WORKSHEET FOR TUTORIAL SESSIONS

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Let $SL(2,\mathbb{R})$ be the set of 2×2 matrices of determinant one. Namely,

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : ad - bc = 1 \right\},$$

where $M_{2\times 2}(\mathbb{R})$ is the space of 2×2 matrices over real numbers.

The aim of this worksheet is to provide some exercises on representation theory of $SL(2,\mathbb{R})$. I hope that the reader will acquire some concrete ideas through the exercises.

1. Iwasawa decomposition of $SL(2,\mathbb{R})$

In representation theory of $SL(2,\mathbb{R})$, we often see the following subgroups:[1]

$$K := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in SL(2,\mathbb{R}) : a, b \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

$$A := \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} : r \in \mathbb{R}_{>0} \right\} = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$P := \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times, x \in \mathbb{R} \right\},$$

where $\mathbb{R}^\times = \mathbb{R}\backslash\{0\}$.

We have $K \simeq S^1$, $A \simeq \mathbb{R}_{>0} \simeq \mathbb{R}$, and $N \simeq \mathbb{R}$. In particular, K is compact and A and N are noncompact. The subgroup P is called a *parabolic subgroup*.

Exercise 1. Define

$$M := Z_K(A) = \{k \in K : ka = ak \text{ for all } a \in A\}.$$

(1) Show that $M = \{\pm I_2\}$, where I_2 denotes the identity matrix.

(2) Show that $P = MAN$ (by this I mean that each element of P is a product of $m \in M$, $a \in A$, and $n \in N$).

Remark 1.1. The decomposition $P = MAN$ is called the *Langlands decomposition.*

[1] Although we do not consider in this worksheet, the subgroup

$$\bar{N} := \{\left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y \in \mathbb{R} \right\} \simeq \mathbb{R}$$

also plays a key role in representation theory of $SL(2,\mathbb{R})$.

Hereafter, we write $G := SL(2,\mathbb{R})$. The aim of the next exercise is to show that each $g \in G$ can be uniquely expressed as $g = kan$ for $k \in K$, $a \in A$, and $n \in N$. The decomposition $G = KAN$ is called the *Iwasawa decomposition.*

Exercise 2. Write $\mathbf{H} := \{x + iy : y > 0\}$, the upper-half plane. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbf{H}$, define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

(1) Show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \in \mathbf{H}$.

(2) Show that, for $g_1, g_2 \in G$ and $z \in \mathbf{H}$, we have $g_1 \cdot (g_2 \cdot z) = (g_1 g_2) \cdot z$. Thus, the group G acts on $\mathbf{H}$.

(3) Show that $\mathrm{Stab}_G(i) = K$, where $\mathrm{Stab}_G(i) = \{g \in G : g \cdot i = i\}$.

(4) Show that for any $g \in G$, there exist $a \in A$ and $n \in N$ such that $g \cdot i = (na) \cdot i$.

(5) Conclude from (3) and (4) that $G = NAK$. (By applying inversion, this shows that $G = KAN$.)

(6) Show that the decomposition $g = kan$ for $k \in K$, $a \in A$, and $n \in N$ is unique.

Exercise 3. Show that the Iwasawa decomposition of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix}.$$

Exercise 4. Show that K is a maximal compact subgroup of G.

Remark 1.2. By the Iwasawa decomposition $G = KAN$, the group $G = SL(2,\mathbb{R})$ is not compact.

Remark 1.3. For $m \in 1 + \mathbb{Z}_{\geq 0}$, define the *special orthogonal group* $SO(m)$ as

$$SO(m) := \{g \in M_{m\times m}(\mathbb{R}) : gg^t = I_m \text{ and } \det g = 1\},$$

where g^t denotes the transpose of g and I_m is the identity matrix. With this notation, the group K is understood as $K = SO(2)$.

2. The classification of $\widehat{K}$ for $K = SO(2)$

The aim of this section is to classify irreducible unitary finite-dimensional representations of $K = SO(2)$. In this worksheet, we mean representations by those defined over complex vector space as usual. We resume the notation from Section 1, unless otherwise specified.

