Lecture 5

Spherical Plancherel decomposition.

Thm (Spherical version of Plancherel).

For $f \in C_c^\infty(\tau : G/H)$:

$$\|f\|^2_{L^2(\tau : G/H)} = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^*] \int_{i\mathfrak{a}_P^*} \| \mathcal{F}f(\lambda)\|^2 \underbrace{d\mu_P}_{\text{Lebesgue}}(\lambda).$$

Aspects of the proof.

Def Schwartz functions à la Harish-Chandra.

Let $\Xi = \varphi_0 = \mathcal{P}_0(1_K) \in C^\infty(K\backslash G/K)$ (for G/K).

Put $\Theta(x) := \Xi(x\sigma(x)^{-1})^{1/2} \in C^\infty(K\backslash G/H)$.

Note:

$$\Theta(a) = \Xi(a^2)^{1/2} \sim a^{-\rho}(1 + |\log a|)^{d/2} \quad (a \xrightarrow{A_q^+} \infty)$$

Def. $\mathcal{C}(G/H) \ni f :\Leftrightarrow$

1) $f \in C^\infty(G/H)$

$|k a h|_\sigma = |\log a|$

2) $\forall_{\substack{u \in U(\mathfrak{g}) \\ N > 0}} \ \exists_{C > 0} \quad |L_u f(x)| \leq C \, (1 + \|x\|_\sigma)^{-N} \, \Theta(x)$

Facts:

- $\mathcal{C}(G/H)$ is Fréchet for obvious seminorms
- $\mathcal{C}(G/H)$ is invariant under left regular L_g $\forall g \in G$
- $(g, f) \mapsto L_g f$, $G \times \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ defines ct^s rep.
- $\mathcal{C}(G/H) = \mathcal{C}(G/H)^\infty$.
- $C_c^\infty(G/H)$ is dense in $\mathcal{C}(G/H)$.

Decay of K-finite matrix coefficients.

Suppose: $(\pi, \mathcal{H})$ irreducible unitary rep of G

Def: For $j \in (\mathcal{H}^{-\omega})^H$ and $v \in \mathcal{H}_K$ put

$$m_{v,j}(x) = \langle \pi(x)^{-1} v, j \rangle, \quad (x \in G).$$ Then $m_{v,j} \in C^\infty(G/H)$.

Charact^n tempered & discrete series w.r.t. G/H:

- $\pi \in (G/H)^\wedge_{temp} :\Longleftrightarrow \exists_j \forall_v \underset{C,N>0}{\exists} \forall_x \; |m_{v,j}(x)| \leq C(1+|x|_r)^N \Theta(x)$
- $\pi \in (G/H)^\wedge_{ds} :\Longleftrightarrow \exists_j \forall_v \; m_{v,j} \in \mathcal{C}(G/H)$

Key result for every $P \in \mathcal{P}_\sigma(A_q)$, $\psi \in \mathcal{A}_{2,P} = \mathcal{A}_{2,P,\tau}$ the normalized τ-spherical Eisenstein integral $E^\circ(P,\psi,\lambda,x)$ is regular for $\lambda \in i\mathfrak{a}_P^*$ and satisfies strong tempered estimates with uniformity in λ. This leads to

Thm $\mathcal{F}_P : \mathcal{C}(\tau : G/H) \longrightarrow \mathcal{S}(i\mathfrak{a}_{Pq}^*) \otimes \mathcal{A}_{2,P}$

continuous linear

usual Eucl^n Schwartz space

Def Inverse (or wave packet) transform

$$\mathcal{J}_P : \mathcal{S}(i\mathfrak{a}_{Pq}^*) \otimes \mathcal{A}_{2,P} \longrightarrow C^\infty(\tau : G/H) \quad \text{by}$$

$$\mathcal{J}_P\varphi(x) = \int_{i\mathfrak{a}_{Pq}^*} E^\circ(P,\varphi(\lambda),\lambda)(x)\, d\mu_P(\lambda)$$

Lebesgue

Thm $\mathcal{J}_p : \mathcal{S}(i\mathfrak{a}_{pq}^*) \otimes \mathcal{A}_{2,P} \longrightarrow \mathcal{C}(\tau : G/H)$ continuous linearly

Pf this requires a theory of the constant term, functions of type $\mathrm{II}(\Lambda)$ à la Harish-Chandra & Maass-Selberg relations

(work of ~, Delorme, Carmona, Schlichtkrull).

