C^*-algebras and tempered representations

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CIRM-IHP research school:
Methods in representation theory and operator algebras

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Our goal in this course

G : real or p-adic reductive group (eg $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{Q}_p)$)

$C_r^*(G)$: reduced group C^*-algebra

Goal: Compute $C_r^*(G)$ in some useful way.

Theorem [A. Wassermann]: If G is a real reductive group, then

$$C_r^*(G) \underset{\text{Morita}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

[For p-adic groups it's a little more complicated. . .]

Why do we want to do this? [one answer] To understand connections between representation theory and operator K-theory.

Some resources

For this computation:

- Penington-Plymen [JFA 1983]
- A. Wassermann [Comptes Rendus, 1987]
- Plymen [JFA, 1990]
- Leung-Plymen [Compositio, 1991]
- Clare-Crisp-Higson [Compositio, 2016]
- Afgoustidis-Aubert [IMRN, 2022]
- Clare-Higson-Song-Tang [Jpn. J. Math, 2024]
- Clare-Crisp [coming soon]

A different approach:

- Bradd-Higson-Yuncken [arXiv:2412.18924]

C^*-algebras: We will use mostly 'classical' theory [Gelfand-Naimark, Segal, Kadison, Kaplansky, ...]. See, eg,

- Dixmier [C^*-algèbres/algebras]
- Blackadar [Operator algebras]
- Rieffel [Kingston proceedings, 1982]

Some resources

We will use lots of deep results from representation theory [Harish-Chandra, Langlands, Knapp-Stein, Arthur, ...].

Read more about those results here:

Real groups:

- Knapp [Overview, 1986]
- Wallach [RRGs vols 1&2, 1988&1992]

p-adic groups:

- Silberger [Intro, 1979]
- Waldspurger [JIMJ, 2003]

Notes for lectures 1 and 2 (under construction)

tinyurl.com/ynprptnf

Plan for these four lectures

1: C^*-algebras, representations, and the Stone-Weierstrass theorem

2: Hilbert modules and Morita equivalence

3: C^*-algebras of real reductive groups, up to isomorphism

4: C^*-algebras of real and p-adic reductive groups, up to Morita equivalence. Coda: K-theory for p-adic groups.

Lecture 1: C^*-algebras, representations, and the Stone-Weierstrass theorem

C^*-algebras : Definitions

A C^*-algebra is an algebra A over $\mathbb{C}$, with

- a conjugate-linear involution $* : A \to A$ satisfying $(ab)^* = b^* a^*$, and
- a norm $\| \ \|$ in which A is complete; $\|ab\| \leq \|a\| \, \|b\|$; and $\|a^* a\| = \|a\|^2$.

A homomorphism of C^*-algebras $\varphi : A \to B$ is a linear map satisfying $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$.

A subalgebra of A is a norm-closed linear subspace closed under multiplication and $*$.

An ideal of A is a norm-closed, two-sided ideal.

C^*-algebras : Facts

- Homomorphisms of C^*-algebras are automatically contractive, and have closed range.
- Injective homomorphisms are automatically isometric.
- So we can often ignore the norm—but we can also use it (thanks to completeness) to solve problems by approximation.
- Ideals in C^*-algebras are automatically closed under $*$.
- The quotient of a C^*-algebra by an ideal is a C^*-algebra in the quotient norm, and all of the expected isomorphism theorems hold.
- C^*-algebras often don't have 1 . . . but in many ways they behave as if they did. (Eg, multiplication $A \times A \to A$ is surjective.)

C^*-algebras : Examples

X : locally compact Hausdorff topological space.

$$C_0(X) := \left\{ f : X \xrightarrow{\text{continuous}} \mathbb{C} \;\middle|\; f(x) \to 0 \text{ at } \infty \right\}$$

(ie, for every $\varepsilon > 0$ the set $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.)

$(f+g)(x) := f(x) + g(x), \quad (cf)(x) := c(f(x)), \quad f^*(x) := \overline{f(x)}$

$\|f\| = \sup_{x \in X} |f(x)|$.

These operations make $C_0(X)$ a commutative C^*-algebra.

Theorem: [Gelfand-Naimark] Every commutative C^*-algebra is isomorphic to one of this form.

C^*-algebras : Examples

H : Hilbert space (Notation: inner product $\langle \eta \,|\, \xi \rangle$ is linear in ξ)

$\mathrm{B}(H)$: algebra of bounded linear operators on H ($st = s \circ t$)

For $t \in \mathrm{B}(H)$, t^* is defined by $\langle \eta \,|\, t^* \xi \rangle = \langle t\eta \,|\, \xi \rangle$

$\|t\| = \sup_{\|\xi\|=1} \|t\xi\|$

These operations make $\mathrm{B}(H)$ into a C^*-algebra.

Theorem: [Gelfand-Naimark] Every C^*-algebra is isomorphic to a subalgebra of some $\mathrm{B}(H)$.

C^*-algebras : Examples

H : Hilbert space

$\mathsf{K}(H) \subseteq \mathsf{B}(H)$: the ideal of compact operators

$$\mathsf{K}(H) = \overline{\mathsf{span}}\Big\{ |\xi\rangle\langle\eta| : \zeta \mapsto \xi\langle\eta \,|\, \zeta\rangle \;\Big|\; \eta, \xi \in H \Big\}.$$

X : locally compact Hausdorff space

$$C_0(X, \mathsf{K}(H)) := \Big\{ f : X \xrightarrow{\text{continuous}} \mathsf{K}(H) \;\Big|\; \|f(x)\| \to 0 \text{ at } \infty \Big\}$$

This is a C^*-algebra under pointwise operations.

C^*-algebras : Examples

A : a C^*-algebra

W : finite group acting on A by automorphisms:

$$\beta_w(ab) = \beta_w(a)\beta_w(b), \quad \beta_w(a^*) = \beta(a)^*, \quad \beta_{w_1 w_2} = \beta_{w_1} \circ \beta_{w_1}$$

Two new C^*-algebras:

Fixed-point algebra: $A^W := \{a \in A \mid \beta_w(a) = a \text{ for all } w \in W\}$

Crossed product: $A \rtimes W := \left\{ \sum_{w \in W} a_w w \;\middle|\; a_w \in A \right\}$

$$wa = \beta_w(a)w \qquad \text{and} \qquad w^* = w^{-1}.$$

(Theorem: a C^*-algebra norm exists.)

C^*-algebras : 2^{nd}-most important example (for us, this week)

X : locally compact Hausdorff space

H : Hilbert space

W : finite group acting on X by homeomorphisms

$\{I_{w,x} \in \mathrm{U}(H) \mid w \in W,\ x \in X\}$: unitary operators satisfying

- $I_{w_1,w_2x} I_{w_2,x} = I_{w_1w_2,x}$ (in particular, $I_{1,x} = \mathrm{id}_H$)
- For each $w \in W$ the map $x \mapsto I_{w,x}$ is continuous in the strong operator topology (ie $x \mapsto I_{w,x}\xi$ cts for each $\xi \in H$.)

Let W act on $C_0(X, \mathrm{K}(H))$ by

$$\beta_w(f)(x) := I_{w,w^{-1}x} f(w^{-1}x) I_{w^{-1},x}.$$

The fixed-point algebra $C_0(X, \mathrm{K}(H))^W$ will be the second-most important example of a C^*-algebra in these lectures.

Examples of $C_0(X, \mathrm{K}(H))^W$

Example 1: $W = \{1, w\}$ acting on $X = \mathbb{R}$ by $wx = -x$.

$H = \mathbb{C}^2$, so $\mathrm{K}(H) = \mathrm{M}_2$ (2×2 matrices).

$I_{w,x} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$

Exercise: $C_0(\mathbb{R}, \mathrm{M}_2)^W \cong C_0([0, \infty), \mathrm{M}_2)$.

Example 2: Same W, X, and H; but now $I_{w,x} = \begin{bmatrix} e^{ix} & 0 \\ 0 & -e^{ix} \end{bmatrix}$.

Exercise: $C_0(\mathbb{R}, \mathrm{M}_2)^W \cong \{f \in C_0([0, \infty), \mathrm{M}_2) \mid f(0) \text{ is diagonal}\}$.

C^*-algebras : Most important example (for us, this week)

G : locally compact group $\qquad$ dg : left Haar measure on G

$C_c(G)$: compactly supported continuous functions $G \to \mathbb{C}$

$$\lambda : C_c(G) \to \mathrm{B}(L^2G) \qquad (\lambda(f)\xi)(g) := \int_G f(h)\xi(h^{-1}g)\, dh$$

Reduced group C^*-algebra: $C^*_r(G) := \overline{\lambda(C_c(G))}^{\|\ \|_{\text{operator}}}$

Note: $\|\lambda(f)\|_{\text{op}} \leq \|f\|_{L^1}$, so L^1-approximation works in $C^*_r(G)$.

Examples: G finite : $C^*_r(G) = \mathbb{C}[G]$, the group algebra; $g^* = g^{-1}$.

