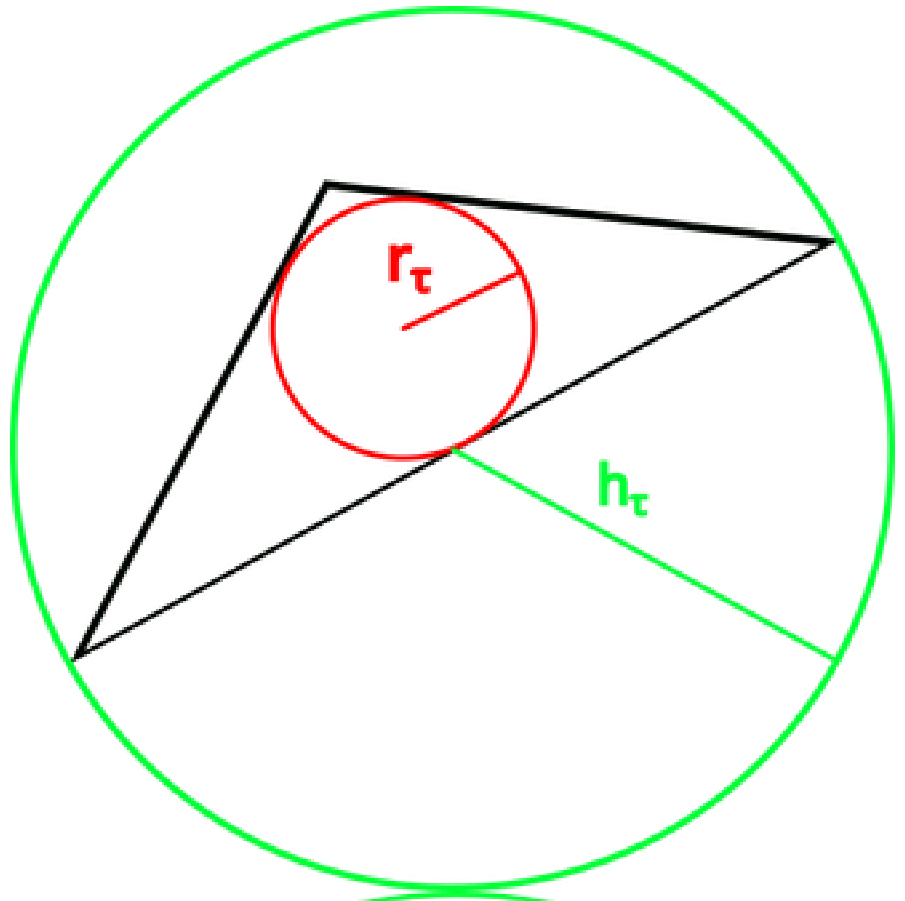
- Requires affine mapping instead of identity.
- The condition only appears through a $d-1$-dimensional space, which need not to be the trace of a d-dimensional one.

Notion of regularity for the mesh sequence

$$\text{Manifold mesh} \;\Rightarrow\; \text{Polytopal (flat) mesh} \;\Rightarrow\; \text{Simplicial mesh.}$$

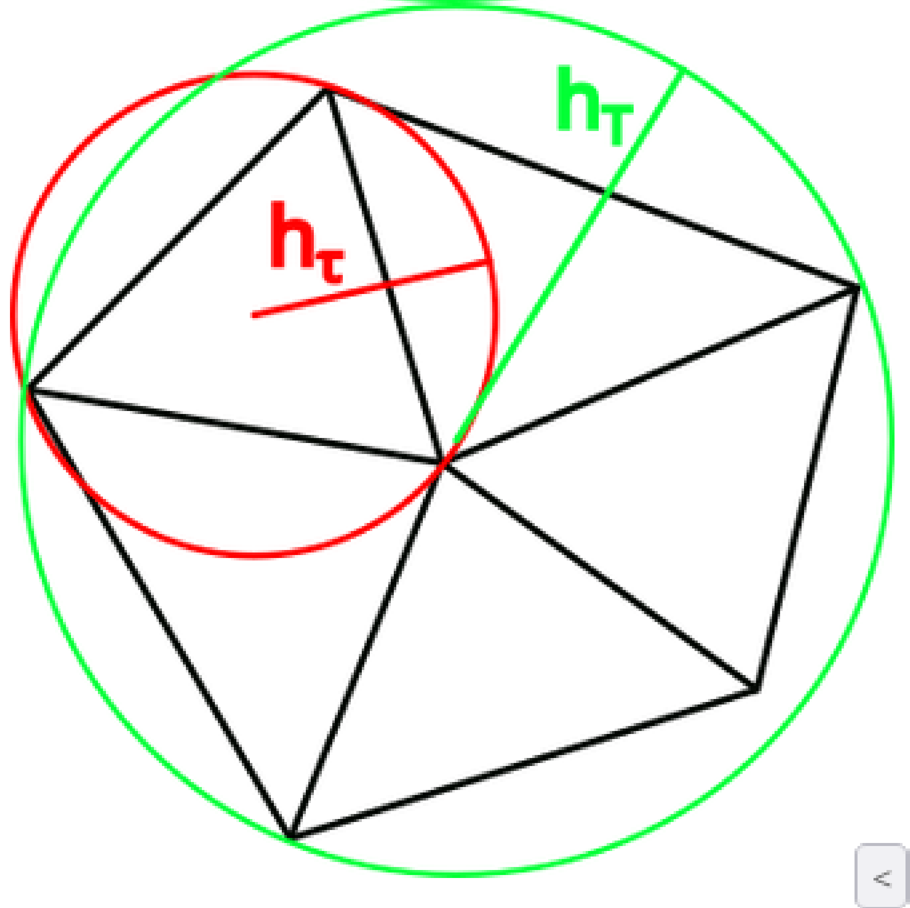
- Simplicial mesh (shape regularity):

$$\rho h_\tau \le r_\tau.$$



- Polytopal mesh (contact regularity):

$$\rho h_T \le h_\tau.$$



Definition: Equivalent polytopal mesh

A manifold mesh $\mathcal{M}_h$ is said to be equivalent to a polytopal mesh M_h is

- For all dimension $0 \leq d \leq n$, there is a bijection $\Phi_d : \Delta_d(\mathcal{M}_h) \to \Delta_d(M_h)$ between their d-cells.
- They are topologically equivalent: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h)$,

$$\Phi_{d-1}(\Delta_{d-1}(f)) = \Delta_{d-1}(\Phi_d(f)).$$

- Their geometry are equivalent: $\forall f \in \mathcal{M}_h$, there is a diffeomorphism $\phi_f : f \to \Phi_d(f)$ such that:

$$\|\nabla \phi_f\|_\infty \approx \|\det(\nabla \phi_f)\|_\infty^{\frac{1}{d}} \approx \left\|\det(\nabla \phi_f^{-1})\right\|_\infty^{-\frac{1}{d}} \approx \left\|\nabla \phi_f^{-1}\right\|_\infty^{-1}.$$