Lemma $\mathcal{F}_p$ and $\mathcal{J}_p$ are transpose to each other, i.e.

$\forall f \in \mathcal{C}(\tau : G/H)$, $\varphi \in \mathcal{S}(i\mathfrak{a}_{pq}^*) \otimes \mathcal{A}_{2,P}$:

$$\langle \varphi, \mathcal{F}_p f \rangle_{L^2(i\mathfrak{a}_{pq}^*) \otimes \mathcal{A}_{2,P}} = \langle \mathcal{J}_p \varphi, f \rangle_{L^2(\tau : G/H)}$$

Thm (equivalent to spherical Plancherel)

For $f \in C_c^\infty(\tau : G/H)$:

$$f = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^*]^{-1} \mathcal{J}_P \mathcal{F}_P f \quad \text{(inversion formula)}$$

$\Downarrow$ 4

Indeed:

$$\langle f, f \rangle = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^*]^{-1} \underbrace{\langle f, \mathcal{J}_P \mathcal{F}_P f \rangle_{L^2(G/H)}}_{= \| \mathcal{F}_P f \|^2_{L^2(\cdot, \cdot\mathfrak{a}_P^*)}}$$

- Delorme proves this by using Bernstein's à priori result that the Planch. measure for G/H is supported by $(G/H)^\wedge_{temp}$
- $\sim$ — Schlichtkrull prove this via a non-unitary inversion formula, involving only $\mathcal{F}_0 = \mathcal{F}_{P_0}$ with $P_0 \in \mathcal{P}_\sigma(A_q)$ minimal.

We focus on the second method.

Adapted notation ($P \in \mathcal{P}_\sigma(A_q)$)

$$E^\circ(P,\psi,\lambda)(x) = E^\circ(P,\lambda,x)\psi \qquad E^\circ(P,\lambda,x) \in \mathrm{Hom}(\mathcal{A}_{2,P}, V_\tau)$$

Put:

$$E^*(P,\lambda,x) := E^\circ(P,-\bar\lambda,x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_{2,P}).$$

Then

$$\mathcal{F}_P f(\lambda) = \int_{G/H} E^*(P,\lambda,y)\, f(y)\, dy \in \mathcal{A}_{2,P}$$

$$\mathcal{J}_P \circ \mathcal{F}_P f(x) = \int_{i\mathfrak{a}_{Pq}^*} \int_{G/H} \underbrace{E^\circ(P,\lambda,x)\, E^*(P,\lambda,y)}_{\mathbb{E}(P,\lambda,x,y)} f(y)\, dy\, d\lambda \in V_\tau$$

Symmetry: $\mathbb{E}(P,-\bar\lambda,x,y)^* = \mathbb{E}(P,\lambda,y,x)$.

This reflects: $(\mathcal{J}_P \circ \mathcal{F}_P)^* = \mathcal{J}_P \circ \mathcal{F}_P$.

Assume $\# W/W_{K\cap H} = 1$ ($\Rightarrow \#\, {}_P\mathcal{W} = 1$) (to simplify exposition)

Non-unitary inversion formula.

Lemma (series expansion $E(P_0)$).

$\exists!\ E_+(\overset{\lambda}{\cdot}, \overset{x}{\cdot}) \in \mathcal{M}_\Sigma(\mathfrak{a}^*_{q\mathbb{C}}, C^\infty(X_+)) \otimes V_\tau$ s.t.

$$E_+(\lambda, kaH) = \tau(k) \sum_{\mu \in \mathbb{N}\Delta} a^{\lambda-\rho-\mu}\, \Gamma_\mu(\lambda) \in \mathrm{End}(V_\tau^{K\cap M_0 \cap H})$$

$(k \in K,\ a \in A_q^+,\ \lambda \in \mathfrak{a}^*_{q\mathbb{C}})$

&

$$E^\circ(P_0, \lambda, x)\psi = \sum_{s \in W} E_+(s\lambda, x)\left[C^\circ_{P_0|P_0}(s, \lambda)\psi\right](e)$$

$(\lambda \in i\mathfrak{a}^*_{q\mathbb{C}},\ x \in X_+,\ \psi \in \mathcal{A}_{2,P_0})$

with $C^\circ_{P_0|P_0}(s, \overset{\lambda}{\cdot}) \in \mathcal{M}_\Sigma(\mathfrak{a}^*_{q\mathbb{C}}) \otimes \mathrm{End}(\mathcal{A}_{2,P_0})$

and $C^\circ_{P_0|P_0}(1, \cdot) \equiv I_{\mathcal{A}_{2,P_0}}$.