G abelian : $C^*_r(G) \cong C_0(\widehat{G})$ via Fourier transform

G compact : $C^*_r(G) \cong \bigoplus_{\rho \in \widehat{G}} \mathrm{K}(H_\rho)$ $\quad$ [Peter-Weyl]

[$\widehat{G}$ = equivalence classes of irreducible unitary representations; see below]

Representations : Definitions

A representation of a C^*-algebra A is a homomorphism of C^*-algebras $\pi : A \to \mathrm{B}(H)$, for some Hilbert space H

An invariant subspace for π is a closed subspace $V \subseteq H$ with $\pi(A)V \subseteq V$.

π is irreducible if $H \neq 0$ and if 0 and H are the only invariant subspaces

Irreducible representations π_1, π_2 are equivalent if there is a unitary $u : H_1 \to H_2$ with $u\pi_1(a) = \pi_2(a)u$.

The spectrum $\widehat{A}$ of A is the set of equivalence classes of irreducible representations ('irreps') of A.

The Jacobson topology on $\widehat{A}$ has one open set

$$O_J = \{\pi \in A \mid \pi(J) \neq 0\}$$

for each ideal $J \subseteq A$.

Gelfand-Naimark-Segal (GNS) construction

A state on A is a bounded linear map $\varphi : A \to \mathbb{C}$ with $\varphi(a^* a) \geq 0$ for all $a \in A$, and $\|\varphi\| = 1$.

GNS : given a state φ, build a representation $\pi : A \to \mathrm{B}(H_\varphi)$:

- $J_\varphi := \{a \in A \mid \varphi(a^* a) = 0\}$
- $H_\varphi := \overline{A/J_\varphi}$, completion in the norm $\|a + J_\varphi\| = \varphi(a^* a)^{1/2}$
- H_φ is a Hilbert space: $\langle a + J_\varphi \mid b + J_\varphi \rangle := \varphi(a^* b)$
- $\pi_\varphi(a) \in \mathrm{B}(H_\varphi)$: $b + J_\varphi \mapsto (ab) + J_\varphi$.

π_φ is irreducible if and only if φ is a pure state (ie, not a convex combination of other states)

Consequences of the GNS construction

Theorem: Let A be a C^*-algebra.

• Every irreducible representation of A is equivalent to a GNS representation.

• If $a \neq b$ in A then $\pi(a) \neq \pi(b)$ for some $\pi \in \widehat{A}$.

• If $B \subseteq A$ is a subalgebra, and $\pi : B \to \mathrm{B}(H)$ is an irreducible representation, then there is an irreducible representation $\pi' : A \to \mathrm{B}(H')$, where $H \subseteq H'$, such that $\pi(b)\xi = \pi'(b)\xi$ for all $b \in B$ and $\xi \in H$.

Proof: Use the Hahn-Banach (states exist) and Krein-Milman (states exist $\Longrightarrow$ pure states exist) theorems. □

Characterisations of irreducibility

Theorem: Let $\pi : A \to \mathrm{B}(H)$ be a representation. Each of the following conditions is equivalent to π being irreducible:

- [GNS] $\pi \cong \pi_\varphi$ for a pure state φ
- [Schur's lemma] $\pi(A)' = \mathbb{C}\,\mathrm{id}_H$, where

$$\pi(A)' := \{t \in \mathrm{B}(H) \mid \pi(a)t = t\pi(a) \text{ for all } a \in A\}$$

- [von Neumann] $\pi(A)$ is dense in $\mathrm{B}(H)$ in the SOT
- [Kadison] H has no A-invariant subspaces, closed or not

Irreducible representations of $C_r^*(G)$

G: locally compact group

- unitary representation: homomorphism $\pi : G \to \mathrm{U}(H)$ (unitary operators), continuous in the SOT
- π is irreducible if H has no proper, nonzero, closed, G-invariant subspaces.
- π extends to a map $C_c(G) \to \mathrm{B}(H)$:

$$\langle \eta \,|\, \pi(f)\xi \rangle = \int_G \langle \eta \,|\, f(g)\pi(g)\xi \rangle \, dg$$

- π extends to $C_r^*(G)$ iff $\|\pi(f)\| \leq \|\lambda(f)\|_{\text{operator}}$ (recall: $\lambda(f) \in \mathrm{B}(L^2(G))$ is convolution with f.)
- $\widehat{G}$: equivalence classes of unitary irreducibles $\widehat{G}_r \subseteq \widehat{G}$: those that extend to $C_r^*(G)$
- Theorem: $\widehat{G}_r \cong \widehat{C_r^*(G)}$.

Examples of $\widehat{G}_r$

G abelian: $\widehat{G}_r = \widehat{G}$ (use the Fourier transform on $L^2(G)$)

G compact: $\widehat{G}_r = \widehat{G}$ (every irrep is a subrep of $L^2(G)$)

Most real/p-adic reductive groups: $\widehat{G}_r \neq \widehat{G}$

Theorem: [Harish-Chandra, Cowling-Haagerup-Howe] Let G be a real or p-adic reductive group. An irreducible unitary representation π lies in $\widehat{G}_r$ if and only if π is tempered—i.e., iff its K-finite matrix coefficients are of class $L^{2+\varepsilon}$ modulo the centre.
[K is a 'good' maximal compact subgroup]

Strategy for computing $C_r^*(G)$: match up tempered representations with representations of simpler C^*-algebras.

Irreducible representations of $C_0(X)$

Schur's lemma $\Longrightarrow$ every irrep of $C_0(X)$ is one-dimensional: $\pi : C_0(X) \to \mathbb{C}$

Riesz rep theorem $\Longrightarrow \pi(f) = \int_X f\, d\mu$ for some measure μ.

$\pi(f_1 f_2) = \pi(f_1)\pi(f_2) \Longrightarrow \mu$ is concentrated at a single $x \in X$.

So:

Theorem: $\widehat{C_0(X)} \cong X$, via the map sending $x \in X$ to the irreducible representation $\mathrm{ev}_x : C_0(X) \to \mathbb{C}$, $f \mapsto f(x)$. $\square$

Irreducible representations of $\mathrm{K}(H)$

Theorem: Every bounded linear map $\mathrm{K}(H) \to \mathbb{C}$ has the form $k \mapsto \mathrm{trace}(tk)$ for some trace-class operator t.

Corollary: the pure states on $\mathrm{K}(H)$ are precisely the maps $\varphi_\xi : k \mapsto \langle \xi \mid k\xi \rangle$ for unit vectors $\xi \in H$.

The map $k + J_{\varphi_\xi} \mapsto k\xi$ gives an isomorphism $H_{\varphi_\xi} \cong H$.

So:

Theorem: Every irreducible representation of $\mathrm{K}(H)$ is equivalent to the identity representation $\mathrm{K}(H) \hookrightarrow \mathrm{B}(H)$. $\square$

Irreducible representations of $C_0(X, \mathsf{K}(H))$

$C_0(X, \mathsf{K}(H))$ is a $C_0(X)$-module

Every irrep $\pi : C_0(X, \mathsf{K}(H)) \to \mathsf{B}(H)$ extends to a rep of $C_0(X)$:

$$\pi(f)\pi(k)\xi := \pi(fk)\xi \qquad (f \in C_0(X),\ k \in C_0(X, \mathsf{K}(H)),\ \xi \in H).$$

Schur $\implies$ $C_0(X)$ acts as scalars $\implies$ $\pi\big|_{C_0(X)} = \mathrm{ev}_x$ for some $x \in X$

π is a $C_0(X)$-module map $\implies$ π factors through
$\mathrm{ev}_x : C_0(X, \mathsf{K}(H)) \to \mathsf{K}(H)$

$\mathsf{K}(H)$ has only one irreducible representation. So:

Theorem: $\widehat{C_0(X, \mathsf{K}(H))} \cong X$, via $x \mapsto \mathrm{ev}_x$. □

Irreducible representations of $C_0(X, \mathrm{K}(H))^W$

Consider X, H, W, $\{I_{w,x} \in \mathrm{U}(H) \mid w \in W,\ x \in X\}$ as before.

Every irrep of $C_0(X, \mathrm{K}(H))^W$ is the restriction of an irrep of $C_0(X, \mathrm{K}(H))$ to a $C_0(X, \mathrm{K}(H))^W$-invariant subspace.

So every irrep factors through $\mathrm{ev}_x : C_0(X, \mathrm{K}(H))^W \to \mathrm{K}(H)$ for some $x \in X$.

Note that $w \mapsto I_{w,x}$ is a unitary rep of $W_x := \{w \in W \mid wx = x\}$.

$$\begin{aligned}\mathrm{ev}_x(C_0(X, \mathrm{K}(H))^W) &= \mathrm{K}(H)^{W_x} \\ &:= \{k \in \mathrm{K}(H) \mid kI_{w,x} = I_{w,x}k \text{ for all } w \in W_x\}\end{aligned}$$

So we need to know $\widehat{\mathrm{K}(H)^{W_x}}$.

