Tempered representations with unipotent parahoric restriction: a noncommutative geometry viewpoint

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The setting of the Harish-Chandra philosophy of cusp forms

Let **G** be a reductive algebraic group over a field $\mathbb{F}$ and let $\mathcal{C}(G)$, where $G := \mathbf{G}(\mathbb{F})$, be an abelian category, equipped with a G-action, such that for every parabolic subgroup P of G, with Levi factor L, we have a parabolic induction functor

$$\mathrm{I}_{L,P}^{G} : \mathcal{C}(L) \to \mathcal{C}(G)$$

satisfying the following properties:

- every simple object π in $\mathcal{C}(G)$ occurs as a quotient of some $\mathrm{I}_{L,P}^{G}(\sigma)$, with σ a simple object in $\mathcal{C}(L)$, (the π's for which such a Levi L is necessarily equal to G are said to be (super)cuspidal),
- if σ is (super)cuspidal, then $G \cdot (L, \sigma)$ is uniquely determined and is called the cuspidal support of π. We will denote by Sc^{G} the map that sends π to $G \cdot (L, \sigma)$, and call it the cuspidal support map.

Examples of categories $\mathcal{C}(G)$

1. the category of complex representations of G, if $\mathbb{F}$ is a finite field (G is called a finite reductive group),
2. the category of smooth complex representations of G, if $\mathbb{F}$ is a non-archimedean local field (G is called a p-adic reductive group),
3. the category of G-equivariant perverse sheaves on the unipotent variety of G, if $\mathbb{F} = \mathbb{C}$ (and G is called a complex reductive group).

In case (2), we assume that **G** is connected.

Decomposition of the category $\mathcal{C}(G)$

In all the above examples, the category $\mathcal{C}(G)$ admits a decomposition (due to Harish-Chandra, Bernstein, Lusztig + A-Moussaoui-Solleveld, respectively) as a direct product of full subcategories:

$$\mathcal{C}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}} \mathcal{C}^{\mathfrak{s}}(G).$$

Notation

In cases (1) and (3), we set $\mathfrak{B} := \{G \cdot (L, \sigma) : \sigma \text{ simple cuspidal}\}$, and define $\mathcal{C}^{\mathfrak{s}}(G)$ to be the fiber of $\mathfrak{s}$ under the cuspidal support map Sc^G. In case (2), in the definition of $\mathfrak{B}$, we replace σ by its orbit under the action of the group of the unramified characters of L.

The ABPS Conjecture (rough form) [A.-Baum-Plymen-Solleveld]

When $\mathbb{F} = F$ is a non-archimedean local field, for every $\mathfrak{s} \in \mathfrak{B}$, the set $\mathrm{Irr}^{\mathfrak{s}}(G)$ of irreducible objects of $\mathcal{C}^{\mathfrak{s}}(G)$ has a very simple geometric structure given by a (possibly twisted by a 2-cocycle) extended quotient $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_\natural$, where $T^{\mathfrak{s}}$ is a complex torus and $W^{\mathfrak{s}}$ is the finite group

$$W^{\mathfrak{s}} := \{w \in \mathrm{N}_G(L)/L : {}^w\mathfrak{s} = \mathfrak{s}\}.$$

Note: If G is a quasi-split classical group [A.-Moussaoui-Solleveld] or the exceptional group G_2 [A.-Xu], no twisting is needed. However, for $G = \mathrm{SL}_n(D)$, with D/F a division algebra, there are cases which require a twisting [A.-Baum-Plymen-Solleveld].

Geometric extended quotient

Let X be a space and Γ a (finite) group acting on X. For $x \in X$, let $\Gamma_x \subset \Gamma$ be the fixator of x: $\Gamma_x := \{\gamma \in \Gamma \,:\, \gamma \cdot x = x\}$.
The quotient X/Γ of X by (the action of) Γ is the set Γ-orbits in X.
We replace X by a bigger space $\widetilde{X}$ on which Γ is still acting, and take the quotient of $\widetilde{X}$ by Γ: the geometric extended quotient of X by Γ is the quotient

$$(X/\!/\Gamma)_{\mathrm{geo}} := \{(x,\gamma) \,:\, x \in X, \gamma \in \Gamma_x\}/\Gamma.$$

Spectral extended quotient

Instead of elements of Γ_x, we can consider irred. repres. of Γ_x.
The (spectral) extended quotient of X by Γ is the quotient

$$X/\!/\Gamma := \{(x,\tau) \,:\, x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\}/\Gamma.$$

The extended quotients $(X/\!/\Gamma)_{\mathrm{geo}}$ and $X/\!/\Gamma$ are in bijection but not in a canonical way in general.