We start with the definition of unitary representation.

Definition 2.1. A representation (π, V) of a Lie group G (need not be $G = SL(2,\mathbb{R})$) defined over a complex vector space V is said to be *unitary* if there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ such that $\langle \pi(g)v, \pi(g)v \rangle = \langle v, v \rangle$ for all $g \in G$ and $v \in V$.

For $K = SO(2)$, put[2]

$$\mathrm{Irr}(K)_{\mathrm{fin}} := \text{the set of equivalence classes of irreducible finite-dimensional representations of } K,$$
$$\widehat{K} := \text{the set of equivalence classes of irreducible unitary finite-dimensional representations of } K.$$

We first show $\mathrm{Irr}(K)_{\mathrm{fin}} = \widehat{K}$. For simplicity, write

$$k_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Exercise 5. Show that any finite-dimensional representation (π, V) of K admits a K-invariant Hermitian inner product. That is, given a representation (π, V) of K, there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V such that $\langle \pi(k)v_1, \pi(k)v_2 \rangle = \langle v_1, v_2 \rangle$. (Hint: Given a Hermitian inner product $(\cdot, \cdot)$, define

$$\langle v_1, v_2 \rangle = \int_0^{2\pi} (\pi(k_\theta)v_1, \pi(k_\theta)v_2)\, d\theta.$$

Then show that $\langle \cdot, \cdot \rangle$ is a desired one.)

As in the lectures by Birgit, irreducible representations in $\widehat{K}$ play a crucial role in the study of admissible representations of $G = SL(2, \mathbb{R})$. Then we next study $\widehat{K}$.

For $n \in \mathbb{Z}$, we define a one-dimensional representation $(\chi_n, \mathbb{C})$ of K by

$$\chi_n(k_\theta)z = e^{in\theta} z.$$

Clearly, $\{\chi_n : n \in \mathbb{Z}_{\geq 0}\} \subset \mathrm{Irr}(K)_{\mathrm{fin}} = \widehat{K}$.

Claim 2.2. We have

$$\widehat{K} = \{\chi_n : n \in \mathbb{Z}_{\geq 0}\} \simeq \mathbb{Z}. \tag{2.3}$$

To prove Claim 2.2, we start with the classification of irreducible finite-dimensional representations of abelian groups.

Exercise 6. Show that every irreducible complex finite-dimensional representation of an abelian group is one-dimensional. (Hint: Use Schur's Lemma below. If you haven't seen it, also consider the proof.)

Fact 2.4 (Schur's Lemma). Let (π, V) be an irreducible representation of a group G, (which need not to be $SL(2, \mathbb{R})$) over a complex finite-dimensional vector space V. If $T : V \to V$ is a linear map such that $T \circ \pi(g) = \pi(g) \circ T$ for all $g \in G$, then there exists a constant $\lambda \in \mathbb{C}$ such that $T = \lambda \cdot \mathrm{Id}_V$, where Id_V denotes the identity map on V.

As K is abelian, Exercise 6 shows that $\widehat{K}$ consists of one-dimensional representations.

Remark 2.5. The analogous statement with Exercise 6 need not be true for representations defined over $\mathbb{R}$. Consider the two-dimensional real representation $(\pi, \mathbb{R}^2)$ of K defined by the standard

[2] The finite-dimensionality of the definition of $\mathrm{Irr}(K)_{\mathrm{fin}}$ and $\widehat{K}$ is unnecessary. In fact, for a compact Lie group K, the density of the space of K-finite vectors forces that any irreducible admissible representation of K is finite-dimensional.

matrix multiplication, namely,

$$\pi \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $(\pi, \mathbb{R}^2)$ is an irreducible representation of K.

As $\chi \in \widehat{K}$ is unitary, it satisfies $|\chi(k)|^2 = 1$ for all $k \in K$. It then follows from Exercise 6 that, to show Claim 2.2, one wishes to classify continuous group homomorphisms $\chi \colon K \to S^1$. As $K \simeq S^1$, this is equivalent to classifying continuous group homomorphisms $\chi \colon S^1 \to S^1$.