Harmonic analysis for the finite group W_x

For each $\rho \in \widehat{W_x}$ set

$\mathrm{HS}(\rho, I_x)^{W_x} := \{ t : H_\rho \to H \mid \text{linear},\ t\rho(w) = I_{w,x} t \text{ for all } w \in W_x \}.$

This is a Hilbert space: $\langle t \mid s \rangle = \mathrm{trace}(t^* s)$.

Theorem: the maps $\xi \otimes t \mapsto (\dim H_\rho)^{\frac{1}{2}} t(\xi)$ give an isomorphism

$$\bigoplus_{\rho \in \widehat{W_x}} H_\rho \otimes \mathrm{HS}(\rho, I_x)^{W_x} \xrightarrow{\cong} H.$$

This isomorphism identifies $I_{w,x}$ with $\bigoplus_\rho \rho(w) \otimes \mathrm{id}$ for each $w \in W_x$; and $\mathrm{K}(H)^{W_x}$ with $\bigoplus_\rho \mathbb{C}\,\mathrm{id} \otimes \mathrm{K}\left(\mathrm{HS}(\rho, I_x)^{W_x}\right)$.

Corollary: The irreps of $\mathrm{K}(H)^{W_x}$ are the maps

$$\mathrm{K}(H)^{W_x} \to \mathrm{K}\left(\mathrm{HS}(\rho, I_x)^{W_x}\right), \qquad k \mapsto (t \mapsto k \circ t)$$

where $\rho \in \widehat{W_x}$, $\mathrm{HS}(\rho, I_x)^{W_x} \neq 0$. [Notation: $\rho \subseteq I_x$.]

Irreducible representations of $C_0(X, \mathsf{K}(H))^W$

Recall: every irrep of $C_0(X, \mathsf{K}(H))^W$ factors through

$$\mathrm{ev}_x : C_0(X, \mathsf{K}(H))^W \to \mathsf{K}(H)^{W_x}.$$

Combining this with what we now know about $\widehat{\mathsf{K}(H)^{W_x}}$:

Theorem: • The irreducible representations of $C_0(X, \mathsf{K}(H))^W$ are

$$\pi_{x,\rho} : C_0(X, \mathsf{K}(H))^W \xrightarrow{\mathrm{ev}_x} \mathsf{K}(H)^{W_x} \xrightarrow{k \mapsto k \circ \_} \mathsf{K}\left(\mathrm{HS}(\rho, I_x)^{W_x}\right)$$

where $x \in X$ and $\rho \in \widehat{W_x}$ with $\rho \subseteq I_x$.

• $\pi_{x,\rho} \cong \pi_{x',\rho'}$ iff $x' = wx$ and $\rho' = \rho(w^{-1} \_ w)$ for some $w \in W$. □

Examples of $C_0(\widehat{X, \mathrm{K}(H)})^W$

Example 1: $W = \{1, w\}$ acting on $X = \mathbb{R}$ by $wx = -x$.

$$H = \mathbb{C}^2, \qquad I_{w,x} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$$

Exercise: $C_0(\widehat{\mathbb{R}, \mathrm{M}_2})^W \cong [0, \infty)$

Example 2: Same W, X, and H; but now $I_{w,x} = \begin{bmatrix} e^{ix} & 0 \\ 0 & -e^{ix} \end{bmatrix}$.

Exercise: $C_0(\widehat{\mathbb{R}, \mathrm{M}_2})^W \xrightarrow{\cong} \{0_+, 0_-\} \sqcup (0, \infty)$ (non-Hausdorff)

CCR/Liminal C^*-algebras

A C^*-algebra A is CCR, aka liminal, if $\pi(A) \subseteq \mathrm{K}(H_\pi)$ for every $\pi \in \widehat{A}$.

[We will soon see that $\pi(A) \subseteq \mathrm{K}(H_\pi)$ implies $\pi(A) = \mathrm{K}(H_\pi)$.]

Examples: $C_0(X)$, $\mathrm{K}(H)$, $C_0(X, \mathrm{K}(H))$, $C_0(X, \mathrm{K}(H))^W$.

Ideals, quotients, and subalgebras of CCR algebras are CCR.

Theorem: [Harish-Chandra, Bernstein] If G is a real or p-adic reductive group, then $C_r^*(G)$ is CCR.

[We will see why later on.]

Kaplansky on CCR/liminal C^*-algebras

6. CCR-algebras

I shall briefly describe how CCR-algebras started, and where they stand today. The material of §5 more or less covers the case where, for every primitive ideal P in the C^*-algebra A, A/P is finite-dimensional. At just about the time this work was completed, the important papers of Gelfand and Naimark on representations of semi-simple Lie groups were beginning to appear. One aspect of their results was the following: for the relevant C^*-algebras every A/P was the algebra of completely continuous operators on a Hilbert space. This encouraged me to see whether some progress was possible on this class of C^*-algebras. The unimaginative name CCR stands for "completely continuous representations". I stand ready to yield to Dixmier's "liminaire", except that I confess that I do not know what it means. At any rate, it turned out that the fibre bundle type of result survived, when appropriately formulated.

[CBMS lecture notes, 1970]

Pedersen on CCR/liminary C^*-algebras

[141]. The original name CCR means 'completely continuous representations' (completely continuous operators being another name for $\boldsymbol{C}(H)$). The next layer in the hierarchy, 'GCR' indicates a generalization of the CCR condition. The modern names 'liminary' and 'postliminary' do not mean anything, which may be more aesthetic. In any case the Anglo-Saxon Habit of

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Condensing Every Formula into its Leading Characters (abbreviated ASHCEFLC) should not be tolerated in mathematics. Theorem 6.2.3 was

[C^*ATAG, 1979]

Kaplansky's Stone-Weierstrass theorem

A subalgebra $B \subseteq A$ is called separating [Dixmier: riche] if:

- irreducible representations of A remain irreducible; and
- inequivalent representations of A remain inequivalent

when restricted to B.

We say that A has the Stone-Weierstrass property (SWP) if
B separating $\implies B = A$.

Theorem: [Kaplansky] Every CCR algebra has the SWP.

Importance for us: a tool for computing the range of a Fourier transform.

Stone-Weierstrass for $C_0(X)$

Recall: $\widehat{C_0(X)} = \{\mathrm{ev}_x \mid x \in X\}$

The one-dimensional rep ev_x is irreducible on B iff it is nonzero; and $\mathrm{ev}_x \cong \mathrm{ev}_y$ on B iff $\mathrm{ev}_x = \mathrm{ev}_y$ on B.

So $B \subseteq C_0(X)$ is separating iff

- for each $x \in X$ there is some $b \in B$ with $b(x) \neq 0$; and
- for all $x \neq y \in X$ there is some $b \in B$ with $b(x) \neq b(y)$.

The usual Stone-Weierstrass theorem thus says that $C_0(X)$ has the SWP.

Stone-Weierstrass for $\mathrm{K}(H)$

Let B be a subalgebra of $\mathrm{K}(H)$.

$\widehat{\mathrm{K}(H)} = \{\mathrm{id}\}$; so B is separating iff $B \hookrightarrow \mathrm{B}(H)$ is irreducible.

Recall:

- $B \hookrightarrow \mathrm{B}(H)$ is irreducible iff $\overline{B}^{\mathrm{SOT}} = \mathrm{B}(H)$.
- Every continuous linear map $\mathrm{K}(H) \to \mathbb{C}$ has the form $k \mapsto \mathrm{trace}(tk)$. Exercise: every such map is SOT-continuous.

So: if $B \subseteq \mathrm{K}(H)$ is separating, then every continuous linear map $\mathrm{K}(H) \to \mathbb{C}$ that vanishes on B vanishes on $\mathrm{K}(H)$.

Hahn-Banach $\implies B = \mathrm{K}(H)$.

So $\mathrm{K}(H)$ has the SWP.

Stone-Weierstrass for $C_0(X, \mathsf{K}(H))$

Let $B \subseteq C_0(X, \mathsf{K}(H))$ be separating

The map $x \mapsto \mathrm{ev}_x \big|_B$ is a homeomorphism $X \cong \widehat{B}$.

Key ingredient [Dauns-Hoffman]: B is a $C_b(X)$-submodule of $C_0(X, \mathsf{K}(H))$. (C_b: bounded continuous functions)

Then a partition-of-unity argument gives $B = C_0(X, \mathsf{K}(H))$.

Stone-Weierstrass for $C_0(X, \mathrm{K}(H))^W$: an exercise

- Suppose J is an ideal in A, such that J and A/J have the SWP. Prove that A has the SWP.

- Suppose $A = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n = 0$, where each J_i is an ideal in J_{i-1} and each J_{i-1}/J_i has the SWP. Prove that A has the SWP.

- Let $X = \mathbb{R}^2$, acted on by the finite group

$$W = \langle s, t\rangle \quad \text{where} \quad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(symmetries of the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$)

Let $H = \mathbb{C}^2$, $I_{w,x} := w \in \mathrm{U}(\mathbb{C}^2)$, and $A = C_0(\mathbb{R}^2, \mathrm{K}(\mathbb{C}^2))^W$.

Prove that A has the SWP by finding a nice composition series.