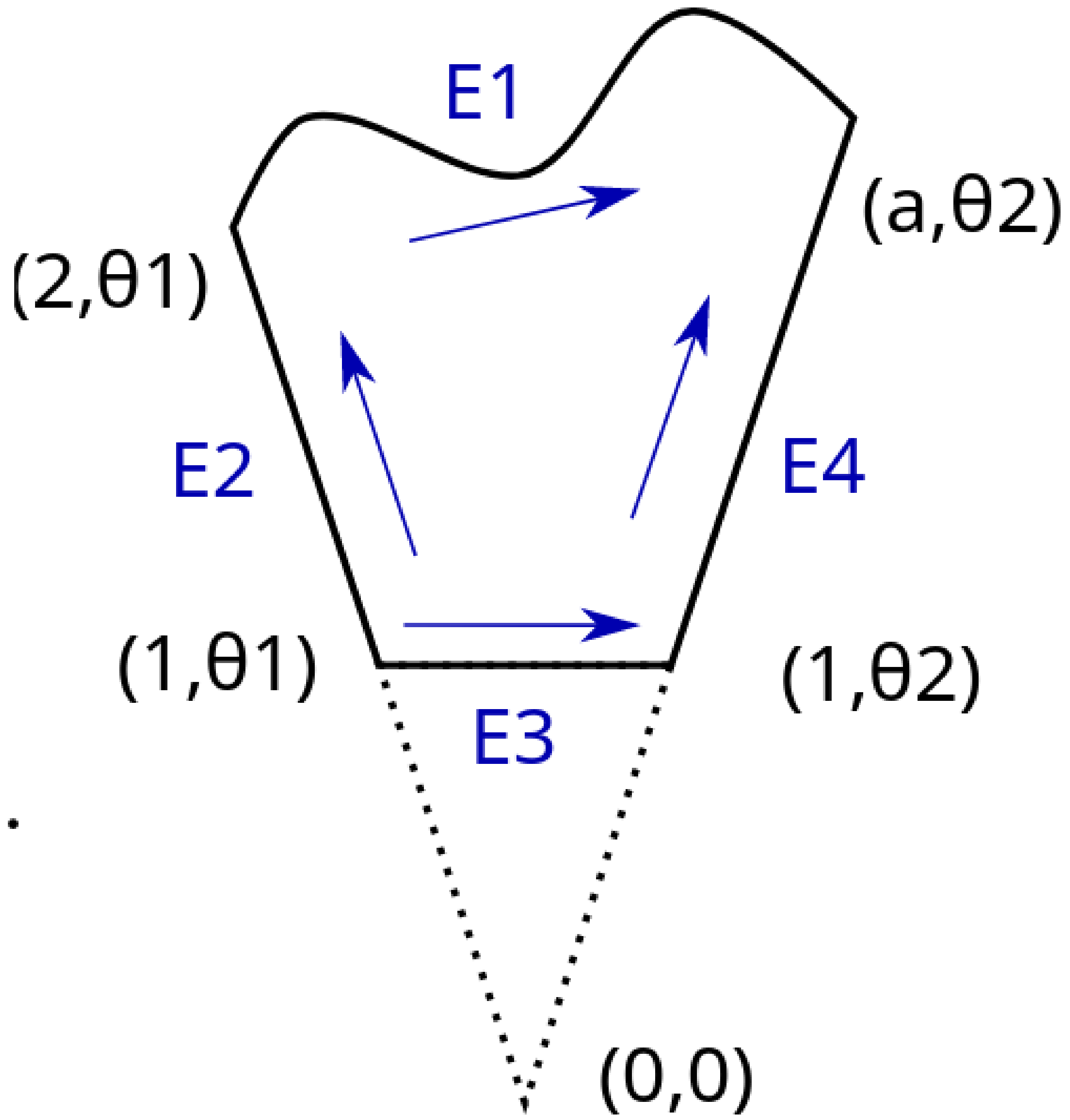
- The geometry is regular on boundaries: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h)$, $\forall f' \in \Delta_{d-1}(f)$,

