Different Aspects of Rankin-Cohen brackets

PART 1½.

$G = SL(2,\mathbb{R})$ acts on $B^2_k(H)$ by H.D.S. representations π_k

$$\left(\pi_k(g) f\right)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right); \quad k = 2,3,4,\dots$$

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathrm{Vect}(h, e^+, e^-)$

- $d\pi_k(e^-) f(z) = k z f(z) + z^2 f'(z)$
- $d\pi_k(e^+) f(z) = -f'(z)$
- $d\pi_k(h) f(z) = -k f(z) - 2z f'(z)$.

where $d\pi_k(X) = \frac{d}{dt}\Big|_{t=0} \pi_k(\exp tX)$.

Let ${}^k v_\ell \equiv v_\ell = \frac{(k+\ell-1)!}{(k-1)!} z^{-k-\ell}$ and $V_k = \mathrm{Vect}\langle v_\ell \rangle_{\ell\in\mathbb{N}}$

Then, $\cdot\ d\pi_k(h)\,v_\ell = (k+2\ell)\,v_\ell, \qquad \forall\, \ell\in\mathbb{N}$

$\cdot\ d\pi_k(e^+)\,v_\ell = v_{\ell+1}, \qquad \forall\, \ell\in\mathbb{N}$

$\cdot\ d\pi_k(e^-)\,v_\ell = -\ell(k+\ell-1)\,v_{\ell-1}; \qquad \forall\, \ell\in\mathbb{N}\setminus\{0\}$

$d\pi_k(e^-)\,v_0 = 0$

$\cdot\ d\pi_k(c)\,v = (k^2-2k)\,v, \qquad \forall\, v\in V_k.$

V_k is a lowest weight module of $\mathfrak{sl}_2(\mathbb{R})$

Consider $V_{k_1}\otimes V_{k_2}$ as a $\mathfrak{sl}(2,\mathbb{R})$-module via diag. embedding.

FACT: $V_{k_1} \otimes V_{k_2} = \bigoplus_{a \in \mathbb{N}} V_{k_1+k_2+2a}$

Identify lowest weight vector in the a-th irreducible component.

An h-eigenvector is of the form $v_l \otimes \tilde{v}_m$ with eigenvalue $k_1+k_2+2(l+m)$. $\Longrightarrow$
The eigenspace of eigenvalue k_1+k_2+2a is generated by elements $v_l \otimes \tilde{v}_{a-l}$; with $l = 0, \dots, a$.

Next step: find l vectors $V_{k_1} \otimes V_{k_2} \ni \sum \lambda_l (v_l \otimes \tilde{v}_{a-l}) \in M \simeq V_{k_1+k_2+2a}$ annihilated by the diagonal action of e^-:

$$\Delta e^- = \left(k_1 z + z^2 \frac{d}{dz}\right) \otimes Id_{V_{k_2}} + Id_{V_{k_1}} \otimes \left(k_2 w + w^2 \frac{d}{dw}\right)$$

with $$\tilde{v}_\ell = \frac{(k_2+\ell-1)!}{(k_2-1)!} w^{-k_2-\ell}$$

$$\Delta e^- \left(\sum_{\ell=0}^{a} \lambda_\ell (v_\ell \otimes \tilde{v}_{a-\ell}) \right) = 0$$

$$\Downarrow$$

$$\lambda_{\ell+1}(\ell+1)(k_1+\ell) + \lambda_\ell (a-\ell)(k_2+a-\ell-1) = 0$$

$\forall \ell = 0,..,a.$

Thus,

$$\lambda_\ell = (-1)^\ell \frac{(k_1+a-1)_{a-\ell}}{(a-\ell)!} \cdot \frac{(k_2+a-1)}{\ell!},$$

where $(x)_n = \frac{x!}{(x-n)!}$ - Pochhammer symbol.