[Hint: take $J_1 = \{f \in A \mid f(0) = 0\}\ldots$]

Stone-Weierstrass for CCR algebras

• If A has a composition series whose quotients all have the SWP, then A has the SWP.

• Every CCR algebra has a composition series whose quotients are CCR algebras with Hausdorff spectrum.

• If A is CCR with $\widehat{A}$ Hausdorff then A is isomorphic to an algebra of compact-operator-valued functions on $\widehat{A}$.

• The proof of SWP for $A = C_0(X, \mathrm{K}(H))$ applies to all CCR A with Hausdorff $\widehat{A}$. □

Looking ahead

- For G a real reductive group: there is a 'Fourier transform'

$$C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$$

for certain Xs, Hs, and Ws.

- Stone-Weierstrass + knowledge of $C_0(\widehat{X, \mathrm{K}(H)})^W$ are used to prove surjectivity.

- For purposes of understanding K-theory and representation theory, the $C_0(X, \mathrm{K}(H))^W$s can be replaced by simpler, Morita equivalent C^*-algebras.

- A similar (but slightly more complicated) picture applies to p-adic reductive groups.

Lecture 2: Morita equivalence

Goal: compute $C_r^*(G)$, the reduced C^*-algebra of a real or reductive p-adic group G.

Why? [one answer] Understand connections between representation theory and operator K-theory.

Plan: replace $C_r^*(G)$ by a simpler C^*-algebra that is Morita equivalent to $C_r^*(G)$.

This is reasonable, because Morita equivalent C^*-algebras have the same K-theory and representations.

This lecture: Morita equivalence for C^*-algebras in general, and for $C_0(X, \mathrm{K}(H))^W$ in particular.

Hilbert modules : Definition

A, B : C^*-algebras

A left Hilbert A-module is:

- a left A-module E, with
- an A-valued inner product $[\xi \mid \eta]$, linear in ξ
- $[a_1\xi \mid a_2\eta] = a_1[\xi \mid \eta]a_2^*$
- $[\xi \mid \eta]^* = [\eta \mid \xi]$
- $[\xi \mid \xi] \geq 0$ in A
  (ie $[\xi \mid \xi] = a^*a$ for some a)
- $\|\xi\| := \|[\xi \mid \xi]\|_A^{1/2}$
  is a complete norm on E

A right Hilbert B-module is:

- a right B-module E, with
- a B-valued inner product $\langle \xi \mid \eta \rangle$, linear in η
- $\langle \xi b_1 \mid \eta b_2 \rangle = b_1^* \langle \xi \mid \eta \rangle b_2$
- $\langle \xi \mid \eta \rangle^* = \langle \eta \mid \xi \rangle$
- $\langle \xi \mid \xi \rangle \geq 0$ in B
  (ie $\langle \xi \mid \xi \rangle = b^*b$ for some b)
- $\|\xi\| := \|\langle \xi \mid \xi \rangle\|_B^{1/2}$
  is a complete norm on E

Hilbert modules : Examples

H : Hilbert space

H is:

- a right Hilbert $\mathbb{C}$-module:

$$\langle \xi \mid \eta \rangle := \langle \xi \mid \eta \rangle$$

- a left Hilbert $\mathrm{B}(H)$-module

$$[\xi \mid \eta] := |\xi\rangle\langle\eta| : \zeta \mapsto \xi\langle \eta \mid \zeta \rangle$$

Fullness: If E is left Hilbert A-module, the set

$$\overline{\mathrm{span}}\{[\xi \mid \eta] \in A \mid \xi, \eta \in E\}$$

is a closed ideal in A. We say that E is full if this ideal is all of A. Similarly for right modules.

Example: H is full over $\mathbb{C}$, but not over $\mathrm{B}(H)$. H is a full left Hilbert $\underline{\mathrm{K}(H)}$-module.

Hilbert modules : Examples

X : locally compact Hausdorff space, H : Hilbert space

$C_0(X, H)$ is:

- A full right Hilbert $C_0(X)$-module:

$$\langle \xi \mid \eta \rangle(x) := \langle \xi(x) \mid \eta(x) \rangle$$

- A full left Hilbert $C_0(X, \mathsf{K}(H))$-module:

$$[\xi \mid \eta](x) := |\xi(x)\rangle\langle\eta(x)|$$

Proof of fullness: if J is an ideal in A with $\pi\big|_J \neq 0$ for all $\pi \in \widehat{A}$, then $J = A$ [because if $J \neq A$ then A/J has an irrep]

Hilbert modules : Examples

A : C^*-algebra, E : left Hilbert A-module

W : finite group acting on A, A^W : fixed points

E is a left Hilbert A^W-module:

$$^{W}[\xi \mid \eta] := \frac{1}{|W|} \sum_{w \in W} \beta_w \left([\xi \mid \eta] \right)$$

If E is full over A, then it is also full over A^W.

Proof that $^{W}[\xi \mid \xi] \geq 0$ and gives a complete norm on E: if $a, b \geq 0$ then $a + b \geq 0$ and $\|a + b\| \geq \|a\|$.

(These facts about positivity are not meant to be obvious!)

Hilbert modules : Examples

W : finite group, $\pi : W \to \mathrm{U}(H)$: unitary rep

$C_r^*(W) = \mathbb{C} \rtimes W = \{\sum_w c_w w \mid c_w \in \mathbb{C}\}$: group (C^*)-algebra

H is a left Hilbert $\mathrm{K}(H)^W$-module:

$$^W[\xi \,|\, \eta] = \frac{1}{|W|} \sum_w \pi(w)[\xi \,|\, \eta]\pi(w^{-1}) = \frac{1}{|W|} \sum_w |\pi(w)\xi\rangle\langle\pi(w)\eta|$$

H is a right Hilbert $C_r^*(W)$-module:

$$\xi \cdot w := \pi(w^{-1})\xi, \qquad \langle \xi \,|\, \eta\rangle_W := \frac{1}{|W|} \sum_{w \in W} \langle \xi \,|\, \pi(w)\eta\rangle w$$

Proof that $\langle \xi \,|\, \xi\rangle_W \geq 0$ and gives a complete norm: this follows from the relation

$$^W[\xi \,|\, \eta]\zeta = \xi\langle \eta \,|\, \zeta\rangle_W.$$

See below.

Aside: induced representations

${}_A E_B^{\langle\,|\,\rangle}$: A-B bimod; right Hilbert B-mod; $\langle a\xi \mid \eta\rangle = \langle \xi \mid a^*\eta\rangle$

$\pi : B \to \mathrm{B}(V)$ a Hilbert-space representation

$\rightsquigarrow$ $E \otimes_B V$ is a Hilbert-space representation of A:

$$\langle \xi_E \otimes \xi_V \mid \eta_E \otimes \eta_V\rangle \coloneqq \left\langle \xi_V \,\middle|\, \pi\left(\langle \xi_E \mid \eta_E\rangle\right) \eta_V \right\rangle$$

Example: [Rieffel] $H \subseteq G$ closed subgroup [unimodular, for simplicity]

$C_c(G)$ is a $C_c(G)$-$C_c(H)$ bimodule; $C_c(H)$-valued inner product

$$\langle \xi \mid \eta\rangle(h) \coloneqq \int_G \overline{\xi}(g)\eta(gh)\, dg$$

Complete to get ${}_{C^*(G)}E_{C^*(H)}^{\langle\,|\,\rangle}$

$E \otimes_{C^*(H)}$: URep$(H) \to$ URep(G): unitary induction

Morita equivalence : Definition

A, B : C^*-algebras

An A-B-bimodule E is a Morita equivalence if:

- E is a left Hilbert A-module (inner product $[\ |\ ]$) and a right Hilbert B-module (inner product $\langle\ |\ \rangle$)
- $\overline{\text{span}}\{[\xi \,|\, \eta]\} = A$ and $\overline{\text{span}}\{\langle \xi \,|\, \eta \rangle\} = B$
- $[\xi b \,|\, \eta] = [\xi \,|\, \eta b^*]$ and $\langle a\xi \,|\, \eta \rangle = \langle \xi \,|\, a^* \eta \rangle$
- $[\xi \,|\, \eta]\zeta = \xi \langle \eta \,|\, \zeta \rangle$

for all $\xi, \eta, \zeta \in E$, $a \in A$, $b \in B$.

(There is some redundancy in this definition; see later.)

If such an E exists then A and B are Morita equivalent ($A \underset{\text{M}}{\sim} B$).

This relation is sometimes called 'strong' Morita equivalence.

Morita equivalence : Properties

- $\underset{\mathrm{M}}{\sim}$ is an equivalence relation [Rieffel]
- $A \underset{\mathrm{M}}{\sim} B \Longrightarrow \widehat{A} \cong \widehat{B}$ and $K_*(A) \cong K_*(B)$
- $A \underset{\mathrm{M}}{\sim} B \Longleftrightarrow A \otimes \mathrm{K}(H) \cong B \otimes \mathrm{K}(H)$ assuming that A and B have countable approximate identities, as do all C^*-algebras of interest in this course [Brown-Green-Rieffel]
- $A \underset{\mathrm{M}}{\sim} B \Longleftrightarrow A$ and B have equivalent categories of operator modules [Blecher]
- If A and B have 1, then $A \underset{\mathrm{M}}{\sim} B \Longleftrightarrow A$ and B have equivalent categories of (algebraic) modules [Beer]
- Equivalence of categories of Hilbert-space representations does not imply (strong) Morita equivalence.