Notation/Definition

- A a k-algebra (with k the coordinate algebra of a complex affine variety)
- $\mathrm{Prim}(A)$ set of primitive ideals of A.
- An ideal I in a k-algebra A is a k-ideal if $\lambda a \in I$ for all $(\lambda, a) \in k \times I$.

In some situations, Morita equivalence can be too strong and we are led to use a weakening of this concept, which we call stratified equivalence. The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A.

Definition

Let A, B two finite type k-algebras. A morphism of k-algebras $f\colon A \to B$ is spectrum preserving if

1. Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $f^{-1}(J) \subset I$.
2. The resulting map $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is a bijection.

Definition

A morphism of k-algebras $f: A \to B$ is spectrum preserving with respect to filtrations if there are k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

in A and k-ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B$$

in B, such that for every $j \in \{1, 2, \ldots, r\}$, we have $f(I_j) \subset J_j$, and

$$I_j/I_{j-1} \to J_j/J_{j-1} \text{ is spectrum preserving.}$$

The primitive ideal spaces of the subquotients I_j/I_{j-1} and J_j/J_{j-1} are called the strata for stratifications of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$.

Note: Each stratum of $\mathrm{Prim}(A)$ is mapped homeomorphically onto the corresponding stratum of $\mathrm{Prim}(B)$.
However, the map $\mathrm{Prim}(A) \to \mathrm{Prim}(B)$ might not be a homeomorphism.

Algebraic variation of k-structure

Let A be a unital $\mathbb{C}$-algebra, and $\psi\colon k \to \mathrm{Z}(A[t, t^{-1}])$ a unital morphism of $\mathbb{C}$-algebras. For $\zeta \in \mathbb{C}^\times$, consider the composition

$$k \xrightarrow{\psi} \mathrm{Z}(A[t, t^{-1}]) \xrightarrow{\mathrm{ev}(\zeta)} \mathrm{Z}(A).$$

and denote by A_ζ the unital k-algebra so obtained. We call such a family $\{A_\zeta\}_{\zeta \in \mathbb{C}^\times}$ an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$.

Definition

With k fixed, we consider the collection of all finite type k-algebras. On this collection, a stratified equivalence is the equivalence relation generated by the two elementary steps:

- ES1 If there is a morphism of k-algebras $f\colon A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.
- ES2 If there is $\{A_\eta\}_{\zeta \in \mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, such that each A_ζ is a unital finite type k-algebra, then for any $\zeta, \zeta' \in \mathbb{C}^\times$, $A_\zeta \sim A_{\zeta'}$.

Equivalent description

Two finite type k-algebras A, B are stratified equivalent if and only if there is a finite sequence A_0, A_1, A_2, ..., A_r of finite type k-algebras with $A_0 = A$, $A_r = B$, and for each $j \in \{0, 1, \ldots, r-1\}$ one of the following three possibilities holds:

- a morphism of k-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k-algebras $A_{j+1} \to A_j$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\eta\}_{\zeta \in \mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, is given such that each A_ζ is a unital finite type k-algebra, and ζ', $\zeta"$ in $\mathbb{C}^\times$ have been chosen such that $A_j = A_{\zeta'}$ and $A_{j+1} = A_{\zeta"}$.

Note

To define a stratified equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology are determined:

$$\mathrm{Prim}(A) \leftrightarrow \mathrm{Prim}(B) \quad \text{and} \quad \mathrm{HP}_*(A) \simeq \mathrm{HP}_*(B).$$

Proposition [A-Baum-Plymen-Solleveld]

If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are stratified equivalent:

$$A \underset{\text{Morita}}{\sim} B \Longrightarrow A \sim B.$$

In contrast, there exist k-algebras that are stratified equivalent but not Morita equivalent, e.g., the affine Hecke algebra $\mathcal{H}_q(W^{\mathrm{aff}})$ (with $q \neq 1$) associated to an affine Weyl group W^{aff} and the group algebra $\mathbb{C}[W^{\mathrm{aff}}]$ are stratified equivalent for almost all q, but they are not Morita equivalent en general.

Representations of p-adic groups

Let G be a connected reductive p-adic group. To study representations of G it is often useful to consider various group algebras of G. There is the Hecke algebra $\mathcal{H}(G)$, defined to be the convolution algebra of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$.
The category $\mathcal{C}(G)$ is equivalent to the category of of nondegenerate $\mathcal{H}(G)$-modules.