Let

$$\varphi_{k_1 k_2}(z,w) = \sum_{\ell=0} (-1) \binom{}{\ell} \frac{\quad}{a!\,(k_1-1)!\,(k_2-1)!}\, z \quad w$$

$$= \frac{1}{a!} \frac{(k_1+a-1)!\,(k_2+a-1)!}{(k_1-1)!\;(k_2-1)!} \cdot \frac{\left(w^{-1} - z^{-1}\right)^a}{z^{k_1}\, w^{k_2}}$$

Is the lowest weight vector in $M \simeq V_{k_1+k_2+2a}$

$\varphi^{2a}_{k_1,k_2}$ is the image of ${}^{k_1}v_0 \otimes {}^{k_2}\tilde{v}_0$ by

$$\sum_{\ell=0}^{a} (-1)^{\ell} \frac{(k_1+a-1)_{a-\ell}}{(a-\ell)!} \cdot \frac{(k_2+a-1)_{\ell}}{\ell!} \left(e^+\right)^{\ell} \otimes \left(e^+\right)^{a-\ell}$$

$$\in \operatorname{diag}\left(\mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2)\right)$$

$\rightsquigarrow \quad RC^{k_1+k_2+2a} \in \operatorname{Hom}\left(V_{k} \otimes V \quad V \right)$

$$\ldots_{K_1,K_2} \in \mathrm{Hom}_{\mathrm{diag}\,\mathfrak{sl}_2}\left(V_{K_1}\otimes V_{K_2}, V_{K_1+K_2+2a}\right).$$

- Combinatorics of coefficients
- Analytic properties of $B^2_K(\mathbb{H})$
- Geometry of $\mathbb{H}$.

Some answers:

$$RC^{K_1+K_2+2a}_{K_1,K_2} = \mathrm{Rest}\Big|_{z=z_1=z_2} \circ \mathbb{P}_a^{K_1-1,K_2-1}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$$

where

$$\mathbb{P}_a^{\alpha,\beta}(x,y) := (-1)^a (x+y)^a \, P_a^{\alpha,\beta}\left(\frac{y-x}{x+y}\right)$$

and

$$P_a^{\alpha,\beta}(t) = \frac{(\alpha+1)_a}{a!} \; {}_2F_1\left(-a \,;\, \alpha+\beta+a+1,\, \alpha+1,\, \frac{1-t}{2}\right)$$

or

$$(1-t)^{\alpha}(1+t)^{\beta}\, P_a^{\alpha,\beta}(t) = \frac{(-1)^a}{2^a a!}\left(\frac{d}{dt}\right)^a \left((1-t)^{a+\alpha}(1+t)^{a+\beta}\right)$$

Remarkable issue :

- Modular forms
- Equivariant Quantization
- Branching & Symmetry breaking
- Conformal geometry

etc...

The Big Three

PART 2

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Let $G' \subset G$ symmetric pair of Lie groups.

$$\begin{tikzcd} & \mathcal{W} \arrow[d] & & \mathcal{V} \arrow[d] & \\ G' \arrow[ur] \arrow[r, phantom, "\curvearrowright"] & Y \arrow[rr, "p"] & & X & G \arrow[ul] \arrow[l, phantom, "\curvearrowleft"] \end{tikzcd}$$

2 mnfds

Assume p is G'-equiv.

$$\mathrm{Diff}_{G'}\big(\mathcal{O}(X,\mathcal{V}), \mathcal{O}(Y,\mathcal{W})\big) =: \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

- If $\mathcal{W} = p^*\mathcal{V} \Rightarrow f \longmapsto f|_Y$ is a 0-order G'-eq. diff. op.
- If $Y \subset X \Rightarrow \mathrm{Rest}|_Y \circ$ Normal deriv. wrt Y are G'-eq. diff. op.