Toward our goal (2.3), we next consider one-dimensional representations of additive group $\mathbb{R}$.

Exercise 7. Show that any continuous homomorphism $\nu \colon \mathbb{R} \to \mathbb{C}^\times$ is of the form

$$\nu \colon x \mapsto e^{sx} \quad \text{for some } s \in \mathbb{C},$$

where $\mathbb{C}^\times$ denotes $\mathbb{C}^\times = \mathbb{C} \backslash \{0\}$. In particular, any continuous homomorphism $\widetilde{\nu} \colon \mathbb{R} \to S^1$ is of the form

$$\widetilde{\nu} \colon x \to e^{iyx} \quad \text{for some } y \in \mathbb{R}.$$

(Hint: Let $\nu \colon \mathbb{R} \to \mathbb{C}^\times$ be a continuous homomorphism of $\mathbb{R}$. Then $\nu(0) = 1$, and so, by continuity, there exists $\delta > 0$ such that $\int_0^\delta \nu(t)\ dx \neq 0$. Then observe that $\int_x^{x+\delta} \nu(t)\, dt = \nu(x) \int_0^\delta \nu(t)\, dt$ and consider $\nu'(x)$. Note: The ν is initially assumed to be merely continuous, but it is in fact differentiable.)

Remark 2.6. Recall from Section 1 that we have $A \simeq \mathbb{R}$, which is given by $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \mapsto \log(r)$. Thus, Exercise 7 shows that any irreducible finite dimensional representation $(\nu, \mathbb{C})$ of A has the form

$$\nu \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = r^s \quad \text{for some constant } s \in \mathbb{C}. \tag{2.7}$$

Exercise 8. Prove Claim 2.2. That is, show that any irreducible unitary finite-dimensional representation $(\xi, \mathbb{C})$ of K is of the form

$$\chi(k_\theta) = e^{in\theta} \quad \text{for some } n \in \mathbb{Z}.$$

Remark 2.8. By Exercise 8, the $(\mathfrak{sl}(2, \mathbb{C}), K)$-module V_K consisting of K-finite vectors of irreducible admissible representation (π, V) of G is of the form

$$V_K = \bigoplus_{n \in \mathbb{Z}} V(n), \quad \text{(algebraic direct sum)}$$

where $V(n)$ is the χ_n-isotypic component of V. Namely,

$$V(n) = \{v \in V : \pi(k)v = \chi_n(k)v \text{ for all } k \in K\}.$$

3. Irreducible finite-dimensional representations of $P = MAN$

In this section we consider irreducible finite-dimensional representations of parabolic subgroup $P = MAN$. We start with the following elementary observation.

Write

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, for $t \in \mathbb{R}$, we have

$$e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A \quad \text{and} \quad e^{tX} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in N.$$

If (π, V) is a finite-dimensional representation of AN, then H and X act on V via the differential $d\pi$ of π, that is,

$$d\pi(H) := \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tH}) \quad \text{and} \quad d\pi(X) := \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tX}).$$

A quick computation shows that

$$[H, E] := HE - EH = 2E.$$

Equivalently,

$$HE = [H, E] + EH = 2E + EH.$$

Thus,

$$d\pi(H)d\pi(E) = d\pi(HE) = 2d\pi(E) + d\pi(E)d\pi(H).$$

In particular, if there exists $\lambda \in \mathbb{C}$ such that $d\pi(H)v = \lambda v$, then

$$d\pi(H)d\pi(E)v = (\lambda + 2)d\pi(E)v. \tag{3.1}$$

Now we consider the following.