Here $\mathcal{M}_\Sigma(\mathfrak{a}^*_{q\mathbb{C}}, V) :=$ the space of weakly meromorphic $\varphi : \mathfrak{a}^*_{q\mathbb{C}} \to V$

s.t. locally $\exists\ \alpha_j \in \Sigma,\ c_j \in \mathbb{R}$: $\prod_j (\langle \alpha_j, \cdot\rangle + c_j)\varphi$ regular.

Theorem (inversion formula) Suppose $\eta \in \mathfrak{a}_{\mathfrak{q}}^*$ sufficiently anti-dominant. Then $\forall f \in C_c^\infty(\tau : G/H)$:

$$f(x) = \int_{i\mathfrak{a}_{\mathfrak{q}}^* + \eta} E_+(\lambda, x)\, \mathcal{F}_\circ f(\lambda)\, d\mu_{P_0}(\lambda), \qquad (x \in X_+).$$

Proof requires:
- Paley-Wiener shift $\eta \to \infty$ radially in $A_{\mathfrak{q}}^-$.
- Holmgren's uniqueness for PDE with C^ω coeffs
- Residue calculus.

Proof of unitary inversion $I = \sum_{P \in \mathcal{P}_\sigma} \mathcal{I}_P \mathcal{F}_P$.

W.l.o.g. may assume $\mathfrak{a}_G = \{0\}$.

Shift $\eta \to 0$, crossing one singular hypersurface $\alpha^\perp + \xi_0$ ($\alpha \in \Sigma$, $\xi_0 \in \mathfrak{a}_{\mathfrak{q}}^*$) at a time.

If η crosses $\alpha^\perp + \xi_0$, a residual integral

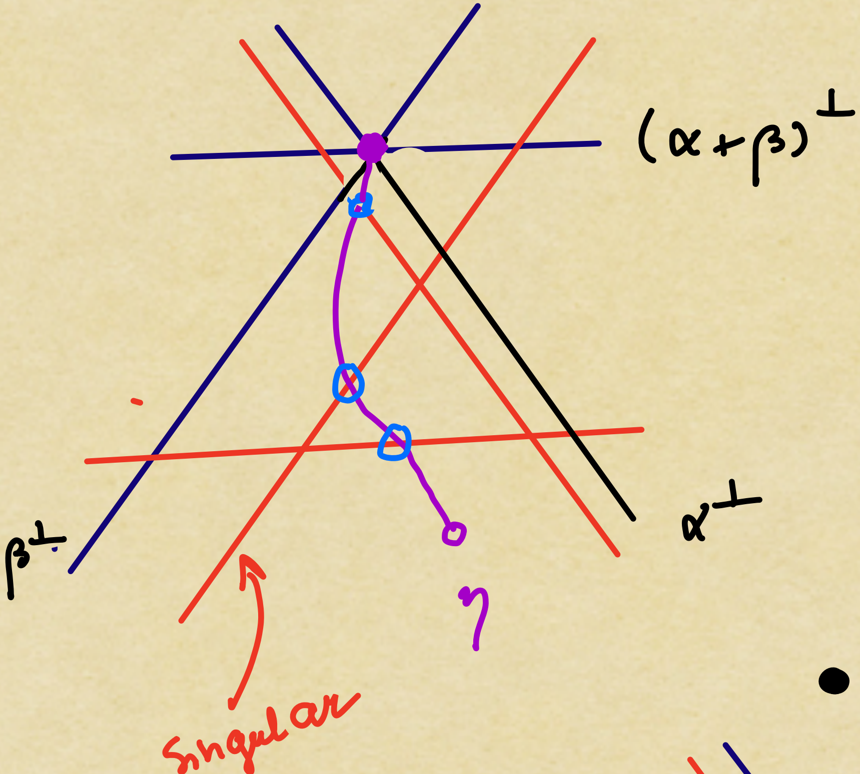
$$\int_{i\alpha^\perp + \xi} \mathrm{Res}(\ldots)$$

with $\xi = \xi_0$ is added. For each such residual integral shift ξ inside the set $\xi_0 + \alpha^\perp$ to the unique point ξ_1 of $\alpha^{\perp\perp} \cap (\alpha^\perp + \xi_0)$. The repetition of this process leads to residual integrals of the form