Aside: Mackey's imprimitivity theorem, selon Rieffel

$H \subseteq G$: closed subgroup $\quad {}_{C^*(G)}E^{\langle\ |\ \rangle}_{C^*(H)}$: induction bimodule

Theorem: [Rieffel] E can be made into a Morita equivalence between $C^*(H)$ and $C_0(G/H) \rtimes G$.

Corollary: [Mackey] Unitary induction gives an equivalence between the category of unitary representations of H, and the category of unitary representations G admitting a compatible rep of $C_0(G/H)$.

Example: $A \rtimes K$: abelian $\rtimes$ compact. $\pi : A \rtimes K \to \mathrm{U}(H)$ irrep

Fourier: $C_0(\widehat{A})$ acts on H; $C_0(\widehat{A}/K)$ acts by intertwiners

Schur: $C_0(\widehat{A}/K)$ acts by by $\mathrm{ev}_{K\varphi}$ for some orbit $K\varphi$.

So π is an irrep of $C(K\varphi) \rtimes K \cong C(K/K_\varphi) \rtimes K$.

Imprimitivity: π is induced from an irrep of K_φ; and conversely, irreps of K_φ induce to irreps of $A \rtimes K$. $\square$

Morita equivalence : Examples

$\mathrm{K}(H) \underset{\mathrm{M}}{\sim} \mathbb{C}$: on H consider the inner products

$$[\xi \mid \eta] := |\xi\rangle\langle\eta| \quad \text{and} \quad \langle \xi \mid \eta \rangle := \langle \xi \mid \eta \rangle.$$

We have

$$[\xi \mid \eta]\zeta = |\xi\rangle\langle\eta|\zeta = \xi\langle\eta \mid \zeta\rangle.$$

(In a Morita equivalence we always have $[\xi \mid \eta] = |\xi\rangle\langle\eta|$.)

$C_0(X, \mathrm{K}(H)) \underset{\mathrm{M}}{\sim} C_0(X)$: on $C_0(X, H)$ consider

$$[\xi \mid \eta](x) := |\xi(x)\rangle\langle\eta(x)| \quad \text{and} \quad \langle \xi \mid \eta \rangle(x) := \langle \xi(x) \mid \eta(x) \rangle.$$

Morita equivalence : $\mathrm{K}(H)^W \underset{\mathrm{M}}{\sim}$?

W : finite group, $\pi : W \to \mathrm{U}(H)$ a unitary representation

$\mathrm{K}(H)^W = \{k \in \mathrm{K}(H) \mid k\pi(w) = \pi(w)k \text{ for all } w \in W\}$

We know:

- H is a full left Hilbert $\mathrm{K}(H)^W$-module under ${}^W[\ |\ ]$
- H is a right $C_r^*(W)$-module, with $C_r^*(W)$-valued inner product $\langle\ |\ \rangle_W$

Easily checked:

- $\langle \xi \,|\, \eta b\rangle_W = \langle \xi \,|\, \eta\rangle_W b$ and $\langle \xi \,|\, \eta\rangle_W^* = \langle \eta \,|\, \xi\rangle_W$
- $\langle a\xi \,|\, \eta\rangle_W = \langle \xi \,|\, a^*\eta\rangle_W$ and ${}^W[\xi b \,|\, \eta] = {}^W[\xi \,|\, \eta b^*]$ for $a \in \mathrm{K}(H)^W$ and $b \in C_r^*(W)$
- ${}^W[\xi \,|\, \eta]\zeta = \xi\langle \eta \,|\, \zeta\rangle_W$.

Theorem: this is enough to imply that H is a Morita equivalence between $\mathrm{K}(H)^W$ and the ideal $J := \overline{\mathrm{span}}\{\langle \xi \,|\, \eta\rangle_W\} \subseteq C_r^*(W)$.

Morita equivalence : $\mathsf{K}(H)^W \underset{\mathsf{M}}{\sim}$?

Theorem: H is a Morita equivalence between $\mathsf{K}(H)^W$ and the ideal $J := \overline{\operatorname{span}}\{\langle \xi \,|\, \eta \rangle_W\} \subseteq C_r^*(W)$.

Proof: Drop the Ws on $[\ |\ ]$ and $\langle\ |\ \rangle$. We need to prove:

(1) $\langle \xi \,|\, \xi \rangle \geq 0$ (2) $\xi \mapsto \|\langle \xi \,|\, \xi \rangle\|^{1/2}$ is a complete norm.

For $j \in J$ let $m_j : H \to H$, $m_j(\eta) = \eta j$.

If $m_j = 0$ then $j'j = 0$ for all $j' \in J$, so $\|j\|^2 = \|j^* j\| = 0$.

$\implies$ m is an injective homomorphism $J^{\mathrm{opp}} \to \mathcal{L}_{\mathsf{K}(H)^W}(H)$.

($\mathcal{L}_{\mathsf{K}(H)^W}(H)$: C^*-algebra of adjointable operators on H wrt $[\ |\ ]$)

So:

(1) $\langle \xi \,|\, \xi \rangle \geq 0$ in J iff $m_{\langle \xi \,|\, \xi \rangle} \geq 0$ in $\mathcal{L}_{\mathsf{K}(H)^W}(H)$

(2) $\|\langle \xi \,|\, \xi \rangle\|_J = \|m_{\langle \xi \,|\, \xi \rangle}\|_{\mathcal{L}_{\mathsf{K}(H)^W}}$

Morita equivalence : $\mathrm{K}(H)^W \underset{\mathrm{M}}{\sim}$?

Theorem: H is a Morita equivalence between $\mathrm{K}(H)^W$ and the ideal $J := \overline{\operatorname{span}}\{\langle \xi \,|\, \eta \rangle_W\} \subseteq C^*_r(W)$.

Proof: We need to prove:

(1) $m_{\langle \xi \,|\, \xi \rangle} \geq 0$ (2) $\xi \mapsto \| m_{\langle \xi \,|\, \xi \rangle} \|^{1/2}$ is a complete norm.

The relation $[\xi \,|\, \eta]\zeta = \xi \langle \eta \,|\, \zeta \rangle$ implies $m_{\langle \xi \,|\, \xi \rangle} = |\xi]^* |\xi]$, where

$$|\xi] : H \xrightarrow{\eta \mapsto [\eta \,|\, \xi]} \mathrm{K}(H)^W \quad \text{and} \quad |\xi]^* : \mathrm{K}(H)^W \xrightarrow{k \mapsto k\xi} H$$

So:

(1) $m_{\langle \xi \,|\, \xi \rangle} = |\xi]^* |\xi] \geq 0$

(2) $\| m_{\langle \xi \,|\, \xi \rangle} \|^{1/2} = \| |\xi]^* |\xi] \|^{1/2} = \| [\xi \,|\, \xi] \|^{1/2}$, a complete norm.

□

Morita equivalence : $\mathrm{K}(H)^W \underset{\mathrm{M}}{\sim}$?

Theorem: H is a Morita equivalence between $\mathrm{K}(H)^W$ and the ideal $J := \overline{\operatorname{span}}\{\langle \xi \,|\, \eta\rangle_W\} \subseteq C_r^*(W)$ ... and $J = \bigoplus_{\rho\in\widehat{W},\ \overline{\rho}\subseteq\pi} \mathrm{K}(H_\rho)$.

$(\overline{\rho} : W \to \mathrm{U}(\overline{H_\rho}),\ \ \overline{\rho}(w)\overline{\xi} := \overline{\rho(w)\xi},\ \ c\overline{\xi} := \overline{\overline{c}\xi},\ \ \langle \overline{\xi} \,|\, \overline{\eta}\rangle := \langle \eta \,|\, \xi\rangle)$

Proof: $C_r^*(W) \cong \bigoplus_{\rho\in\widehat{W}} \mathrm{K}(H_\rho)$.

J and $\bigoplus_{\overline{\rho}\subseteq\pi} \mathrm{K}(H_\rho)$ are ideals in $C_r^*(W)$

They are equal iff for all $\rho\in\widehat{W}$ we have $\rho(J)\neq 0 \iff \overline{\rho}\subseteq\pi$.