$$\|\nabla \phi_f\|_\infty \approx \|\nabla \phi_{f'}\|_\infty.$$

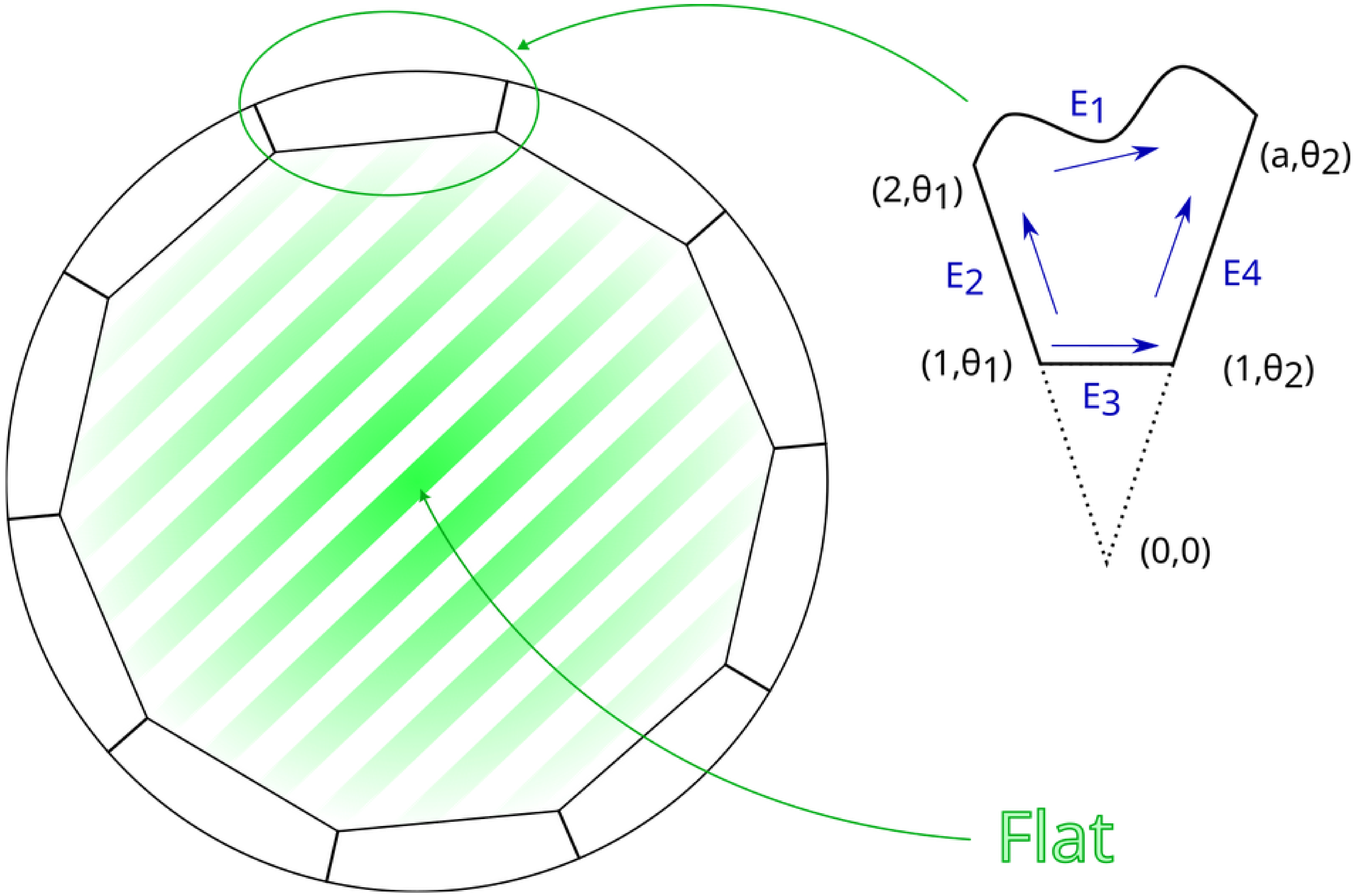
Example of construction

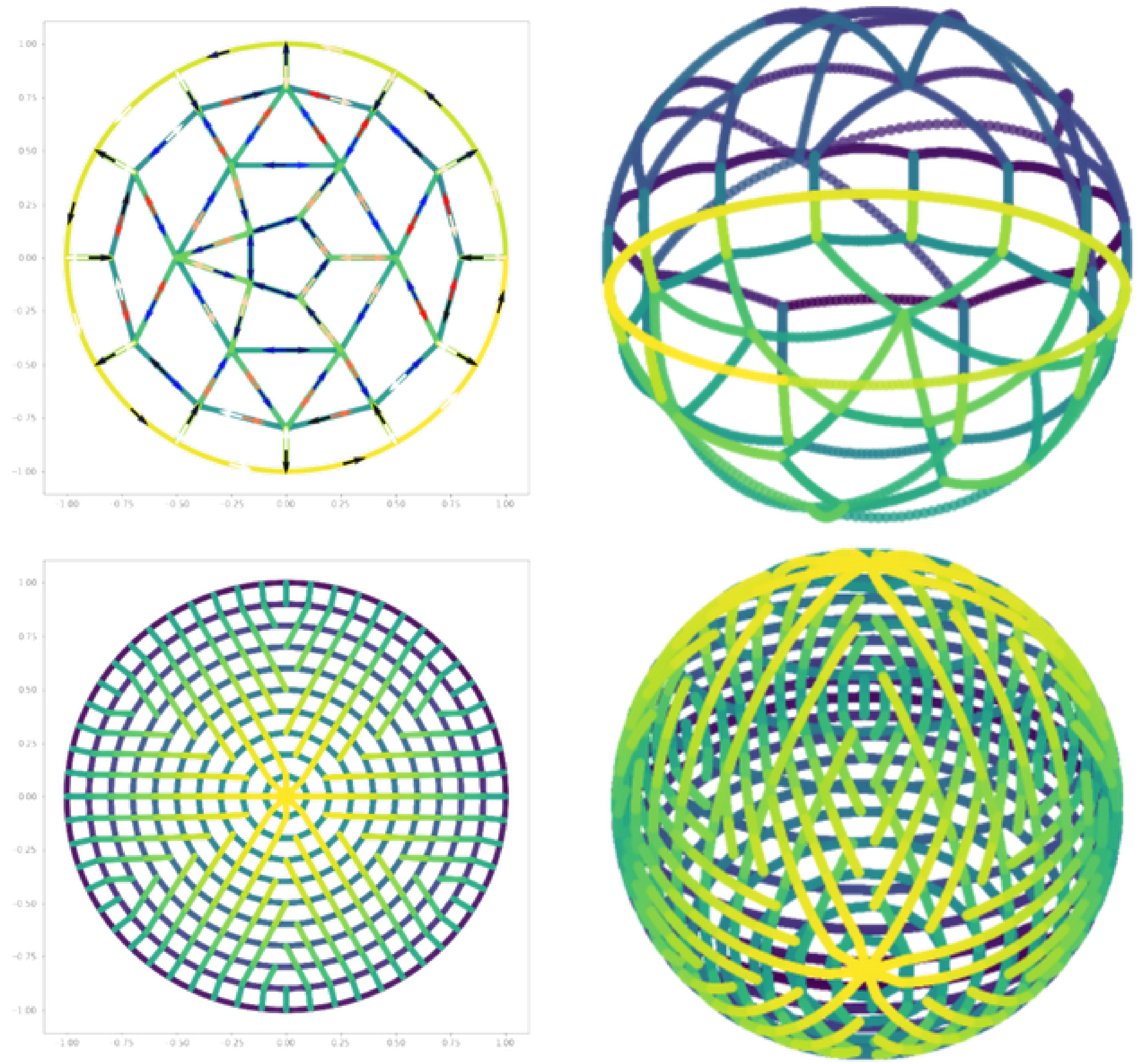
$$\begin{aligned} I_1(t) &= (g_1(t), h(t)), & h(0) &= \theta_1, \quad h(1) = \theta_2, \\ I_2(t) &= (t+1, \theta_1), & g_1(0) &= 2, \quad g_1(1) = a, \\ I_3(t) &= (g_3(t), h(t)), & g_3(0) &= 1, \quad g_3(1) = 1. \\ I_4(t) &= (t(a-1)+1, \theta_2), \end{aligned}$$

$$I_f(t,p) := \begin{pmatrix} pg_1(t) + (1-p)g_3(t) \\ h(t) \end{pmatrix}, J_f(a,b) := \begin{pmatrix} \frac{a - g_3(h^{-1})(b))}{g_1(h^{-1}(b)) - g_3(h^{-1}(b))} \\ h^{-1}(b) \end{pmatrix}.$$



Building a mesh on the sphere





Discrete De Rham on Manifolds

- The unknows lives on elements of various dimension, and are fully independents.
- The differential operators are defined to mimic integration by parts formulas.
- Potential reconstruction operators combines information from a cell and its boundary to give an higher degree, cell-wise polynomial.
- Interpolators commute with the continuous and discrete differential opeators.

Discrete objects

An element $\underline{\boldsymbol{u}}$ of the discrete space $\underline{X}_h$ is a collection of local functions over the cells, edges and vertices:

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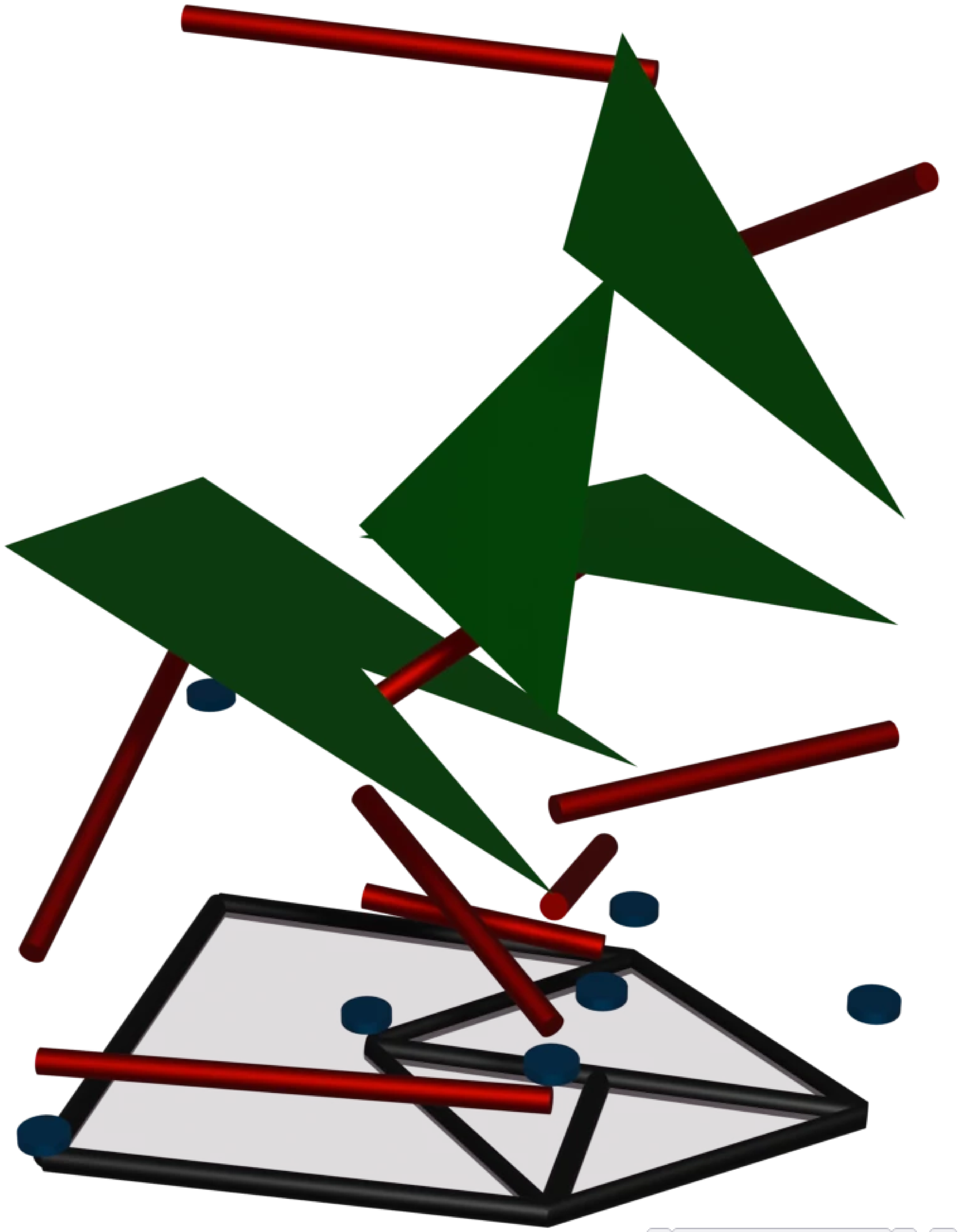
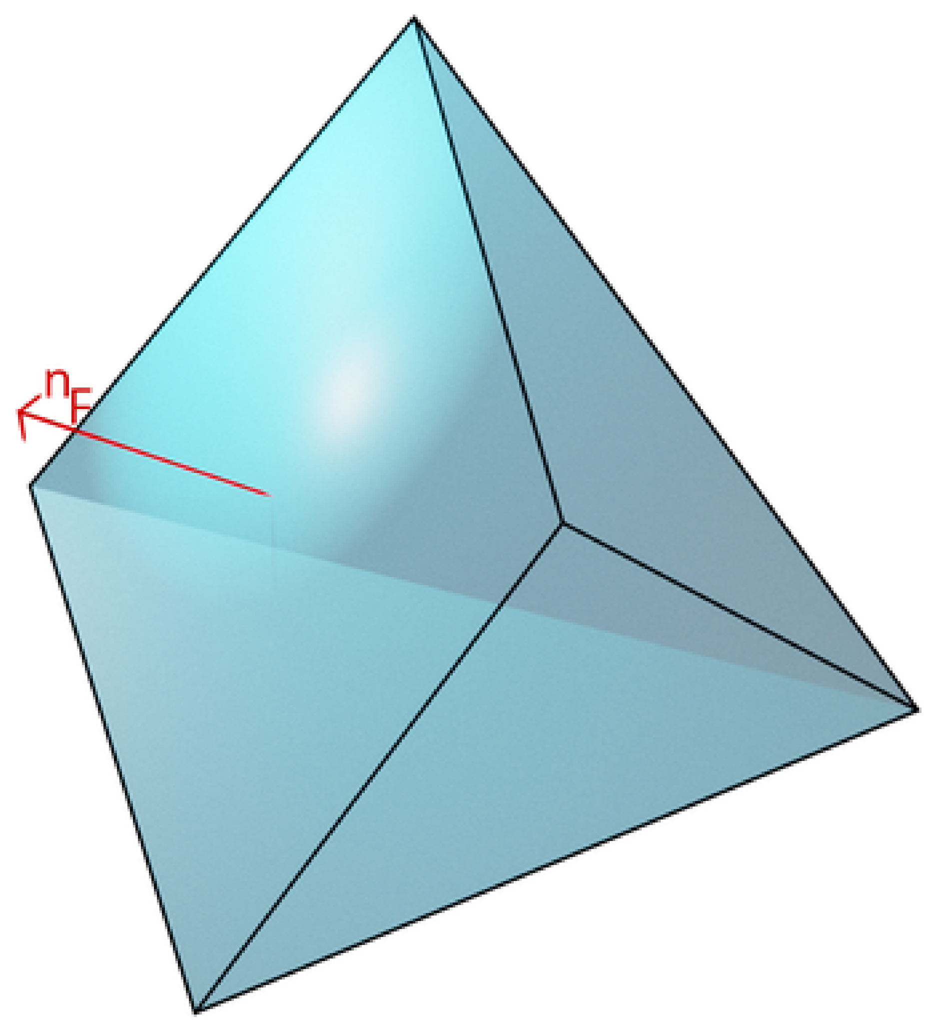


Discrete $\boldsymbol{H}(\mathrm{div})$ space



- Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};T)$. For any $q \in \mathcal{P}^k(T)$, we have

$$\int_T \mathrm{div}\,\boldsymbol{v} \underbrace{q}_{\in\mathcal{P}^k(T)} = -\int_T \boldsymbol{v}\cdot\underbrace{\mathbf{grad}\,q}_{\in\boldsymbol{\mathcal{G}}^{k-1}(T)} + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F(\boldsymbol{v}\cdot\boldsymbol{n}_F)\underbrace{q}_{\in\mathcal{P}^k(F)}$$

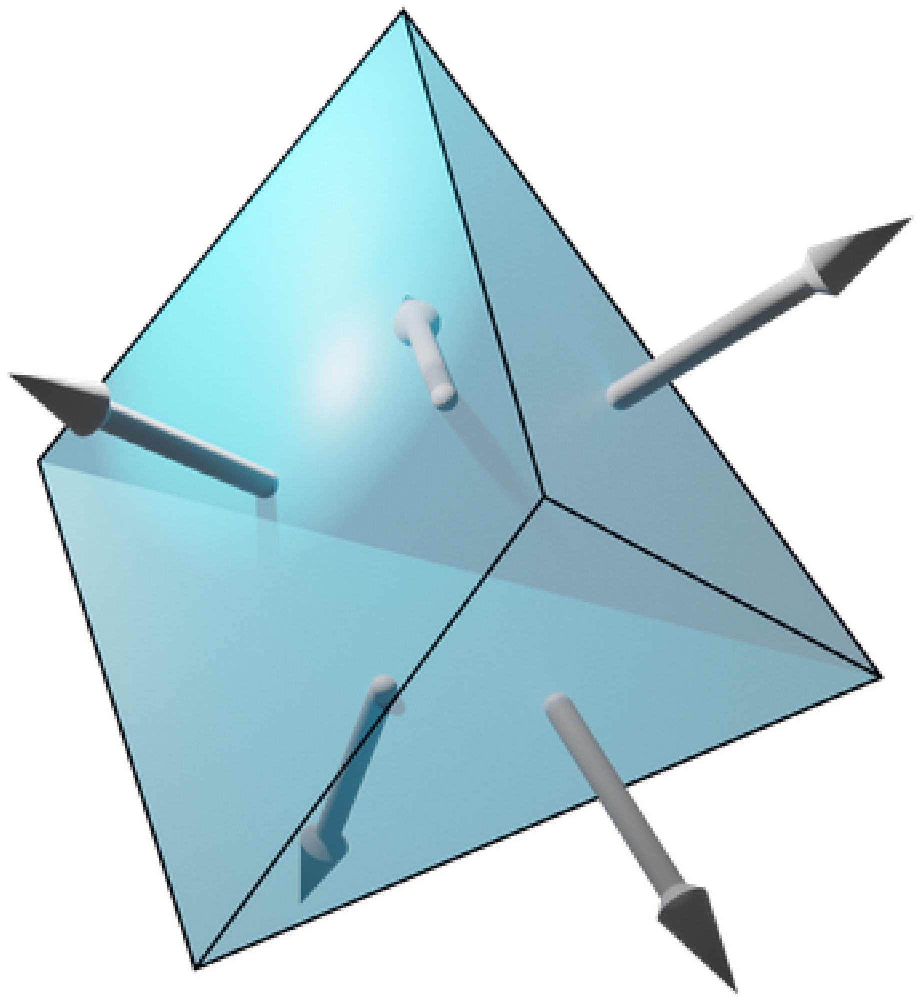
- Hence, $\pi^k_{\mathcal{P},T}\,\mathrm{div}\,\boldsymbol{v}$ can be computed given $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{N}},T}\boldsymbol{v}$ and $\pi^k_{\mathcal{P},F}\boldsymbol{v}\cdot\boldsymbol{n}_F$.