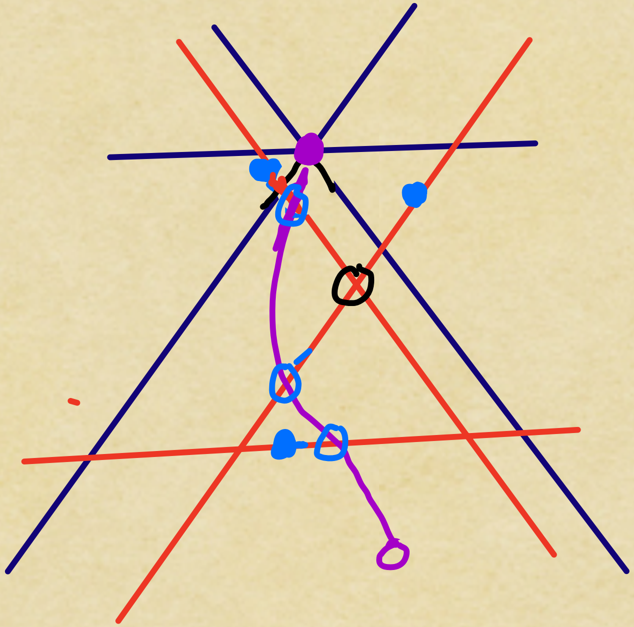
$$\int_{i\mathfrak{a}^*_{Rq} + \xi'} \mathrm{Res}'(\ldots)$$

with $\xi' = \xi'_0$. If $\mathfrak{a}_{Rq} = \{0\}$, no further shift is required. If $\mathfrak{a}_{Rq} \neq 0$, shift ξ' inside $\mathfrak{a}^*_{Rq} + \xi'_0$ to the unique point $\xi_1' \in \mathfrak{a}_{Rq}^{*\perp} \cap (\mathfrak{a}_{Rq}^* + \xi_0')$. Each crossing of a singular hyperplane then leads to a residual integral over a set of the form $i\mathfrak{a}_{Sq}^* + \xi_0''$, where $\mathfrak{a}_{Sq}^* \subset \mathfrak{a}_{Rq}^*$, $\dim \mathfrak{a}_R^* / \mathfrak{a}_S^* = 1$. See figure below.

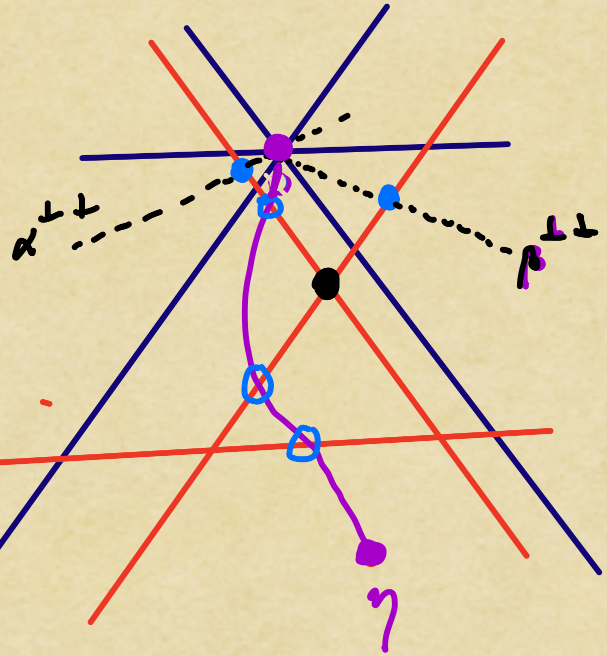
• dim 2 ○ dim 1



• dim 1 ○ dim 0



• point residue



End result:

$$f(x) = \sum_{P \in \mathbb{P}_\sigma} T_P(f)(x)$$

where $T_P(f)$ comes from residual integrals over the spaces $i\mathfrak{a}^*_{R_\mathfrak{q}} + \xi$, $\xi \in \mathfrak{a}^{*\perp}_{R\mathfrak{q}}$, where $R \in \mathcal{P}_\sigma(A_\mathfrak{q})$, $\mathfrak{a}_{R\mathfrak{q}} \overset{W}{\sim} \mathfrak{a}_{P\mathfrak{q}}$.