Take $\rho,\rho'\in\widehat{W}$, $t\in \mathrm{HS}(\rho,\pi)^W$ and $\xi,\eta\in H_\rho$. We have:

$$\rho'\big(\langle t\xi \,|\, t\eta)\rangle_W\big) = \frac{t^*t}{|W|}\sum_{w\in W}\langle \xi \,|\, \rho(w)\eta\rangle \rho'(w).$$

Schur orthogonality: this is 0 for all ξ,η iff $\rho'\not\cong\overline{\rho}$. □

The second-most-important Morita equivalence

X : locally compact Hausdorff space $\qquad H$: Hilbert space

W : finite group acting on X

$I_{w,x} \in \mathrm{U}(H)$: SOT-cts family of unitaries, $I_{w_2,w_1x} I_{w_1,x} = I_{w_1w_2,x}$

$C_0(X, \mathrm{K}(H))^W = \{f \in C_0(X, \mathrm{K}(H)) \mid f(x) = I_{w^{-1},wx} f(wx) I_{w,x}\}$

Example: $X = \bullet$: $C_0(X, \mathrm{K}(H))^W = \mathrm{K}(H)^W$.

Recall: H is a Morita equivalence between $\mathrm{K}(H)^W$ and an ideal in $C_r^*(W) = C_0(\bullet) \rtimes W$ ... and we can say which ideal.

A similar argument shows:

Theorem: $C_0(X, H)$ is a Morita equivalence between $C_0(X, \mathrm{K}(H))^W$ and an ideal in $C_0(X) \rtimes W$ ...

... and (under some additional assumptions) we can say which ideal.

Additional assumptions on $I_{w,x}$

$I_{w,x} \in \mathrm{U}(H)$: SOT-cts family of unitaries, $I_{w_1,w_2x} I_{w_2,x} = I_{w_1 w_2,x}$

$W_x := \{w \in W \mid wx = x\}$ and $W'_x = \{w \in W_x \mid I_{w,x} \in \mathbb{C}\,\mathrm{id}_H\}$

Normalisation: for all x and all $w \in W'_x$, $I_{w,x} = \mathrm{id}_H$

Completeness: for all x, the unitary representation $I_x : W_x \to \mathrm{U}(H)$ contains every $\rho \in \widehat{W_x/W'_x}$

Example: $X = \mathbb{R}$, $W = \{1, w\}$, $wx = -x$, $I_{w,x} = \begin{bmatrix} e^{ix} & 0 \\ 0 & -e^{ix} \end{bmatrix}$

$W'_0 = \{1\}$, $I_{w,0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ normalisation ✓ completeness ✓

Example: $X = \mathbb{R}^2$, $W = D_4$ (dihedral), $I_{w,x} = w \in \mathrm{U}(\mathbb{C}^2)$

$W'_0 = \{\pm 1\}$, $I_{-1,0} = -\mathrm{id}$, $|W_0/W'_0| = 4 > 2$

normalisation x completeness x

The second-most-important Morita equivalence

X, W, H, $I_{w,x}$ as above; assume normalisation and completeness.

Theorem: $C_0(X, H)$ can be made into a Morita equivalence between $C_0(X, \mathrm{K}(H))^W$ and the ideal

$$C(X, W, I) := \left\{ \sum_{w \in W} f_w w \in C_0(X) \rtimes W \;\middle|\; \begin{array}{c} \forall x \in X,\ \forall w' \in W'_x,\ \forall w \in W : \\ f_{w'w}(x) = f_w(x) \end{array} \right\}$$

in $C_0(X) \rtimes W$. □

Note that $C(X, W, I)$ depends on H and I only through the system of subgroups W'_x: ie, it depends only the answer to the question "which $I_{w,x}$s are scalars?"

$C(X, W, I)$ in terms of representations

$$C(X,W,I) :=$$
$$\left\{ \sum_{w \in W} f_w w \in C_0(X) \rtimes W \;\middle|\; \begin{matrix} \forall x \in X,\ \forall w' \in W'_x,\ \forall w \in W : \\ f_{w'w}(x) = f_w(x) \end{matrix} \right\}$$

For each $x \in X$:

$$\begin{matrix} C_0(X) \rtimes W & \twoheadrightarrow & C(Wx) \rtimes W & \underset{\mathrm{M}}{\sim} & C^*_r(W_x) & \cong & \bigoplus_{\rho \in \widehat{W_x}} \mathrm{K}(H_\rho) \\ \cup\!\!\mid & & & & & & \cup\!\!\mid \\ C(X,W,I) & & & & & & \bigoplus_{\rho \in \widehat{W_x/W'_x}} \mathrm{K}(H_\rho) \end{matrix}$$

$$C(X,W,I) = \bigcap_{x \in X} \left(\text{preimage of } \bigoplus_{\rho \in \widehat{W_x/W'_x}} \mathrm{K}(H_\rho) \right).$$

A special case

Corollary: Suppose that $W = W' \rtimes R$, where for each $x \in X$ we have $W'_x = W_x \cap W'$. Then

$$C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R.$$

A special case

Corollary: Suppose that $W = W' \rtimes R$, where for each $x \in X$ we have $W'_x = W_x \cap W'$. Then

$$C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R.$$

Proof:

$$\begin{array}{ccccc} C_0(X, \mathrm{K}(H))^W & \underset{\mathrm{M}}{\sim} & C(X, W, I) & \cong & C(X, W', I) \rtimes R \\ & & \cap\!| & & \cap\!| \\ & & C_0(X) \rtimes W & \cong & \big(C_0(X) \rtimes W'\big) \rtimes R \end{array}$$

For the action of W' on X, which $I_{w,x}$s are scalars? All of them! This is the same as for the operators $\mathrm{id}_{w,x} := \mathrm{id}_{\mathbb{C}}$; so

$$C(X, W', I) = C(X, W', \mathrm{id}) \underset{\mathrm{M}}{\sim} C_0(X, \mathrm{K}(\mathbb{C}))^{W'} \cong C_0(X/W').$$

The equivalence bimodule $C_0(X, \mathbb{C})$ is R-equivariant, so we get

$$C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C(X, W', I) \rtimes R \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R. \quad \square$$

Examples

Example 1: $W = \{1, w\}$ acting on $X = \mathbb{R}$ by $wx = -x$.

$$H = \mathbb{C}^2, \qquad I_{w,x} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$$

$$W_0 = W_0' = W$$

$$C(X, W, I) = \{f_1 1 + f_w w \in C_0(\mathbb{R}) \rtimes W \mid f_1(0) = f_w(0)\}$$

$$W = W \rtimes 1 \Longrightarrow C_0(\mathbb{R}, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(\mathbb{R}/W) \rtimes 1 \cong C_0([0, \infty))$$

Example 2: Same W, X, and H; but now $I_{w,x} = \begin{bmatrix} e^{ix} & 0 \\ 0 & -e^{ix} \end{bmatrix}$.

$$W_0 = W,\ W_0' = 1 \qquad C(X, W, I) = C_0(\mathbb{R}) \times W$$

$$W = 1 \rtimes W \quad \Longrightarrow \quad C_0(\mathbb{R}, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(\mathbb{R}) \rtimes W$$

Looking ahead

For G a real reductive group:

$$C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$$

where the normalisation, completeness, and $W = W' \rtimes R$ ($W'_x = W_x \cap W'$) conditions all hold; so

$$C_r^*(G) \underset{\mathrm{M}}{\sim} \bigoplus C_0(X/W') \rtimes R.$$

For G a p-adic reductive group:

$$C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$$

where:

- normalisation does not (?) always hold
- completeness (appropriately modified) does hold
- $W = W' \rtimes R$, $W'_x = W_x \cap W'$ does not always hold
- $I_{w_1, w_2 x} I_{w_2, x} = \gamma(w_1, w_2) I_{w_1 w_2, x}$ for a 2-cocycle γ

Lecture 3: C^*-algebras of real reductive groups, up to isomorphism

Goal: compute $C_r^*(G)$, the reduced C^*-algebra of a real or reductive p-adic group G.

Why? [one answer] Understand connections between representation theory and operator K-theory.

The story so far:

- $C_r^*(G)$ is a C^*-algebra whose irreducible reprsentations are precisely the tempered irreducible representations of G.
- The Stone-Weierstrass theorem that tells us when a homomorphism $A \to C_0(X, \mathrm{K}(H))^W$ is surjective.
- Under certain conditions, $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R$.

This lecture: $C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$, via a kind of Fourier transform.

Resources

Background: Knapp (*Overview*); Wallach (*RRGs*)

This C^*-algebra computation:

Clare, Crisp, Higson : *Parabolic induction and restriction via C^*-algebras and Hilbert C^*-modules* (Compositio, 2016)

Real reductive groups

A real reductive group is (for us) ~~a closed subgroup $G \subseteq GL(n, \mathbb{R})$, closed under transpose, that is the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$.~~

$\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{R})$, or $\mathrm{SL}(n, \mathbb{C})$

Langlands decomposition: $G = M_G \times A_G$ where M_G has compact centre, and $\exp : \mathfrak{a}_G \to A_G$ is a group isomorphism

Note: A_G is not the 'A' in $G = KAN$!

Eg: $\mathrm{GL}(n, \mathbb{R})$. $M_G = \{g \mid \det(g) = \pm 1\}$, $A_G = \{a \cdot 1 \mid a > 0\}$

Eg: $\mathrm{SL}(n, \mathbb{R})$. $M_G = G$, $A_G = \{1\}$.

Square-integrable representations

A unitary irrep σ of M is square-integrable if for all $\xi, \eta \in H_\sigma$, the function

$$c_{\xi,\eta} : m \mapsto \langle \sigma(m)\xi \mid \eta \rangle$$

is in $L^2(M)$.