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \overset{?}{\subset} \mathrm{Hom}_{G'}\big(\mathcal{O}(X,\mathcal{V}), \mathcal{O}(Y,\mathcal{W})\big)$$

$X = G/H$; $Y = G'/H'$

Theorem 1 (DUALITY Theorem). T. Kobayashi - M.P. 2016

Let $H' \subset H$ be closed s/groups of a Lie group G with Lie alg. $\mathfrak{h}' \subset \mathfrak{h}$ resp. Suppose W & V are fin. dim^l repr. of H' and H resp.

Let G' be a subgroup of G containing H'.

$$\mathcal{V}_X := G \times_H V \,; \quad \mathcal{W}_Y := G' \times_{H'} W$$

corr. hom. vector bndls.

Then, $\exists$ a linear isomorphism

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \xrightarrow[D_{X \to Y}]{\sim} \mathrm{Hom}_{H'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$$

$$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V^\vee$$

<u>Rem</u> If $G = G'$; $X = Y = G/B$ the flag variety
$\Rightarrow$ Theorem is due to B. Kostant.

- If $\dim \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) = 1$
$\Rightarrow$ generator is "special"

- If $\mathfrak{q}$ is a parabolic s/alg. with an abelian nilradical
  $\Rightarrow$ "often" $\dim_{G'} \mathcal{D}(\mathcal{V}_X, \underline{\underline{\mathcal{W}_Y}}) = 1$

Let $G' \subset G$ be reductive, Let K', K be their max. compacts

Let G/K, G'/K' are hermitian and $\exists$ $Z \in \text{center}(\mathfrak{k})$ s. that

- $Z \in \mathfrak{k}'$
- $\mathfrak{g} = \underbrace{\mathfrak{k} + \mathfrak{n}_+}_{:= \mathfrak{p}} + \mathfrak{n}_-$: $ad(Z)$-eigen space.

Let V be a f.dim^l. repr. of K and W —"— of K'.

Theorem 2 (Localness theorem). T. Kobayashi – M.P. 2016.

Any G'-homomorphism from $\mathcal{O}(G/K, \mathcal{V}) \to \mathcal{O}(G'/K', \mathcal{W})$ is given by a holomorphic diff operator.

$$\mathrm{Hom}_{G'}\big(\mathcal{V}(G/K, \mathcal{V}), \mathcal{V}(G'/K', \mathcal{V})\big) = \mathrm{Diff}^{hol}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

Let $P'_{\mathbb{C}} := P_{\mathbb{C}} \cap G'$; $\quad Y_{\mathbb{C}} := G'_{\mathbb{C}}/P'_{\mathbb{C}}$; $\quad X_{\mathbb{C}} := G_{\mathbb{C}}/P_{\mathbb{C}}$

$$\begin{tikzcd}
G'/K' \arrow[r, hook, "\underline{\text{holom. embed.}}"'] \arrow[d, phantom, "\text{open}\ \cap"] & G/K \arrow[d, phantom, "\cap\ \text{open}"] \\
Y_{\mathbb{C}} = G'_{\mathbb{C}}/P'_{\mathbb{C}} \arrow[r] & X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}
\end{tikzcd}$$

Theorem 3 (Extension Theorem). T. Kobayashi – MP. 2016

Any diff. operator from Theorem 2 <u>extends</u> to a $G'_{\mathbb{C}}$-equiv. hol. D.O. NAMELY

$$\mathrm{Diff}^{hol}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}) \simeq \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$$

This injection is bijective.

How to find, to describe elements of $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$.

$\Rightarrow$ F – method.

Another example.

Let f, g 2 f^{ons} on $\mathbb{R}^2 \setminus \{0,0\}$

$\rightsquigarrow \quad F = \begin{pmatrix} f \\ g \end{pmatrix}$

$$\rightsquigarrow \quad F_\lambda^\vee(r\cos\theta, r\sin\theta) := r^{-2\lambda} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} F\left(\frac{-\cos\theta}{r}, \frac{\sin\theta}{r}\right)$$

twisted inversion. $\qquad \lambda \in \mathbb{C}$.