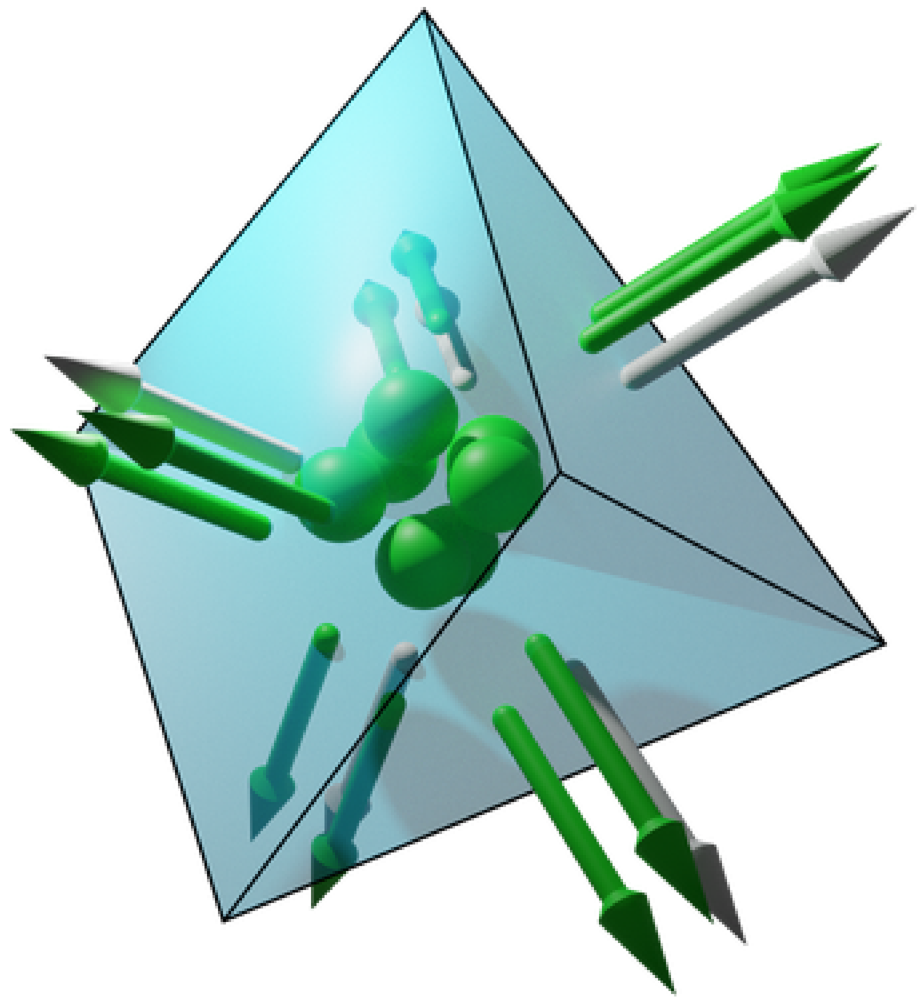
- Based on this remark, we take as discrete counterpart of $\boldsymbol{H}(\mathrm{div};T)$

$$\underline{\boldsymbol{X}}_{\mathrm{div},T}^{k} := \{\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (v_F)_{F\in\mathcal{F}_T}) : \boldsymbol{v}_T \in \boldsymbol{\mathcal{N}}^k(T) \text{ and } v_F \in \mathcal{P}^k(F)\}$$

- Let $\underline{\boldsymbol{I}}_{\mathrm{div},T}^{k} : \boldsymbol{C}^0(\overline{T}) \to \underline{\boldsymbol{X}}_{\mathrm{div},T}^{k}$ be such that $\forall \boldsymbol{v} \in \boldsymbol{C}^0(\overline{T})$,

$$\underline{\boldsymbol{I}}_{\mathrm{div},T}^{k}\boldsymbol{v} := (\boldsymbol{\pi}_{\boldsymbol{\mathcal{N}},T}^{k}\boldsymbol{v}, (\pi_{\mathcal{P},F}^{k}\boldsymbol{v}\cdot\boldsymbol{n}_F)_{F\in\mathcal{F}_T})$$





- The discrete divergence $D_T^k : \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \to \mathcal{P}^k(T)$, such that $\forall q \in \mathcal{P}^k(T)$

$$\int_T D_T^k \underline{\boldsymbol{v}}_T \, q = -\int_T \boldsymbol{v}_T \cdot \mathbf{grad}\, q + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F v_F \, q$$

- The potential reconstruction $P_{\mathrm{div},T}^k : \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \to \mathcal{P}^k(T)$, such that $\forall q \in \mathcal{P}^{0,k+1}(T)$

$$\int_T P_{\mathrm{div},T}^k \underline{\boldsymbol{v}}_T \cdot \mathbf{grad}\, q = -\int_T D_T^k \underline{\boldsymbol{v}}_T \, q + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F v_F \, q$$

- These constructions are consistent on polynomials

$$D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{v}) = \mathrm{div}\, \boldsymbol{v} \text{ and } P_{\mathrm{div},T}^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) = \boldsymbol{w} \text{ for all } \boldsymbol{w} \in \boldsymbol{\mathcal{RT}}^{k+1}(T)$$

Proof of the commutation property

$$\int_T D_T^k \underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{v}\, q = -\int_T \boldsymbol{\pi}_{\mathcal{N},T}^k \boldsymbol{v} \cdot \mathbf{grad}\, q + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \pi_{\mathcal{P},F}^k \boldsymbol{v} \cdot \boldsymbol{n}_F\, q$$

$$= -\int_T \boldsymbol{v} \cdot \mathbf{grad}\, q + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{v} \cdot \boldsymbol{n}_F\, q$$

$$= \int_T \mathrm{div}\, \boldsymbol{v}\, q.$$

Notion of Polynomial

As there is no natural notion of polynomial on manifolds, we seek spaces mimicking their properties.

- $$\cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r\Lambda^l(f) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^{l+1}(f) \xrightarrow{\mathrm{d}} \dots$$
  $$\cdots \xleftarrow{\kappa} \mathcal{P}_r\Lambda^l(f) \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^{l+1}(f) \xleftarrow{\kappa} \dots$$
- $\mathcal{P}_r\Lambda^l(f) := \bigoplus_{s\leq r} \mathcal{H}_s\Lambda^l(f)$ such that, $\forall s, l, \exists \lambda_{s,l} \in \mathbb{R}, \forall p \in \mathcal{H}_s\Lambda^l(f), (\mathrm{d}\,\kappa + \kappa\,\mathrm{d})p = \lambda_{s,l}p$
- The eigenvalues identify the degree: $\forall d > 0, 0 \leq l \leq d$, $\lambda_{s,l} = \lambda_{s',l} \Leftarrow s = s'$. Moreover, $\lambda_{s,l} = 0 \Leftrightarrow (s,l) = (0,0)$.
- $\mathrm{d}\,\mathcal{H}_s\Lambda^l(f) \neq \{0\}$ when $l < d$ and $s > 0$.