$\widehat{M}_{L^2}$: set of iso classes of square-integrable irreps of M

Theorem: $\widehat{M}_{L^2} \subseteq \widehat{M}_r$.

Proof: Schur orthogonality relations show that for every $\sigma \in \widehat{M}_{L^2}$ there is some $d_\sigma > 0$ making

$$(\star) \qquad \overline{H_\sigma} \otimes H_\sigma \to L^2(M), \qquad \overline{\xi} \otimes \eta \mapsto d_\sigma^{1/2} c_{\xi,\eta}$$

an isometry. So σ is a subrep of the regular representation. □

Theorem: If $\sigma \in \widehat{M}_{L^2}$ then $\sigma(C_r^*(M)) = \mathrm{K}(H_\sigma)$.

Proof: Use $(\star)$ to show that for each $f \in C_c(M)$ the operator $\sigma(f)$ is Hilbert-Schmidt. □

A partial Fourier transform

Take $G = MA$ $(M = M_G,\, A = A_G)$ and $\sigma \in \widehat{M}_{L^2}$

$\mathfrak{a}^* \cong \widehat{A}$: given $\chi : \mathfrak{a} \to \mathbb{R}$ let $\chi : A \to \mathrm{U}(\mathbb{C})$ be $e^x \mapsto e^{i\chi(x)}$.

For each $\chi \in \mathfrak{a}^*$ we get an irreducible unitary representation

$$\sigma \otimes \chi : G \to \mathrm{U}(H_\sigma), \qquad ma \mapsto \sigma(m)\chi(a).$$

Theorem: For $f \in C_c(G)$ and $\chi \in \mathfrak{a}^*$ let

$$\pi_{G,\sigma}(f)(\chi) := (\sigma \otimes \chi)(f) = \int_M \int_A f(ma)\sigma(m)\chi(a)\, da\, dm.$$

The map $\pi_{G,\sigma}$ extends to a homomorphism of C^*-algebras

$$\pi_{G,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}^*, \mathrm{K}(H_\sigma)).$$

A partial Fourier transform

Theorem: For $f \in C_c(G)$ and $\chi \in \mathfrak{a}^*$ let

$$\pi_{G,\sigma}(f)(\chi) := (\sigma \otimes \chi)(f) = \int_M \int_A f(ma)\sigma(m)\chi(a)\, da\, dm.$$

The map $\pi_{G,\sigma}$ extends to a homomorphism of C^*-algebras

$$\pi_{G,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}^*, \mathrm{K}(H_\sigma)).$$

Proof:

- Each $\pi_{G,\sigma}(\cdot)(\chi)$ extends to $C_r^*(G)$, because $\sigma \otimes \chi$ is L^2 modulo centre
- Functions of the form $ma \mapsto f_M(m) f_A(a)$ span a dense subspace of $C_r^*(G)$, and $\pi_{G,\sigma}(f_M f_A)(\chi) = \sigma(f_M)\widehat{f_A}(\chi)$. So:
- $\pi_{G,\sigma}(f)(\chi)$ is compact (because $\sigma(f_M)$ is); and
- $\pi_{G,\sigma}(f)$ is a C_0 function of χ (because $\widehat{f_A}$ is.) □

The $\pi_{G,\sigma}$s are not enough

Not every tempered irreducible representation of G is a $\sigma \otimes \chi$.

Indeed, sometimes (eg $\mathrm{SL}(2,\mathbb{C})$; $\mathrm{SL}(3,\mathbb{R})$) G has no $\sigma \otimes \chi$s.

But: every $\pi \in \widehat{G}_r$ can be obtained from a $\sigma \otimes \chi$ of a parabolic subgroup of G, via parabolic induction.

Parabolic subgroups

G : ~~real reductive group~~ $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$

$P = L_P N_P = M_P A_P N_P$: a parabolic subgroup of G

Example: In $G = \mathrm{GL}(3,\mathbb{R})$:

$$P = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right] \quad L_P = \left[\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & * \end{array}\right] \quad N_P = \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$A_P = \left[\begin{array}{cc|c} a & 0 & 0 \\ 0 & a & 0 \\ \hline 0 & 0 & b \end{array}\right] \qquad M_P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{m}} & 0 \\ & & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right]$$

$$(a, b > 0) \qquad\qquad (\det m = \pm 1)$$

Parabolic subgroups

Parabolic subgroup $P = L_P N_P = M_P A_P N_P \subseteq G$

Properties:

- L_P is a real reductive group
- $P \cong L_P \rtimes N_P$
- $L_P \cong M_P \times A_P$
- eg: $P = G$ (then $N_G = \{1\}$; $L_G = G$; M_G, A_G are as above)
- $\exp : \mathfrak{a}_P \xrightarrow{\cong} A_P$ $\left(\Longrightarrow \mathfrak{a}_P^* \cong \widehat{A_P}\right)$
- $G = KP$ for a maximal compact K
- There are only finitely many P, up to conjugacy

Parabolic induction

$P = M_P A_P N_P$ parabolic subgroup of G

$\sigma \in \widehat{(M_P)}_{L^2}$, $\chi \in \mathfrak{a}_P^* \rightsquigarrow \sigma \otimes \chi \in \widehat{P}$: $\sigma \otimes \chi(man) = \sigma(m)\chi(a)$

Parabolic induction: $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is the unitary representation of G induced from $\sigma \otimes \chi$.

Hilbert space: P-equivariant (with a ρ-shift) functions $G \to H_{\sigma \otimes \chi}$

Inner product: $\langle \xi \,|\, \eta \rangle = \int_K \langle \xi(k) \,|\, \eta(k) \rangle \, dk$

G-action: translation

Compact picture: $G = KP \Longrightarrow \mathrm{Ind}_P^G(\sigma \otimes \chi) \cong \mathrm{Ind}_{K \cap P}^K(\sigma)$ over K

$\Longrightarrow$ all $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ can be realised on the same space $\mathrm{Ind}_P^G H_\sigma$, and are all isomorphic as representations of K.

Another partial Fourier transform

Theorem: For $f \in C_c(G)$, $\sigma \in \widehat{(M_P)}_{L^2}$, and $\chi \in \mathfrak{a}_P^*$, define

$$\pi_{P,\sigma}(f)(\chi) := \mathrm{Ind}_P^G(\sigma \otimes \chi)(f).$$

The map $\pi_{P,\sigma}$ extends to a homomorphism of C^*-algebras

$$\pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Proof: Similar to the case $P = G$. Use the fact that G/P is compact. □

The complete Fourier transform

Theorem: We have an injective homomorphism of C^*-algebras

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

$[P, \sigma]$ ranges over the set of equivalence classes of pairs (P, σ) (equivalence: conjugacy of $M_P A_P$ and σ).

Proof: Two main parts. Each one relies on a big theorem from representation theory.

The complete Fourier transform

Theorem: We have an injective homomorphism of C^*-algebras

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Part 1: The map goes into the direct sum. We use:

Theorem [Harish-Chandra]: Each irrep of K occurs in only finitely many $\mathrm{Ind}_P^G \sigma$. □

For each $\rho \in \widehat{K}$ consider $e_\rho \in C(K)$, $e_\rho(k) = \frac{\dim H_\rho}{\mathrm{vol}(K)} \mathrm{trace}(\rho(k^{-1}))$.

Schur orthogonality: for $\pi \in \widehat{G}$, $\pi(e_\rho) = 0$ if $\rho \not\subseteq \pi\big|_K$.

So $\pi_{P,\sigma}(e_\rho C_c(G)) \neq 0$ for only finitely many $[P, \sigma]$.

Harmonic analysis: $\sum_{\rho \in \widehat{K}} e_\rho C_c(G)$ is dense in $C_r^*(G)$. □

The complete Fourier transform

Theorem: We have an injective homomorphism of C^*-algebras

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Part 2: The map is injective. We use:

Theorem [Langlands, Trombi]: Each $\pi \in \widehat{G}_r$ occurs in some $\mathrm{Ind}_P^G(\sigma \otimes \chi)$. □

Recall: $\widehat{G}_r = \widehat{C_r^*(G)}$, and the irreducible representations of a C^*-algebra separate points.

Conclusion: if $\pi_{P,\sigma}(f)(\chi) = 0$ for all P, σ, χ, then $f = 0$. □

The complete Fourier transform

Theorem: We have an injective homomorphism of C^*-algebras

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Next question: What is the image of this Fourier transform?

Observation: if $\pi_{P,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$ is surjective, then each $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is an irreducible representation of G.

In general $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is not irreducible. We need to understand its decomposition into irreducibles.

We also need to understand when an irreducible representation appears in two different $\mathrm{Ind}_P^G(\sigma \otimes \chi)$s.

So we need to understand the intertwining operators between $\mathrm{Ind}_P^G(\sigma \otimes \chi)$s.

Intertwining operators

Intertwining operator $\pi_1 \to \pi_2$: bounded linear map of Hilbert spaces with $t\pi_1(g) = \pi_2(g)t$ for all $g \in G$.