Exercise 9. Let (π, V) be an irreducible finite-dimensional representation of P. Show that N acts on V trivially. Namely,

$$\pi(n)v = v \quad \text{for all } n \in N \text{ and } v \in V.$$

As N is a normal subgroup of P, it follows from Exercise 9 that any irreducible finite-dimensional representation (π, V) of $P = MAN$ is regarded as a representation of $MAN/N \simeq MA$. Then, for $H \in \{P, M, A\}$, put

$$\mathrm{Irr}(H)_{\text{fin}} := \text{the set of equivalence classes of irreducible finite-dimensional representations of } H.$$

As $M \simeq \{\pm I_2\}$, we have

$$\mathrm{Irr}(M)_{\text{fin}} = \{\mathrm{triv}, \mathrm{sgn}\} \simeq \mathbb{Z}/2\mathbb{Z},$$

where $\mathrm{triv}(-I_2) = 1$ (*trivial representation*) and $\mathrm{sgn}(-I_2) = -1$ (*sign representation*).

Let $\chi_n \in \widehat{K}$. Since $M = Z_K(A) \subset K$, the restriction of χ_n to M is well-defined. Let$\chi_n|_M$ denote the restriction of χ_n to M.

Exercise 10. Check that

$$\chi_n|_M = \begin{cases} \mathrm{triv} & \text{if } n \text{ is even,} \\ \mathrm{sgn} & \text{if } n \text{ is odd.} \end{cases}$$

As we observed in Remark 2.6, the set $\mathrm{Irr}(A)_{\mathrm{fin}}$ is given by

$$\mathrm{Irr}(A)_{\mathrm{fin}} = \{\nu_s : s \in \mathbb{C}\} \simeq \mathbb{C},$$

where ν_s is defined as in (2.7). Thus, we have

$$\mathrm{Irr}(P)_{\mathrm{fin}} \simeq \mathrm{Irr}(M)_{\mathrm{fin}} \times \mathrm{Irr}(A)_{\mathrm{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}.$$

4. Principal series representation $(l, \mathrm{Ind}_P^G(\sigma \otimes \nu_s))$

For $(\sigma, \nu_s) \in \mathrm{Irr}(P)_{\mathrm{fin}} \simeq \mathrm{Irr}(M)_{\mathrm{fin}} \times \mathrm{Irr}(A)_{\mathrm{fin}}$, define

$$\mathrm{Ind}_P^G(\sigma \otimes \nu_s) := \{f \in C^\infty(G) : f(gman) = \nu_s(a^{-1})\sigma(m^{-1})f(g) \text{ for all } m \in M,\ a \in A, \text{ and } n \in N\}.$$

The group G acts on $\mathrm{Ind}_P^G(\sigma \otimes \nu_s)$ by left-translation l, that is,

$$l(g)f(x) = f(g^{-1}x) \text{ for } g, x \in G.$$

The representation $(l, \mathrm{Ind}_P^G(\sigma \otimes \nu_s))$ is called a *parabolically induced representation* or *principal series representation* of G.

It is known that it follows from the so-called Peter–Weyl theorem that the space $\mathrm{Ind}_P^G(\sigma \otimes \nu_s)_K$ of K-finite vectors decomposes into

$$\mathrm{Ind}_P^G(\sigma \otimes \nu_s)_K \simeq \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{in\theta} \otimes \mathrm{Hom}_M(\chi_n|_M, \sigma). \quad \text{(algebraic direct sum)}$$

It then follows from Exercise 10 that

$$\mathrm{Ind}_P^G(\sigma \otimes \nu_s)_K \simeq \begin{cases} \bigoplus_{n \in 2\mathbb{Z}} \mathbb{C}e^{in\theta} & \text{if } \sigma = \mathrm{triv}, \\ \bigoplus_{n \in 1+2\mathbb{Z}} \mathbb{C}e^{in\theta} & \text{if } \sigma = \mathrm{sgn}. \end{cases} \tag{4.1}$$

Remark 4.2. The decomposition of (4.1) is independent of the complex parameter $s \in \mathbb{C}$ for the character ν_s for A. The parameter $s \in \mathbb{C}$ is strongly related to the $\mathfrak{sl}(2, \mathbb{C})$-module structure of $\mathrm{Ind}_P^G(\sigma \otimes \nu_s)_K$ such as irreducibility and composition factors.