T_P has kernel $K_P(\lambda, x, y)$ determined by mentioned residues.

Claim $\quad K_P = [W_P : W_P^*]\, \mathbb{E}_P$, $\quad (P \in \mathbb{P}_\sigma)$.

Note $\quad$ The claim implies the spherical inversion formula

$$I = \sum_{P \in \mathbb{P}_\sigma} [W : W_P^*] \cdot \mathcal{J}_P \circ \mathcal{F}_P$$

We will prove the claim by induction over $\dim(G)$.

0 • Reduce to $\mathfrak{a}_G = 0$ (using Eucl F.T.), so may assume $G = M_G$.

1 • By induction on $\dim G$, $K_P = E_P$, for $P \subsetneq G$.

This follows from 'transitivity of residues' comparing the P-residues with point residues in $\mathfrak{a}^{*\perp}_{Pq}$ (the $\mathfrak{a}^*_q$ of M_P)

2 • In particular T_P is symmetric.

3 • T_G is symmetric. This follows from $T_G = I - \sum_{P \subsetneq G} T_P$ on $C_c^\infty(\tau : G/H)$.

$$\Rightarrow \quad K_G(x,y) = K_G(y,x).$$

4 • $K_G(x,y)$ is a sum of point residues at finitely many places in $\mathfrak{a}^*_{q\mathbb{C}}$.

5 • $K_G \in \mathcal{E}(\tau : G/H)^{\Lambda} \otimes C^{\infty}(\tau : G/H)$. Here Λ is a fixed cofinite ideal in $\mathbb{D}(G/H)$, and $\mathcal{E}(\tau : G/H)^{\Lambda}$ $= \{ f \in \mathcal{E}(\tau : G/H) \mid \Lambda \cdot f = 0 \}$. This follows from the residue process.

6 • $\mathcal{E}(\tau : G/H)^{\Lambda} \subset \mathcal{E}(\tau : G/H)_{\mathbb{D}(G/H)} \subset L^2_d(\tau : G/H)$

$\mathbb{D}(G/H)$ – finite

7 • $K_G \in \mathcal{E}(\tau : G/H)^{\Lambda} \otimes \mathcal{E}(\tau : G/H)^{\Lambda}$ (by symmetry & (5))

8 • $\forall_{P \subsetneq G}$ $\mathcal{F}_P = 0$ on $\mathcal{E}(\tau : G/H) \cap L^2_d(\tau : G/H)$

• (from separation of infinitesimal characters).
(from separation of infinitesimal characters).

9 • $I - T_G = 0$ on $\mathcal{E}(\tau : G/H) \cap L^2_d(\tau : G/H)$

10 • $T_G : \mathcal{E}(\tau : G/H) \longrightarrow \mathcal{E}(\tau : G/H)^\wedge$

11 • $I : \mathcal{E}(\tau : G/H) \cap L^2_d(\tau : G/H) \longrightarrow \mathcal{E}(\tau : G/H)^\wedge$

12 • By 6), $\mathcal{E}(\tau : G/H)^\wedge = L^2_d(\tau : G/H)$

13 • $\dim L^2_d(\tau : G/H) < \infty$ (!)

14 • $\# \{ \pi \in \widehat{(G/H)}_{ds} \mid H_\pi[\delta] \neq \phi \} < \infty$.

15 • $T_G = \mathrm{pr}^\perp [L^2(\tau : G/H) \longrightarrow L^2_d(\tau : G/H)] \Big|_{C_c^\infty(\tau : G/H)}$

Hence $T_G = \mathcal{I}_G \circ \mathcal{F}_G$ ☒.

Thank you.

References see also My personal webpage, Publications, for full texts.

[1] E. vd Ban. The Plancherel formula for a reductive symmetric space pp 1–97 in: Lie theory, Harmonic analysis on Symmetric Spaces — General Plancherel Theorems, PM 230 Birkhäuser

[2] E.P. vd Ban & H. Schlichtkrull. The Plancherel decomposition for a reductive symmetric space I, II. Inv. Math. 161 (3), 453–566, 615–712, 2005