This ensures

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \kappa\,\mathcal{P}_{r-1}\Lambda^1(f)$$
$$\mathcal{P}_r\Lambda^l(f) = \mathrm{d}\,\mathcal{P}_{r+1}\Lambda^{l-1}(f) \oplus \kappa\,\mathcal{P}_{r-1}\Lambda^{l+1}(f), \quad l \geq 1$$

Trimmed polynomials

$$\mathcal{P}_r^-\Lambda^0(f) := \mathcal{P}_r\Lambda^0(f)$$
$$\mathcal{P}_r^-\Lambda^l(f) := \mathrm{d}\,\mathcal{P}_{r}\Lambda^{l-1}(f) \oplus \kappa\,\mathcal{P}_{r-1}\Lambda^{l+1}(f), \quad l \geq 1$$

Assumption

$$\mathrm{tr}_{f'}\,\mathcal{P}_r^-\Lambda^l(f) \subset \mathcal{P}_r^-\Lambda^l(f'), \quad \forall f' \in \partial f.$$

Possible to construct from pullback using appropriate parametrizations.

General case

The divergence formula is a special case of the Stokes formula:

$$\int_f \mathrm{d}\,\alpha = \int_{\partial f} \iota\alpha,$$

where f is a d-dimensional manifold and α a $d-1$-form. Combined with the Leibniz rule

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\,\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \mathrm{d}\,\beta,$$

we obtain the integration by part formula

$$\int_f \mathrm{d}\,\alpha \wedge \beta = (-1)^{k+1} \int_f \alpha \wedge \mathrm{d}\,\beta + \int_{\partial f} \iota\alpha \wedge \iota\beta,$$

where α is a k-form and β a $d-1-k$-form.

Discrete spaces

Spaces definition

Given a polynomial degree $r \geq 0$, the discrete counterpart $\underline{\boldsymbol{X}}^k_{r,h}$ of the space $H\Lambda^k(\Omega)$, $0 \leq k \leq n$, is defined as

$$\underline{\boldsymbol{X}}^k_{r,h} := \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1} \mathcal{P}^-_r \Lambda^{d-k}(f).$$

Interpolator definition

The interpolator $\underline{\boldsymbol{I}}^k_{r,h} : C^0\Lambda^k(\overline{\Omega}) \to \underline{\boldsymbol{X}}^k_{r,h}$ is defined by projecting the traces on the polynomial spaces: for all $\omega \in C^0\Lambda^k(\overline{\Omega})$,

$$\underline{\boldsymbol{I}}^k_{r,h}\omega := (\star^{-1}\pi^{-,d-k}_{r,f} \star \mathrm{tr}_f\, \omega)_{f \in \Delta_d(\mathcal{M}_h), d \in [k,n]}$$

Discrete differential and potential

Let $f \in \Delta_d(\mathcal{M}_h)$. We define $\mathrm{d}^k_{r,f} : \underline{\boldsymbol{X}}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k-1}(f)$
$P^k_{r,f} : \underline{\boldsymbol{X}}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k}(f)$ as follows:

- If $d = k$:

$$P^k_{r,f}\underline{\omega}_f := \omega_f \in \star^{-1}\mathcal{P}_r\Lambda^0(f).$$

- If $d \geq k+1$:
  - $\forall \mu_f \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\,\underline{\omega}_f \wedge \mu_f := (-1)^{k+1}\int_f \omega_f \wedge \mathrm{d}\,\mu_f + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_f \wedge \mathrm{tr}_{\partial f}\,\mu_f,$$

  - $\forall \mu_f \in \kappa\,\mathcal{P}_r\Lambda^{d-k}(f), \nu_f \in \kappa\,\mathcal{P}_{r-1}\Lambda^{d-k+1}(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge (\mathrm{d}\,\mu_f + \nu_f) := \int_f \mathrm{d}^k_{r,f}\,\underline{\omega}_f \wedge \mu_f - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu_f + (-1)^{k+1}\int_f \omega_f \wedge v_f.$$

The full discrete differential is given by:

$$\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h := \left(\star^{-1}\pi^{-,d-k-1}_{r,f} \star \mathrm{d}^k_{r,f}\,\underline{\omega}_f\right)_{f\in\Delta_d(\mathcal{M}_h),d\in[k+1,n]}.$$

The discrete L^2-like inner product is given by:

$$\langle\underline{\omega}_h,\underline{\mu}_h\rangle := \sum_{f\in\Delta_n(\mathcal{M}_h)}\Bigg(\int_f \omega_f\wedge\star\mu_f + \sum_{d=k}^{n-1} h_f^{n-d}\sum_{f'\in\Delta_d(f)}\int_{f'}(\omega_{f'} - \mathrm{tr}_{f'}\,P^k_{r,f}\underline{\omega}_f)\wedge\star(\mu_{f'} - \mathrm{tr}_{f'}\,P^k_{r,f}\underline{\mu}_f)\Bigg).$$

About the Hodge star

- The expression are the same as in the flat case.
- The Hodge star is trivial on flat spaces, but does not preserve polynomial spaces on manifolds.
- The test functions must be polynomials.

$$\int_f \alpha \wedge \beta = (\alpha, \star^{-1}\beta)_f = (\star\alpha, \beta)_f$$

- We are actually solving for $\star\,\mathrm{d}$ and $\star P$.

Algebraic properties

- $$P^k_{r,f}\underline{\mathrm{d}}^{k-1}_{r,f}\underline{\omega}_f = \mathrm{d}^{k-1}_{r,f}\,\underline{\omega}_f$$
- $$\underline{\mathrm{d}}^k_{r,f}\underline{\mathrm{d}}^{k-1}_{r,f}\underline{\omega}_f = 0$$
- $$\star^{-1}\pi^{-,d-k}_{r,f}\star P^k_{r,f}\underline{\omega}_f = \omega_f$$
- $$\underline{\mathrm{d}}^k_{r,f}(\underline{\boldsymbol{I}}^k_{r,f}\omega) = \underline{\boldsymbol{I}}^{k+1}_{r,f}(\mathrm{d}\,\omega) \quad \forall\omega\in C^1\Lambda^k(\overline{f})$$
- $$\frac{\operatorname{Ker}\underline{\mathrm{d}}^{k+1}_{r,h}}{\operatorname{Im}\underline{\mathrm{d}}^k_{r,h}} \approx \frac{\operatorname{Ker}\mathrm{d}^{k+1}}{\operatorname{Im}\mathrm{d}^k}$$

Consistency

Under the assumption

$$\left\|\star^{-1}\pi_{\mathcal{P},f}^{d-k}\star\omega-\omega\right\|_f\lesssim h_f^{r+1}|\omega|_{H^{r+1}\Lambda^k(f)}\quad\forall\omega\in H^{r+1}\Lambda^k(f),$$