Application of the Bernstein decomposition

By letting G act on $\mathcal{H}(G)$ by left translation, we obtain a decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}(G)$.

Conjecture [A.-Baum-Plymen-Solleveld]

Assume that G is quasi-split (to simplify the exposition). Let $\mathfrak{s}$ be a point in the Bernstein spectrum of G. There is a canonical bijection

$$\nu^{\mathfrak{s}} \colon T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathfrak{s}}(G),$$

which

1. comes from a canonical stratified equivalence of the two unital finite-type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$–algebras $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$ and $\mathcal{H}(G)^{\mathfrak{s}}$,
2. maps $T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}}$ onto $\mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}^{\mathrm{temp}}(G)$, where $T^{\mathfrak{s}}_{\mathrm{cpt}}$ is the maximal compact subgroup of $T^{\mathfrak{s}}$,

and, there is an algebraic family $\vartheta_z \colon T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$ of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^\times$, such that ϑ_1 is the natural projection and $\vartheta_{\sqrt{q}} = \mathrm{Sc}^{\mathfrak{s}} \circ \nu^{\mathfrak{s}}$, where q is the order of the residue field of F.
Moreover, $\nu^{\mathfrak{s}}$ is compatible the local Langlands correspondence.

Remarks

- In the above form, the conjecture is only known for $G = \mathrm{GL}_n(F)$ [A.-Baum-Plymen-Solleveld] and $G = \mathrm{G}_2$ [A.-Baum-Plymen].
- Related versions have been established in numerous situations.
- Galois analogue was formulated and proved [A.-Moussaoui-Solleveld].

Decomposition of $C^*_{\mathrm{r}}(G)$

From the point of view of noncommutative geometry, for the study of tempered representations of G, it is interesting to use the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$, whose spectrum coincides with the tempered dual $\mathrm{Irr}^{\mathrm{temp}}(G)$ of G. Analogously, we have the following decomposition of $C^*_{\mathrm{r}}(G)$:

$$C^*_{\mathrm{r}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} C^*_{\mathrm{r}}(G)^{\mathfrak{s}}.$$

K-theoretical version of the ABPS conjecture

Let $\mathfrak{s} \in \mathfrak{B}(G)$. There exists a canonical isomorphism

$$K^*_{W_{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{cpt}}) \longrightarrow K_*(C^*_{\mathrm{r}}(G)^{\mathfrak{s}}).$$

Decomposition of $C^*_{\mathrm{r}}(G)$ and the theta correspondence [A.]

Let $(G_n, G'_{n'})$ be an irreducible reductive dual pair, formed of a p-adic symplectic group $\mathrm{Sp}_{2n}(F)$ and a p-adic orthogonal group $\mathrm{O}_{2n'}(F)$, such that $n' = n$ or $n' = n + 1$.

Then the theta correspondence for the reductive dual pair $(G_n, G'_{n'})$ (which preserves temperedness) induces a correspondence between subsets of simple modules of $C^*_{\mathrm{r}}(G)^{\mathfrak{s}}$ and $C^*_{\mathrm{r}}(G)^{\theta(\mathfrak{s})}$.

The map $\mathfrak{s} \mapsto \theta(\mathfrak{s})$ can be explicitly described.

Notation

- $\beta(G) :=$ Bruhat-Tits building of G, and $\beta_{\mathrm{red}}(G) :=$ reduced BT building. $\beta(G) = \beta_{\mathrm{red}}(G) \times (X_*(\mathrm{Z}_G) \otimes_{\mathbb{Z}} \mathbb{R})$, where $X_*(\mathrm{Z}_G)$ is the set of F-algebraic cocharacters of the center Z_G of G.
- For $x \in \beta(G)$, let $G_{x,0}$ be the associate parahoric subgroup, and $G_{x,0+}$ the pro-p unipotent radical of $G_{x,0}$.
- Let $\mathbb{G}_{x,0} := G_{x,0}/G_{x,0+}$ a connected reductive group over $\mathbb{F}_q$.

Definition

An irreducible smooth representation (π, V) of G is said to have unipotent parahoric restriction (or, for short, to be unipotent) if there is $x \in \beta(G)$ such the $G_{x,0+}$-invariants in V contain an irreducible cuspidal unipotent representation π_x in the sense of Deligne-Lusztig theory, i.e., such that

$$\langle \pi_x, R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1) \rangle \neq 0,$$

for a maximal torus $\mathbb{T}$ of $\mathbb{G}_{x,0}$, with $R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1)$ a Deligne-Lusztig character.

