Mackey Analogy in Periodic Cyclic Homology

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I) Setup

Let G be a Lie group over $\mathbb{R}$, and A a complex Fréchet algebra endowed with a smooth action of G. We define *the crossed product of* A *by* G as the following convolution algebra:

$$A \rtimes G := \mathcal{C}_c^\infty(G, A) \quad (f_1 \star f_2)(g) = \int_G f_1(h)h \cdot f_2(h^{-1}g)dh \in A.$$

Crossed products are powerful algebraic tools to understand covariant representations of the pair (G, A).

To study $A \rtimes G$, we will use periodic cyclic homology (HP), see [Lod] for more details. HP is a $\mathbb{Z}/2\mathbb{Z}$-graded theory such that :
→ If M is a compact manifold:

$$HP_0(\mathcal{C}^\infty(M)) = H_{DR}^{\text{even}}(M)$$
$$HP_1(\mathcal{C}^\infty(M)) = H_{DR}^{\text{odd}}(M)$$

→ It corresponds to the codomain of the Chern character:

$$K_i(A) \xrightarrow{\text{Ch.}} HP_i(A)$$

which leads to a paining $\langle -, - \rangle : K \times HP \to \mathbb{C}$.
→ It is computed using a $\mathbb{Z}/2\mathbb{Z}$-graded complex:

$$\widehat{CC}(A) \Rightarrow HP(A) = HP(\widehat{CC}(A))$$

Aim of the talk: Description of $HP(A \rtimes G)$.

II) Expectations and results:

Theorem: (Green, 1976, [Gre]) $\forall H < G$ closed subgroup:

$$A \rtimes H \underset{\text{Morita}}{\sim} \mathcal{C}_0(G, A)^H \rtimes G.$$

In other words, the understanding of the crossed product by a subgroup corresponds to the understanding of the associated homogeneous space. When $K < G$ is a maximal compact sugroup, $G/K \sim \mathbb{R}^q$ is contractible so one may expect to obtain results relating G and the maximal compact subgroup K.

Theorem: (Nistor, 1993, [Nis]) $\forall x \in G : HP_*(A \rtimes G)_x \simeq HP_{\star+q}(A \rtimes K)_x$.

The subscript means that we localize at a conjugacy class $\langle x \rangle$ of G (see [Bur] for the discrete case). More presicely, $C(A \rtimes G)_x$ corresponds to the subcomplex of smooth complactly supported functions lying on tuples of G whose product is closed to $\langle x \rangle$. It is the localization of the complex $C(A \rtimes G)$ at the prime ideal of functions wanishing at x of the algebra of central functions in G.
Remark: In particular, the theorem proves that $HP_\star(A \rtimes G)_x \simeq 0$ for any element x that doesn't belong to any maximal complact subgroup. One can recover informations about orbital integrals in this framework (see [BB] for Selberg principle).
Problem: There isn't any copatibility of these "local" isomorphisms while changing conjugacy classes. It comes from the fact that the centralizers G_x are do not deform

continuously with respect to x (even jumps of dimensions sometimes).

Theorem: (G., Work in progress 2025) There exists a canonical isomorphism :

$$HP_\star(A \rtimes G) \simeq HP_{\star+n}(A \rtimes K).$$

Ideas of the proof:
→ V. Nistor proposed the idea to replace the periodic complex $\widehat{CC}(A \rtimes G)$ by a G-acyclic resolution, and then take coinvariants.

$$L(G, A) := (L_n(G, A))_{n \geqslant 0}$$
$$L_n(G, A) := \mathcal{C}_c^\infty(G, (\mathcal{C}_c^\infty(G, A), \widetilde{\star})^{\otimes_\pi n+1})$$

Where $\widetilde{\star}$ is a product on the space $\mathcal{C}_c^\infty(G, A)$ which differ from the classical convolution. See [Pus] and [Nis] for more details. We have an action of G on this graded vector space:

$$(g \cdot \varphi)(\gamma, g_0, \cdots, g_n) := g^{\otimes n+1} \cdot \varphi\left(g\gamma g^{-1}, gg_0, \cdots, gg_n\right).$$