Primal consistency

$$\left\|P_{r,f}^k\underline{\boldsymbol{I}}_{r,f}^k\omega-\omega\right\|_f\lesssim h_f^{r+1}|\omega|_{r,f,\Delta}\quad\forall\omega\in H^{r+1}\Lambda^k(f;\Delta),$$

$$\left\|\mathrm{d}_{r,f}^k\underline{\boldsymbol{I}}_{r,f}^k\omega-\mathrm{d}\,\omega\right\|_f\lesssim h_f^{r+1}|\mathrm{d}\,\omega|_{r,f,\Delta}\quad\forall\omega\in C^1\Lambda^k(\overline{f})\text{ s.t. }\mathrm{d}\,\omega\in H^{r+1}\Lambda^{k+1}(f;\Delta).$$

$$|\omega|_{r,f,\Delta}:=\sum_{d'=k}^{d}h_f^{\frac{d-d'}{2}}|\mathrm{tr}_{f'}\,\omega|_{H^{r+1}\Lambda^k(f')}\quad\forall\omega\in H^{r+1}\Lambda^k(f;\Delta).$$

$$H^{r+1}\Lambda^k(f;\Delta):=\left\{\omega\in H^{r+1}\Lambda^k(f)\,:\,\mathrm{tr}_{f'}\,\omega\in H^{r+1}\Lambda^k(f')\quad\forall f'\in\Delta_{d'}(f)\,,\forall d'\in[k,d-1]\right\}$$

Partial differential equations on manifolds

How does the physic translate on curved space ?

Partial differential equations on manifolds

How does the physic translate on curved space ?

One easy way is to consider intrinsic formulations.

Maxwell equations

Letting $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ denote the electromagnetic field tensor, the Maxwell equations are given by:

$$\begin{aligned} \mathrm{d}\, F &= 0, \\ \mathrm{d} \star F &= \epsilon^{-1} \star \underline{j}, \end{aligned}$$

where ϵ is the permittivity of the meidum, and $\underline{j}$ the electric 4-current.

In 3 dimensions

$$F = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_z & B_x & 0 \end{pmatrix}$$

Follation and $2+1$ decomposition

We decompose a $2+1$ space-time manifold into spatial slices corresponding to level sets of a time function t. The metric is given by

$$(g_{\mu\nu}) := \begin{pmatrix} -N^2 + |\beta|_\gamma^2 & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix},$$

where γ is the metric on the spacial surfaces, N is the lapse function, and β the shift.

Remark

β and N reflect a gauge choice in the parametrization of the manifold. While the shift β is important in numerical relativity to avoid degeneracy, it is not necessary for the Maxwell equations.

We consider the case $\beta = 0$.

We define the future pointing unit normal to the spatial surfaces by

$$n := (-N \, \mathrm{d}\, t)^{\sharp}.$$

The magnetic and electric fields

We retrieve the magnetic field B and the electric field E as

$$\begin{aligned} E &:= -\iota_n F, \\ B &:= \iota_n(\star F). \end{aligned}$$

Explicit decomposition

The electromagnetic tensor field can be written as

$$F = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

Compared with the 3 dimensional case, the magnetic field becomes a scalar field.

Some identities

Denoting with a tilde ˜ the respective operator acting on the 2 dimensionnal surfaces:

- $$F = n^\flat \wedge E + B\iota_n \,\mathrm{vol}.$$

- $$\mathrm{d}\,F = \mathrm{d}\,t \wedge \left[\tilde{\mathrm{d}}(N\tilde{E}) + \mathcal{L}_{Nn}(B\tilde{\mathrm{vol}})\right],$$
$$\mathrm{d}\star F = \tilde{\mathrm{d}}\tilde{\star}\tilde{E} + \mathrm{d}\,t \wedge \left[\mathcal{L}_{Nn}(\tilde{\star}\tilde{E}) - \tilde{\mathrm{d}}(NB)\right].$$

- $$\star\underline{j} = \rho\tilde{\mathrm{vol}} - \mathrm{d}\,t \wedge \tilde{\star}\tilde{J}.$$

Equations

In the case of a time independant metric and for $N \equiv c$ constant, the Maxwell equations become

$$\tilde{\mathrm{d}}\tilde{E} = -\partial_t B'$$
$$-\tilde{\delta}\tilde{E} = \frac{\rho}{\epsilon_0}$$
$$\tilde{\delta}B' = \mu_0 \tilde{J} + \frac{1}{c^2}\partial_t \tilde{E}$$

where $B' := \frac{1}{c} B \tilde{\mathrm{vol}}$, and $\mu_0 := \frac{1}{c^2\epsilon_0}$.

To simplify, we work in geometric units and take $c = \epsilon_0 = 1$.

Scheme

The discrete scheme is built on $X_h := \underline{\boldsymbol{X}}^1_{r,h} \times \underline{\boldsymbol{X}}^2_{r,h}$ and reads: find $(\underline{\boldsymbol{E}}_h, \underline{B}_h) \in C^1([0,T], X_h)$ such that for all $(\underline{\boldsymbol{v}}^1_h, \underline{v}^2_h) \in X_h$,

$$\begin{aligned}
\langle \underline{\mathrm{d}}^1_{r,h} \underline{\boldsymbol{E}}_h(t), \underline{v}^2_h \rangle &= -\langle \partial_t \underline{\boldsymbol{B}}_h(t), \underline{v}^2_h \rangle, \\
\langle \underline{\boldsymbol{B}}_h(t), \underline{\mathrm{d}}^1_{r,h} \underline{\boldsymbol{v}}^1_h \rangle &= \langle \underline{\boldsymbol{I}}^1_{r,h} \widetilde{J}(t), \underline{\boldsymbol{v}}^1_h \rangle + \langle \partial_t \underline{\boldsymbol{E}}_h(t), \underline{\boldsymbol{v}}^1_h \rangle.
\end{aligned}$$

Discrete charge density

Defining the discrete electric charge as $\forall t \in [0,T], \forall \underline{v}_h^0 \in \underline{\boldsymbol{X}}_{r,h}^0$,

$$\langle \rho_h(t), \underline{v}_h^0 \rangle = -\langle \underline{\boldsymbol{E}}_h(0), \underline{\mathrm{d}}_{r,h}^0 \underline{v}_h^0 \rangle + \int_0^t \langle \underline{\boldsymbol{I}}_{r,h}^1 \widetilde{J}(s), \underline{\mathrm{d}}_{r,h}^0 \underline{v}_h^0 \rangle \, ds$$

we have

$$\langle \underline{\boldsymbol{E}}_h(t), \underline{\mathrm{d}}_{r,h}^0 \underline{v}_h^0 \rangle = -\langle \rho_h(t), \underline{v}_h^0 \rangle \qquad \forall t \in [0,T], \forall \underline{v}_h^0 \in \underline{\boldsymbol{X}}_{r,h}^0.$$

Definition of ρ_h

The definition is equivalent to $(\underline{\mathrm{d}}_{r,h}^1)^\dagger \underline{\boldsymbol{E}}_h(t) = -\rho_h(t)$.

Energy preservation

In the absence of a current density, the solution satisfies

$$\langle \partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{E}}_h \rangle + \langle \partial_t \underline{\boldsymbol{B}}_h, \underline{\boldsymbol{B}}_h \rangle = 0.$$

This ensure the preservation of the total energy of the system when using time discretisation scheme preserving quadratic invariants.

