Branching Problems and Symmetry-Breaking Introductory lectures at IHP Paris, January 2025 trisemester program on Representation Theory and Noncommutative Geometry

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This lecture is part of the sequence

- Introduction to Representation Theory (3h), Birgit Speh
- Different Aspects of Rankin–Cohen Operators (2h), Michael Pevzner
- Branching Problems in Representation Theory (3h) Toshiyuki Kobayashi
- Geometric and Analytic Aspects of Branching Laws (2h) Bent Ørsted

with the purpose of introducing key concepts and indicating recent advances in representation theory of Lie groups, with a special emphasis on branching problems for infinite-dimensional representations of real reductive Lie groups. Below is included a list of references for further reading.

Overview and motivation

In these lectures will be explained some of the basic mathematical theory of group representations, mostly of reductive Lie groups, and their structure via the theory of branching problems and intertwining operators. Thus the aim is to consider a group G and a subgroup H and a representation π of G in some vector space V (depending on the category, say Hilbert space for a continuous unitary representation, or a Fréchet space for a smooth representation). Now for a representation π^H of H in V^H of a similar nature we wish to consider the restriction of π to H and look for a linear $H-$equivariant map between them, i.e.

$$T : V \mapsto V^H,\ \ \pi^H(h)T = T\pi(h),\ \ (h \in H).$$

Again, depending on the category, we may want T to be say a partial isometry or just continuous. Such an operator we call *symmetry-breaking*, and in the special case of $H = G$ it is just an *intertwining operator*, already an interesting class of operators. See [7] for an overview. Constructing such operators is closely connected to the problem of decomposing the restriction of π to H as a kind of sum or integral of irreducible representations of H, the so–called *branching problem.* If π is a unitary (always continuous) representation of a Lie group G the restriction to a one-parameter subgroup $U(t) = \pi(exp(tX))$ is the Fourier transform of a spectral measure on the real line, and the generator $d\pi(X) = d/dt|_{t=0}\pi(exp(tX))$ is a skew-adjoint operator. In some applications in quantum mechanics this spectral measure and its support has a meaning in terms of observable quantities, and the corresponding decomposition of the Hilbert space is thought of as a breaking of the symmetries from G.

Now the branching problem for a unitary representation is to find in a similar way the spectrum of H as explicitly as possible. One is usually interested in the following aspects of intertwining operators and symmetry-breaking:

- Intertwining differential operators
- Knapp-Stein operators and their residues
- Symmetry-breaking operators and their construction as integral and differential operators

and we shall give some examples, in particular relating to the Dirac operator in [2] and the wave operator with generalizations as in [8], [9], and [10].

Many important differential equations in physics, such as Maxwell's equations, have symmetries, and these reflect the relation between the space of solutions and a representation of the group of symmetries, often a Lie group G. Hence as in Galois theory G permutes the solutions of the equation. Thus one studies

- natural differential operators in Riemannian and conformal geometry
- differential operators on induced representations
- intertwining integral operators, Knapp–Stein operators, see [4] Chapter VII.
- differential operators as residues of integral operators

with differential forms as a main example, see [3]. On spheres we will then have concrete operators that can be analyzed using the analogue of spherical harmonics as in [1]. It is an important point that conformal differential geometry and canonical differential operators here are closely connected with the representation theory of the conformal group of the sphere. Other geometries arise as well for other groups. For a special case related to the Heisenberg group see [11]. In particular there are natural first–order operators as in [16]. There is also a natural class of integral symmetry–breaking operators for principal series representations, see [12].

For a major background article on symmetry–breaking see [7], and for an important special case with many details [6]; see also [5].

Time permitting we shall also give examples of branching laws for discrete series, see [13], [14], [15], and [17].

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