Since

$$C^\infty(S^1)_K = L^2(S^1)_K = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{in\theta} \quad \text{(algebraic direct sum)},$$

it follows from (4.1) that, as K-representations, we have

$$C^\infty(S^1)_K \simeq \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)_K \oplus \mathrm{Ind}_P^G(\mathrm{sgn} \otimes \nu_s)_K. \tag{4.3}$$

Define

$$C^\infty(S^1)^+ := \{f \in C^\infty(S^1) : f(-x) = f(x)\},$$
$$C^\infty(S^1)^- := \{f \in C^\infty(S^1) : f(-x) = -f(x)\}.$$

Then the space $C^\infty(S^1)_K^\alpha$ of K-finite vectors decomposes into

$$C^\infty(S^1)_K^\alpha = \begin{cases} \bigoplus_{n \in 2\mathbb{Z}} \mathbb{C}e^{in\theta} & \text{if } \alpha = +, \\ \bigoplus_{n \in 1+2\mathbb{Z}} \mathbb{C}e^{in\theta} & \text{if } \alpha = -. \end{cases}$$

Thus, we have

$$C^\infty(S^1)^+_K \simeq \mathrm{Ind}_P^G(\mathrm{triv}\otimes\nu_s)_K \quad \text{and} \quad C^\infty(S^1)^-_K \simeq \mathrm{Ind}_P^G(\mathrm{sgn}\otimes\nu_s)_K.$$

In the rest of this note, we are going to show explicitly that

$$C^\infty(S^1)^+ \simeq \mathrm{Ind}_P^G(\mathrm{triv}\otimes\nu_s) \quad \text{and} \quad C^\infty(S^1)^- \simeq \mathrm{Ind}_P^G(\mathrm{sgn}\otimes\nu_s)$$

as G-representations. (Thus the equivalences (4.3) are indeed as $(\mathfrak{sl}(2,\mathbb{C}),K)$-modules.)

Exercise 11. For $g\in G$ and $x\in S^1\subset\mathbb{R}^2$, we have $gx\neq 0$, where gx is the standard matrix multiplication of g and $x\in\mathbb{R}^2$. Thus $gx/|gx|\in S^1$. Write

$$\varepsilon(g,x) := \frac{gx}{|gx|}.$$

(1) Show that, for $g_1,g_2\in G$ and $x\in S^1$, we have $\varepsilon(g_1g_2,x)=\varepsilon(g_1,\varepsilon(g_2,x))$.

(2) For $s\in\mathbb{C}$, $g\in G$, and $f\in C^\infty(S^1)^+$, define $\varpi_s(g)f\in C^\infty(S^1)^+$ by

$$\varpi_s(g)f(x) := |g^{-1}x|^{-s}f(\varepsilon(g^{-1},x)).$$

Show that $\varpi_s(g_1g_2)f(x)=\varpi_s(g_1)\varpi_s(g_2)f(x)$. Thus, $(\varpi_s, C^\infty(S^1)^+)$ is a representation of G.

Write

$$\Xi := \mathbb{R}^2\backslash\{0\}. \tag{4.4}$$

For $s\in\mathbb{C}$, we put

$$C_s^\infty(\Xi)^+ := \{F\in C^\infty(\Xi) : F(t\xi)=t^sF(\xi) \text{ and } F(-\xi)=F(\xi) \text{ for } t\in\mathbb{R}_{>0}\}. \tag{4.5}$$

The group G acts on $C_s^\infty(\Xi)^+$ by

$$\delta(g)F(\xi) := F(g^{-1}\xi).$$

Then $(\delta, C_s^\infty(\Xi)^+)$ is a representation of G.