Numerical test

Exact solution on the sphere

On stereographic projections, the metric is given by

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda := \frac{4}{(1 + X^2 + Y^2)^2}.$$

The differential are given by

$$\tilde{\mathrm{d}}\tilde{E} = \frac{1}{\lambda}(\partial_X E_Y - \partial_Y E_X)\tilde{\mathrm{vol}}, \quad -\tilde{\delta}\tilde{E} = \frac{1}{\lambda}(\partial_X E_X + \partial_Y E_Y),$$
$$\tilde{\delta}(B\tilde{\mathrm{vol}}) = \partial_Y B \,\mathrm{d}X - \partial_X B \,\mathrm{d}Y.$$

Reference solution

North:

$$B = \left[(X^2+Y^2-1)\cos(t) + X^2 + Y^2 + 1 - 2X\sin(t)\right] \mathrm{d}\,X \wedge \mathrm{d}\,Y,$$

$$\begin{aligned} E = {} & Y\big((2-X^2-Y^2)\sin(t) - 2X\cos(t)\big)/4\,\mathrm{d}\,X \\ & + X\big((X^2+Y^2-2)\sin(t) + (3X^2+Y^2-3)\cos(t)\big)/4\,\mathrm{d}\,Y, \end{aligned}$$

$$\rho = 0,$$

$$\begin{aligned} J = {} & \Big[Y\big(3(X^2+Y^2)^2/2(1+\cos(t)) + (X^2+Y^2)(3+5/4\cos(t)) \\ & +1/2-\cos(t)\big) - XY(2X^2+2Y^2+5/2)\sin(t)\Big]\,\mathrm{d}\,X \\ & + \Big[-X\big(3(X^2+Y^2)^2/2(1+\cos(t)) + (X^2+Y^2)(3+5/4\cos(t)) + 1/2 \\ & -\cos(t)\big) + (10X^4+12X^2Y^2+15X^2+2Y^4+5Y^2-1)/4\sin(t)\Big]\,\mathrm{d}\,Y \end{aligned}$$

South:

$$B = \left[(X^2+Y^2-1)\cos(t) - X^2 - Y^2 - 1 + 2X\sin(t)\right] \mathrm{d}\,X \wedge \mathrm{d}\,Y,$$

$$\begin{aligned} E = {} & Y\big((2-X^2-Y^2)\sin(t) + 2X\cos(t)\big)/4\,\mathrm{d}\,X \\ & + X\big((X^2+Y^2-2)\sin(t) - (3X^2+Y^2-3)\cos(t)\big)/4\,\mathrm{d}\,Y, \end{aligned}$$

$$\rho = 0,$$

$$\begin{aligned} J = {} & \Big[-Y\big(3(X^2+Y^2)^2/2(1-\cos(t)) + (X^2+Y^2)(3-5/4\cos(t)) \\ & +1/2+\cos(t)\big) + XY(2X^2+2Y^2+5/2)\sin(t)\Big]\,\mathrm{d}\,X \\ & + \Big[X\big(3(X^2+Y^2)^2/2(1-\cos(t)) + (X^2+Y^2)(3-5/4\cos(t)) + 1/2 \\ & +\cos(t)\big) - (10X^4+12X^2Y^2+15X^2+2Y^4+5Y^2-1)/4\sin(t)\Big]\,\mathrm{d}\,Y. \end{aligned}$$

Remark

The solution is merely C^0.

Results

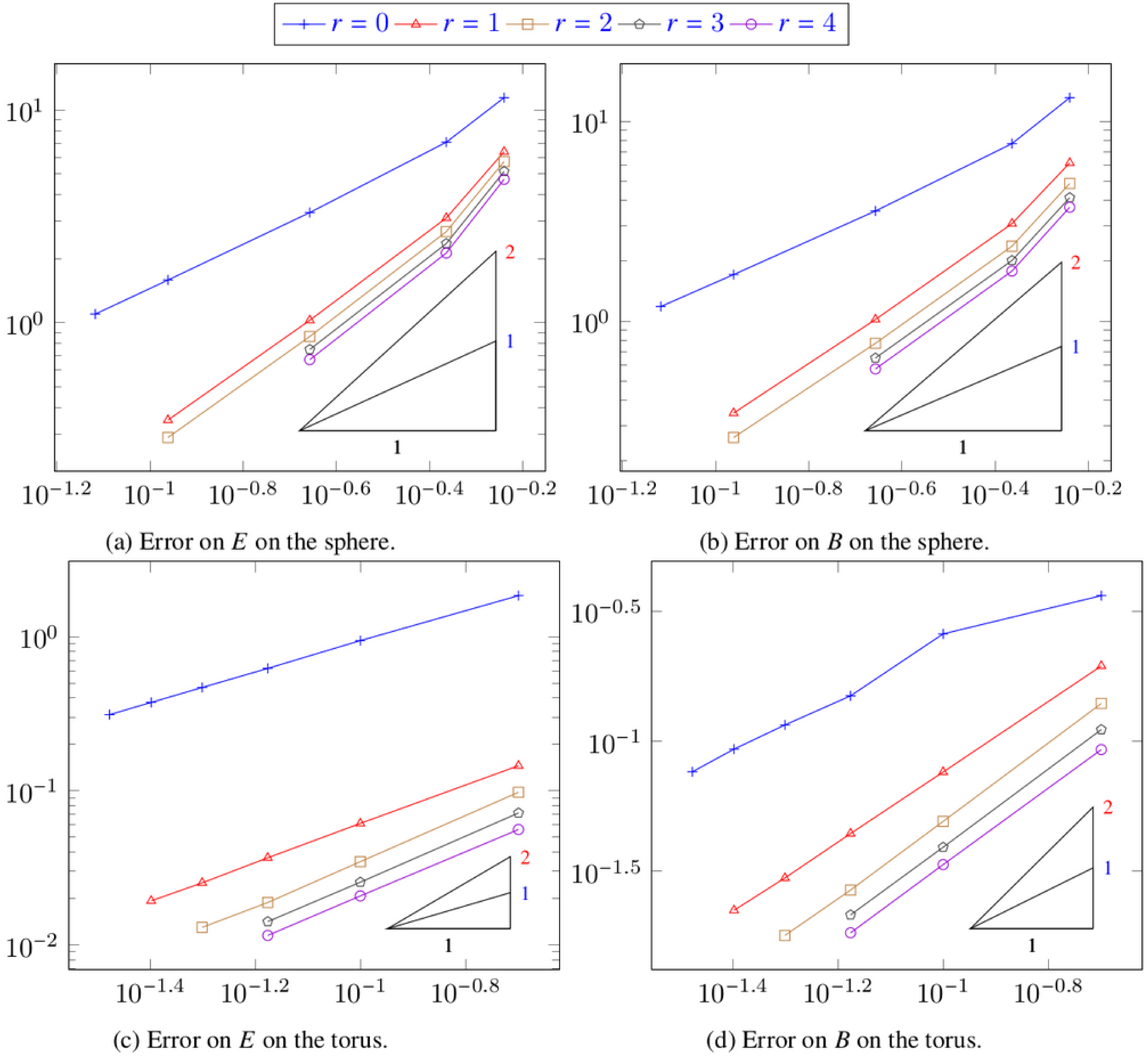


Figure 2: Absolute error L^2 integrated over time and space vs. mesh size.


2.5e-01
0.1
0.0e+00
Evect Magnitude
5.0e-01
0
-5.0e-01
Y
Z
X
2.5e-01
0.2
0.1
0.0e+00
5.0e-01
0
-5.0e-01

Results2

Sources

Library: https://mlhanot.github.io/Manicore/

- General structure inspired by the HArDCore library.
- Implement the de Rham sequence in any dimension $n \geq 2$.
- New mesh format given in two parts:
    1. A json file giving the topological information.
    2. A shared library giving the geometrical information with user defined functions.

Thank you