Also, this graded vector space is endowed with an increasing differential B and a decreasing differential b. It is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with the differential $b + B$. *Warning:* The space $L(G, A)$ is not a cyclic object. The cyclic operator is not periodic and then $(b+B)^2 \neq 0$. But it becomes periodic after taking coinvariants over the action of G and we have :

Lemma: (Nistor, 1993, [Nis]) $\quad L(G, A)_G \simeq \widehat{CC}(A \rtimes G)$.

→ We want to set up a diagram of complexes :

$$\widehat{CC}(A \rtimes G)[q] \underset{\text{qis}}{\longleftarrow} \mathfrak{B} \underset{\text{qis}}{\longrightarrow} \widehat{CC}(A \rtimes K)$$

We would like to choose for $\mathfrak{B}$ the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\Omega_c^\star(G/K) \otimes_G L(G, A)$, where the group G acts by translations on $\Omega_c^\star(G/K)$, and endowed it with the differential $\delta = d \otimes id + id \otimes (b + B)$. But $\delta^2 \neq 0$, so we need to modify this object to obtain a complex. To do that, we introduce the G-equivariant operator T defined as $T(\overline{g}, x) = (\overline{xg}, x)$ on the product of the homogeneous space G/K and the copy of G in $L(G, A)$.

Lemma/Definition: We take for $\mathfrak{B}$ the following which is a complex, i.e. $\delta^2 = 0$:

$$\left(\left([\Omega_c^\star(G/K) \otimes L(G, A)]_{(1-T)}\right)_G, \delta = d \otimes id + id \otimes (b + B)\right).$$

Proposition: The G-equivariant contraction of G/K gives:

$$\left([\Omega_c^\star(G/k) \otimes L(G, A)]_{(1-T)}\right)_G \xrightarrow[\text{qis}]{\int_{G/K}} L(G, A)_G[q] \simeq \widehat{CC}(A \rtimes G)[q].$$

Proposition: The restriction of $\mathfrak{B}$ to the submanifold $Y = \{(\overline{g}, x) \in G/K \times G \mid x \in gKg^{-1}\}$ of T-fixed points gives:

$$\left([\Omega_c^\star(G/K) \otimes L(G, A)]_{(1-T)}\right)_G \xrightarrow[\text{qis}]{\text{Res. to } Y} L(K, A)_K \simeq \widehat{CC}(A \rtimes K)$$

The restriction kills any element that does not belong to a maximal compact subgroup. Also, after taking G-coinvariants, any element that belongs to a certain maximal compact subgroup can be sent to K via a translation in the homogeneous space G/K.

Taking the restriction to Y corresponds to the integration along the fiber of $Y \to G$ and it remains only to divide by the action of K on $L(K, A)$.

□

Remarks: → The identification is *canonical*. Its proof relies only on geometrical arguments, and follows closely the arguments of Nistor.
→ The isomorphism happens in HP but not in HC because we used the *Goodwillie's theorem* and the *homotopy invariance*.
We recover with this method the classical Connes-Thom isomorphism:

Corollary: $HP_\star(A \rtimes \mathbb{R}^q) \simeq HP_{\star+q}(A)$.

III) Consequences for tempered representations
Let us denote by $G_0 := K \rtimes \mathbb{R}^q$ the motion group associated to the Lie group G. In the 70's, Mackey showed (see [Mac]) that tempered irreducible unitary reprentations of G and irreducible unitary representations of G_0 are related :

$$\widehat{G_t} \longleftrightarrow \widehat{G_0}$$

From classical algebraic constructions one may expect some links between the (reduced) crossed product algebras associated to G and G_0.

Theorem: (Mackey Analogy in HP) $HP_\star(A \rtimes G) \simeq HP_{\star+q}(A \rtimes K) \simeq HP_\star(A \rtimes G_0)$.

References

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