Exercise 12. Define a continuous linear map

$$T\colon C_s^\infty(\Xi)^+ \longrightarrow C^\infty(S^1)^+$$

by

$$F\longmapsto f_F := F|_{S^1}.$$

Show that T is an intertwining operator, that is to show that

$$T(\delta(g)F)(x) = \varpi_s(g)T(F)(x) \quad \text{for all } g\in G \text{ and } F\in C_s^\infty(\Xi)^+.$$

The intertwining operator $T\colon C_s^\infty(\Xi)^+\to C^\infty(S^1)^+$ has inverse

$$T^{-1}\colon C^\infty(S^1)^+\to C_s^\infty(\Xi)^+, \quad f\mapsto F_f,$$

where

$$F_f(x) := |x|^{-s}f(x/|x|).$$

It then follows from Exercise 12 that the two representations $(\delta, C_s^\infty(\Xi)^+)$ and $(\varpi_s, C^\infty(S^1)^+)$ are equivalent.

Now we aim to show that

$$(l, \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)) \simeq (\varpi_s, C^\infty(S^1)^+). \tag{4.6}$$

To do so, it suffices to show $(l, \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)) \simeq (\delta, C_s^\infty(\Xi)^+)$.

Exercise 13. Recall that we have $\Xi = \mathbb{R}^2 \backslash \{0\}$.

(1) Show that G acts on Ξ transitively, that is to show that, for any $\xi \in \Xi$, there exists $g \in G$ and $\xi' \in \Xi$ such that $\xi = g\xi'$.

(2) As usual, write $e_1 = (1,0)^t$ (the transpose of the row vector $(1,0)$). By (1), any $\xi \in \Xi$ has the form $\xi = ge_1$. Check that $\mathrm{Stab}_G(e_1) = N$.

(3) We define a continuous linear map

$$\Gamma \colon \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s) \longrightarrow C_s^\infty(\Xi)^+$$

by

$$f \longmapsto \Gamma(f)(\xi) := f(g_\xi),$$

where g_ξ is some element in G such that $\xi = g_\xi e_1$. Show that Γ is well-defined, that is to show the following.

(a) For $\xi = g_1 e_1 = g_2 e_1$, we have $f(g_1) = f(g_2)$ for $f \in \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)$.

(b) $\mathrm{Im}(\Gamma) \subset C_s^\infty(\Xi)^+$.

(4) Define a continuous linear map

$$\Lambda \colon C_s^\infty(\Xi)^+ \longrightarrow \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)$$

by

$$F \longmapsto \Lambda(F)(g) := F(ge_1).$$

Show that $\mathrm{Im}(\Lambda) \subset \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)$.

(5) Show that Λ and Γ are inverse to each other.

(6) Show that Λ and Γ are G-intertwining operators to conclude that $(l, \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s))$ is equivalent to $(\delta, C_s^\infty(\Xi)^+)$.

Remark 4.7. For $s \in \mathbb{C}$, put

$$C_s^\infty(\Xi)^- := \{F \in C^\infty(\Xi) : F(t\xi) = t^s F(\xi) \text{ and } F(-\xi) = -F(\xi) \text{ for all } t \in \mathbb{R}_{>0}\}.$$

One can similarly show that

$$(l, \mathrm{Ind}_P^G(\mathrm{sgn} \otimes \nu_s)) \simeq (\delta, C_s^\infty(\Xi)^-) \simeq (\varpi_s, C^\infty(S^1)^-).$$

Remark 4.8. Observe that we have

$$C^\infty(S^1) = C^\infty(S^1)^+ \oplus C^\infty(S^1)^-.$$

Indeed, for $f(x) \in C^\infty(S^1)$, define

$$f(x)^+ := \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f(x)^- := \frac{f(x) - f(-x)}{2}.$$

Then,

$$f(x) = f(x)^+ + f(x)^-$$

and

$$f(x)^\alpha \in C^\infty(S^1)^\alpha \quad \text{for } \alpha \in \{\pm\}.$$

It then follows from the above argument that we have

$$(\varpi_s, C^\infty(S^1)) \simeq (l, \mathrm{Ind}_P^G(\mathrm{triv} \otimes \nu_s)) \oplus (l, \mathrm{Ind}_P^G(\mathrm{sgn} \otimes \nu_s)).$$

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