D_1, D_2 diff. operators on $\mathbb{R}^2$

$$\rightsquigarrow \quad (f,g) \xmapsto{D} (D_1 f)(x,0) + (D_2 g)(x,0)$$

$$D := \mathrm{Rest}|_{y=0} \circ (D_1, D_2)$$

QUESTION

① Find $(\lambda,\nu)\in\mathbb{C}^2$: $\exists$ D_1 D_2 satisf

- D_1, D_2 have const. coeff.
- $\forall F\in C^\infty(\mathbb{R}^2)\oplus C^\infty(\mathbb{R}^2)$

$$\boxed{(DF_\lambda^\nu)(x)=|x|^{-2\nu}(DF)\left(-\frac{1}{x}\right)} \quad (\star)$$

Find such D expl.

Ex.

- $\nu=\lambda \Rightarrow \mathrm{Rest}|_{y=0}\circ(\mathrm{id},0)$

$$(f,g)\longrightarrow f(x,0)$$

- $\nu=\lambda+1$; $D_{\lambda,\nu}:=\mathrm{Rest}|_{y=0}\left(\frac{\partial}{\partial x},\lambda\cdot\frac{\partial}{\partial y}\right)$

$$(f,g)\to\frac{\partial f}{\partial x}(x,0)+\lambda\frac{\partial g}{\partial y}(x,0)$$

- $\nu = \lambda + 2$

$$(f,g) \mapsto 2(2\lambda+1)\frac{\partial^2 f}{\partial x \partial y}(x,0) + (\lambda-1)\frac{\partial^2 g}{\partial x^2}(x,0) +$$

$$+ \ (\lambda+1)(2\lambda+1)\ \frac{\partial^2 g}{\partial y}\ (x,0)$$

$$D = \mathrm{Rest}\big|_{y=0} \circ (D_1, D_2)$$

$$\rightsquigarrow \quad D^\vee := \mathrm{Rest}\big|_{y=0}(-D_2, D_1)$$

Let $C_\ell^\alpha(t)$ be the Gegenbauer polynomial.

$$C_\ell^\alpha(s,t) = s^{\frac{\ell}{2}}\, C_\ell^\alpha\left(\frac{t}{\sqrt{s}}\right)$$

$$\rightsquigarrow$$

$$\mathcal{C}_\ell^\alpha := C_\ell^\alpha\left(-\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial y}\right) \quad \text{on } \mathbb{R}^2.$$

<u>Theorem</u> T. Kobayashi, T. Kubo, M.P.

Let $a := \nu - \lambda \in \mathbb{N}$. For $a > 0$

$$D_1 = a(2\lambda + a - 1)\frac{\partial}{\partial x} \circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}}$$

$$D_2 = \left(2\lambda^2 + 2(a-1)\lambda + a(a-1)\right)\frac{\partial}{\partial y} \circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}} + (\lambda-1)(2\lambda+1)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \circ \mathcal{C}_{a-2}^{\lambda+3/2}$$

$D, D^\vee$ sat. $\star$.

If $2\lambda \notin \{0, -1, -2, \dots\}$ then

$\exists$ a nontrivial sol. to $\star \iff \nu - \lambda \in \mathbb{N}$.

Question' Describe equiv. maps

$$T : \mathcal{E}^i_\lambda(X) \longrightarrow \mathcal{E}^j_\nu(Y)$$

resp. $\mathrm{Conf}(X, Y)$

$X = S^2$; $Y = S^1$

$SL_2(\mathbb{C}) = O(3,1)_0$

$SL_2(\mathbb{R}) = O(2,1)_0$

$i = 1$; $j = 0$.

$$\mathcal{E}^1(S^2) \to \mathcal{E}^0(S^1).$$

$\leftarrow$ $(*)$

Question "

$$\mathrm{Hom}_{\mathfrak{g}}\left(M(\mu), M(\lambda_1) \otimes M(\lambda_2)\right) \neq \{0\}.$$

Duality Theorem connects $\nearrow$ with solutions of $(*)$