Example

Every Iwahori-spherical representation of G is unipotent.
Iwahori-spherical representations belongs to $\mathcal{C}^{\mathfrak{s}}(G)$, where $\mathfrak{s} = [T, 1]_G$, with T a maximal torus of G.

Supercuspidal unipotent representations

These are the representations π of G such that there exist a vertex $x \in \beta_{\mathrm{red}}(G)$ and an irreducible unipotent cuspidal representation π_x of $\mathbb{G}_{x,0}$, such π is compactly induced from $\widetilde{\pi}_x$, an extension to $\mathrm{N}_G(G_{x,0})$ of the inflation of π_x to $G_{x,0}$:

$$\pi = \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_G(G_{x,0})}^{G}(\pi_x),$$

where $\mathrm{N}_G(G_{x,0})$ is the normalizer of $G_{x,0}$ in G (a totally disconnected group that is compact mod center). It coincides with the fixator under the action of G on $\beta_{\mathrm{red}}(G)$ of the image of x in $\beta_{\mathrm{red}}(G)$.

From parahorics to parabolics

1. For a facet $\mathcal{F}$ of $\beta(G)$, let $\Phi_{\mathcal{F}}^{\mathrm{aff}}$ be the set of affine roots vanishing on $\mathcal{F}$, and $\Phi_{\mathcal{F}}$ the associate set of roots.
2. $\Phi_{\mathcal{F}}$ is a not necessarily closed root system, and we denote its closure by ${}^{\mathrm{c}}\Phi_{\mathcal{F}}$.
3. There is a connected reductive F-subgroup $\mathbf{H}$ of $\mathbf{G}$ containing $\mathbf{T}$, which has the relative root system ${}^{\mathrm{c}}\Phi_{\mathcal{F}}$ with respect to $\mathbf{T}$.
4. Let $\mathbf{L}$ be the centralizer of the F-split component of the center of $\mathbf{H}$. Then $\mathbf{L}$ is the Levi factor of a parabolic F-subgroup of $\mathbf{G}$.
5. Moreover, $G_{x,0} \cap L = L_{y,0}$ for some vertex $y \in \beta(L)$, i.e., it is a maximal parahoric subgroup of L, and $\mathbb{L}_{y,0} = \mathbb{G}_{x,0}$.

The category $\mathcal{C}^{\mathrm{u}}(G)$ of unipotent representations of G

We have

$$\mathcal{C}^{\mathrm{u}}(G) := \prod_{\substack{\mathfrak{s}=[L,\sigma] \\ \sigma \text{ unipotent}}} \mathcal{C}^{\mathfrak{s}}(G).$$

Parametrization of $\mathrm{Irr}^{\mathrm{u}}(G)$ [Lusztig, Feng-Opdam-Solleveld]

$$\mathrm{Irr}^{\mathrm{u}}(G) \xleftrightarrow{1-1} (s, u, \rho)_{G^\vee}$$

where $G^\vee$ is the complex reductive group dual to G, $s, u \in G^\vee$, with s semisimple, u unipotent, such that $sus^{-1} = u^q$, and $\rho \in \mathrm{Irr}(A(s, u))$, with $A(s, u) := \mathrm{Z}_{G^\vee}(s, u)/\mathrm{Z}_{G^\vee} \cdot \mathrm{Z}_{G^\vee}(s, u)^\circ$. We decompose s into its compact and hyperbolic parts: $s = s_{\mathrm{c}} s_{\mathrm{h}}$.The tempered unipotent representations are those such that $s_{\mathrm{h}} = 1$, and the tempiric ones are those such that $s = 1$.

How is it related to the Bernstein decomposition of $\mathrm{Irr}^{\mathrm{u}}(G)$?

Given a pair (s, u), we may choose a homomorphism $\varphi_u \colon \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ such that

$$\varphi_u \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u \quad \text{and such that } s_u := \varphi_u \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \text{ commutes with } s.$$

Then $t := s_u^{-1} s \in \mathrm{Z}_{G^\vee}(u)$. We set $\mathcal{G}_t := \mathrm{Z}_{G^\vee}(t)$ and decompose the category of $\mathcal{G}_t$-equivariant perverse sheaves on the unipotent variety of $\mathcal{G}_t$ into blocks via the generalized Springer correspondence [Lusztig, A.-Moussaoui-Solleveld]. We have a notion of cuspidality, and the corresponding blocks match.

Thank you very much for your attention!