Different Aspects of Rankin-Cohen Operators

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$\mathbb{H} = \{ z = x+iy \mid x \in \mathbb{R};\ y > 0 \}$ + Poincaré $ds^2 = \frac{dx^2+dy^2}{y^2}$

$G = SL(2,\mathbb{R}) \curvearrowright \mathbb{H} : \quad z \longmapsto \frac{az+b}{cz+d} =: g.z; \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$G \curvearrowright \mathcal{O}(\mathbb{H})$.

A holom. function $f \in \mathcal{O}(\mathbb{H})$ is hol. modular form of weight $k \in 2\mathbb{N}$ wrt $\Gamma = SL_2(\mathbb{Z})$ if:

- $\underbrace{f\left(\frac{az+b}{cz+d}\right) = f(z)\cdot(cz+d)^k}$; $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
- growth conditions at ∞.

Ex. $\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}$ — mod. form of weight 12.

Observation. Product of 2 modular forms of weight k_1 & k_2 is modular of weight $k_1 + k_2$.

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$$\frac{\partial f}{\partial z}(\gamma z) = ?$$

$$\frac{\partial f}{\partial z} = -k \cdot c\,(cz+d)^{-k-1} f\left(\frac{az+b}{cz+d}\right) + (cz+d)^{-k-2}\,\frac{\partial f}{\partial z}\left(\frac{az+b}{cz+d}\right).$$

$k_1 f_1 \cdot f_2' - k_2 f_1' \cdot f_2$ – modular form of weight $\underline{k_1+k_2+2}$

There exist an infinite family of such bi-differential operators

$$\boxed{RC}$$

$$RC_{k_1 k_2}^{k_3}(f_1, f_2)(z) := \sum_{l=0}^{a} (-1)^{l} \binom{k_1+a-1}{l}\binom{k_2+a-1}{a-l} f_1^{(a-l)}(z)\cdot f_2^{(l)}$$

where $\boxed{k_3 = k_1 + k_2 + 2a}$

Prop. If f_1 is hol. mod. form of weight k_1
f_2 ——— // ——— k_2

Then $RC_{k_1 k_2}^{k_2}(f_1, f_2)$ – hol. mod. form of weight $\underbrace{k_1+k_2+2a}_{k_3}$.

$$G = SL_2(\mathbb{R}) \curvearrowright \mathcal{O}(H) \cap L^2(H, y^{k}\, dx\,dy) =: B_k(H)$$

by $$\pi_k(g)\, f(z) := (cz+d)^{-k}\, f\left(\frac{az+b}{cz+d}\right); \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$k = 2, 3, 4, 5, \dots$

$\{\pi_k\}_{\substack{k\in\mathbb{N}\\ k\geq 2}}$ – Hol. discr. series repr.
unitary & irred..

<u>Thm</u> $$RC^{k_3}_{k_1 k_2}(\pi_{k_1}(g) f_1, \pi_{k_2}(g) f_2) = \pi_{k_3}(g)\, RC^{k_3}_{k_1 k_2}(f_1, f_2) \quad \forall g \in SL_2(\mathbb{R})$$

$$\mathfrak{g} = sl_2(\mathbb{R}) = \mathrm{Vect}\langle \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{h}, \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{e^+}, \overset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{e^-} \rangle$$

$$[h, e^\pm] = \pm 2 e^\pm$$
$$[e^+, e^-] = h.$$

$$Z(\mathfrak{g}) \ni c = h^2 + 2(e^+e^- + e^-e^+).$$

For $X \in \mathfrak{g}$, $$d\pi_k(X) := \frac{d}{dt}\Big|_{t=0} \pi_k(\exp tX)$$

$X = h, e^+, e^-$.

$$d\pi_k(e^-) f(z) = kz \cdot f(z) + z^2 f'(z)$$

$$d\pi_k(e^+)\, f(z) = -f'(z)$$

$$d\pi_k(h)\, f(z) = -k f(z) - 2z f'(z).$$

Consider $\quad {}^{k}v_l = \frac{(k+l-1)!}{(k-1)!} \cdot z^{-k-l}$; $\quad$ Let $V_k = \langle v_l \rangle_{l \in \mathbb{N}}$.

$$d\pi_k(e^+)\, v_l = v_{l+1}\,; \quad \forall\, l \in \mathbb{N}.$$

$$d\pi_k(e^-)\, v_l = -\,l\,(k+l-1)\, v_{l-1}\,; \quad l \in \mathbb{N}\setminus\{0\}.$$

$$d\pi_k(e^-)\, v_0 = 0.$$

$$d\pi_k(h)\, v_l = (k+2l)\, v_l\,; \quad \forall\, l.$$

$$d\pi_k(C)\, v = (k^2 - 2k)\, v\,; \quad \forall\, v \in V_k.$$

Consider $\quad V_{k_1} \otimes V_{k_2} \quad$ as a $\mathfrak{sl}_2$-module by diag. embed.

$$\Rightarrow \quad V_{k_1} \otimes V_{k_2} = \bigoplus_{a=0}^{\infty} \underbrace{V_{k_1+k_2+2a}}_{?}$$

comes from the second copy.

Irred. components contain: $\displaystyle\sum_l \lambda_l \,(v_l \otimes \tilde{v}_{a-l})$

which are killed by the diagonal action of e^-

which are killed by the diagonal action of [illegible].

Indeed, an h-eigenvector is of the form $v_\ell \otimes \tilde{v}_m \in V_{k_1} \otimes V_{k_2}$ with $k_1 + k_2 + 2(\ell + m)$, so the eigenspace corresp. to the eigenvalue $k_1 + k_2 + 2a$ is generated by $v_\ell \otimes \tilde{v}_{a-\ell}$ with $\underline{\ell = 0, \dots, a}$

To find a lowest weight vector we need to solve a system of PDE. But instead, we may reduce computations to a recursion relation on the <u>coefficients</u>.