Abstract for "Algebraic Quantum Field Theory and causal homogeneous spaces"

Karl-Hermann Neeb

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Lorentzian manifolds and their conformal compactifications provide the most symmetric models of spacetimes. The structures studied on such spaces in Algebraic Quantum Field Theory (AQFT) are so-called nets of operator algebras, i.e., to each open subset $\mathcal{O}$ of the space-time manifold one associates a von Neumann algebra $\mathcal{M}(\mathcal{O})$ in such a way that a certain natural list of axioms is satisfied.

We report on an ongoing project concerned with the construction of such nets on general causal homogeneous spaces $M = G/H$. The rough plan for the lectures is as follows:

Lect. 1: **Nets of operator algebras and AQFT.** We start with the translation from nets of operator algebras to nets of real subspaces, based on modular theory. We introduce real standard subspaces, discuss the Tomita-Takesaki Theorem as a key result from the modular theory of operator algebras and then describe axioms for nets of real subspaces $\mathsf{H}(\mathcal{O})$ in a unitary representation of a Lie group. These are structures than can be explored completely from the perspective of the geometry of homogeneous spaces and unitary representations.

Lect. 2: **Euler elements and causal homogeneous spaces.** We explore which specific structures we need on the homogeneous space $M = G/H$ and the Lie group G, so that a rich supply of nets may exist. In particular, we explain how Euler elements of Lie algebras (elements defining 3-gradings) enter the picture as candidates of generators of modular groups. This leads to several families of causal homogeneous spaces such as compactly and non-compactly causal symmetric spaces and causal flag manifolds.

Lect. 3: **Analytic continuation of orbit maps and crown domains.** The construction of interesting nets of real subspaces rests on the existence of holomorphic extension of orbit maps in unitary representations. For semisimple groups, complex crowns of Riemannian symmetric spaces G/K provide a natural context for this extension process. We explain how this can be set up on general Lie groups whose Lie algebra contains an Euler element.

Lect. 4: **Constructing nets of real subspaces.** Finally, we arrive at rather general characterizations of unitary representations and homogeneous spaces for which a rich supply of nets exists. Many classification results are still open and more bridges to Physics have to be built, but the overall structure of the theory takes shape.

Lecture notes are available under
`https://en.www.math.fau.de/wp-content/uploads/sites/3/2025/03/qft-lect.pdf`