Theorem: [Bruhat] The intertwining operators between $\mathrm{Ind}_P^G(\sigma \otimes \chi)$s are controlled by a certain finite group.

'Weyl' groups: fix a parabolic $P = MAN$

- $W_P := \mathrm{Norm}_G(A_P)/\,\mathrm{Cent}_G(A_P)$; a finite group, acting by conjugation on $\mathfrak{a}_P^*$ and on $\widehat{M_P}$.

Example: $G = \mathrm{GL}(n, \mathbb{R})$, $P =$ upper-triangular matrices, $M_P = \{\pm 1\}^n$, $A_P = \mathbb{R}^n_{>0}$, $W_P \cong S_n$ (permutation matrices), acting on $\widehat{M_P} = \{\mathrm{triv}, \mathrm{sign}\}^n$ and $\mathfrak{a}_P^* \cong \mathbb{R}^n$ by permuting coordinates.

- For each $\sigma \in \widehat{M}_{L^2}$: $W_\sigma := \{w \in W_P \mid w\sigma \cong \sigma\}$
- For each $\chi \in \mathfrak{a}_P^*$: $W_{\sigma,\chi} := \{w \in W_\sigma \mid w\chi = \chi\}$

Intertwining operators

Fix $P = MAN$ and $\sigma \in \widehat{M}_{L^2}$

$W_\sigma = \{w \in W_P \mid w\sigma \cong \sigma\}, \quad W_{\sigma,\chi} = \{w \in W_\sigma \mid w\chi = \chi\}$

Theorem: [Knapp-Stein] There are unitary operators

$$I_{w,\chi} \in \mathrm{U}(\mathrm{Ind}_P^G H_\sigma) \quad (w \in W_\sigma,\ \chi \in \mathfrak{a}_P^*)$$

satisfying:

- $\chi \mapsto I_{w,\chi}$ is continuous in the strong operator topology
- $I_{w_1,w_2\chi} I_{w_2,\chi} = I_{w_1 w_2,\chi}$
- $I_{w,\chi}$ is an intertwiner $\mathrm{Ind}_P^G(\sigma \otimes \chi) \to \mathrm{Ind}_P^G(\sigma \otimes w\chi)$. □

Intertwiners and the image of the Fourier transform

$I_{w,\chi} \in \mathrm{U}(\mathrm{Ind}_P^G H_\sigma)$ for $w \in W_\sigma \rightsquigarrow W_\sigma$ acts on $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$:

$$\beta_w(f)(\chi) := I_{w,w^{-1}\chi} f(w^{-1}\chi) I_{w^{-1},\chi}.$$

Since $I_{w,\chi}$ is an intertwiner $\mathrm{Ind}_P^G(\sigma \otimes \chi) \to \mathrm{Ind}_P^G(\sigma \otimes w\chi)$, we have

$$\pi_{P,\sigma}\big(C_r^*(G)\big) \subseteq C_0(\mathfrak{a}^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}.$$

Theorem: the Fourier transform

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$$

is an isomorphism of C^*-algebras.

Proof: We only need to show surjectivity. Use Stone-Weierstrass + two more big theorems from representation theory.

Some reminders

Theorem: • The irreducible representations of $C_0(X, \mathsf{K}(H))^W$ are

$$\pi_{x,\rho} : C_0(X, \mathsf{K}(H))^W \xrightarrow{\mathrm{ev}_x} \mathsf{K}(H)^{W_x} \xrightarrow{k \mapsto k \circ \_} \mathsf{K}\left(\mathrm{HS}(\rho, I_x)^{W_x}\right)$$

where $x \in X$ and $\rho \in \widehat{W_x}$ with $\rho \subseteq I_x$.

• $\pi_{x,\rho} \cong \pi_{x',\rho'}$ iff $x' = wx$ and $\rho' = \rho(w^{-1} \_ w)$ for some $w \in W$. □

Theorem: Let B be a subalgebra of a CCR algebra A. Suppose:

- $\pi\big|_B$ is irreducible for all $\pi \in \widehat{A}$; and
- $\pi\big|_B \cong \rho\big|_B$ iff $\pi \cong \rho$ (π, ρ irreps of A).

Then $B = A$.

$C_r^*(G)$, up to isomorphism

Step 1: Each irrep of $C_0(\mathfrak{a}^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$ remains irreducible over $C_r^*(G)$.

Proof: The representations in question are

$$C_r^*(G) \to \mathrm{K}\left(\mathrm{HS}(\rho, \mathrm{Ind}_P^G(\sigma \otimes \chi))^{W_{\sigma,\chi}}\right), \quad f \mapsto \pi_{P,\sigma}(f)(\chi) \circ \_$$

for $\rho \in \widehat{W_{\sigma,\chi}}$, $\overline{\rho} \subseteq I_\chi$.

If t is an intertwiner of this representation then $\mathrm{id}_{H_\rho} \otimes t$ is an intertwiner of

$$(\dagger) \qquad H_\rho \otimes \mathrm{HS}(\rho, \mathrm{Ind}_P^G(\sigma \otimes \chi))^{W_{\sigma,\chi}}.$$

This tensor product is a G-subrep of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$.

Theorem: [Harish-Chandra] The space of intertwiners of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is $\mathrm{span}\{I_{w,\chi} \mid w \in W_{\sigma,\chi}\}$. □

The $I_{w,\chi}$s act only on the H_ρ factor in $(\dagger)$, so t is a scalar. □

$C_r^*(G)$, up to isomorphism

Theorem: the Fourier transform

$$\bigoplus \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$$

is an isomorphism of C^*-algebras.

Step 1: Irreps of the RHS remain irreducible over $C_r^*(G)$ ✓

Step 2: Inequivalent irreps of the RHS remain inequivalent over $C_r^*(G)$.

This follows immediately from:

Theorem [Langlands]: The only coincidences between irreducible subreps of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$s are the ones coming from conjugacy. □

Looking ahead

Known: for G a real reductive group:

$$C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}.$$

Next: the normalisation, completeness, and $W = W' \rtimes R$ conditions all hold; so

$$C_r^*(G) \underset{\mathrm{M}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

For G a p-adic reductive group:

$$C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$$

where:

- normalisation does not (?) always hold
- $W = W' \rtimes R$ does not always hold
- $I_{w_1, w_2 x} I_{w_2, x} = \gamma(w_1, w_2) I_{w_1 w_2, x}$ for a 2-cocycle γ ...

Looking ahead

Known: for G a real reductive group:

$$C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}.$$

Next: the normalisation, completeness, and $W = W' \rtimes R$ conditions all hold; so

$$C_r^*(G) \underset{\mathrm{M}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

For G a p-adic reductive group:

$$C_r^*(G) \cong \bigoplus C_0(X, \mathrm{K}(H))^W$$

where:

- normalisation does not (?) always hold
- $W = W' \rtimes R$ does not always hold
- $I_{w_1, w_2 x} I_{w_2, x} = \gamma(w_1, w_2) I_{w_1 w_2, x}$ for a 2-cocycle γ ...
- but we can still compute $K_*(C_r^*(G))$.

Lecture 4: C^*-algebras of real and p-adic reductive groups, up to Morita equivalence

Goal: compute $C_r^*(G)$, the reduced C^*-algebra of a real or reductive p-adic group G.

Why? [one answer] Understand connections between representation theory and operator K-theory.

The story so far:

- For real reductive G: $C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$
- $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim}$ a certain ideal in $C_0(X) \rtimes W$.
- Under certain conditions ($W = W' \rtimes R$, etc) we have $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R$.

This lecture:

- For real reductive G: the conditions ($W = W' \rtimes R$, etc) are satisfied by $C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$.
- See how this plays out for p-adic reductive groups.

Reminders about the component C^*-algebras

G : real reductive group $\qquad P = MAN$: parabolic subgroup

$\sigma \in \widehat{M}_{L^2}$: square-integrable irreducible representation

$\operatorname{Ind}_P^G H_\sigma$: Hilbert space for the parabolically induced representations $\operatorname{Ind}_P^G(\sigma \otimes \chi)$, where $\chi \in \mathfrak{a}^* \cong \widehat{A}$.

W_σ : finite group acting on A and M, fixing σ

$I_{w,\chi} : \operatorname{Ind}_P^G(\sigma \otimes \chi) \to \operatorname{Ind}_P^G(\sigma \otimes w\chi)$ unitary intertwiners

W_σ acts on $C_0(\mathfrak{a}^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))$:

$$\beta_w(f)(\chi) := I_{w,w^{-1}\chi} f(w^{-1}\chi) I_{w^{-1},\chi}.$$

$C_0(\mathfrak{a}^*, \mathrm{K}(\operatorname{Ind}_P^G H_\sigma))^{W_\sigma}$ is the fixed-point C^*-algebra.

The R-group

Fix P and σ. Recall : $W'_{\sigma,\chi} = \{w \in W_\sigma \mid I_{w,\chi} \in \mathbb{C}\,\mathrm{id}_{\mathrm{Ind}_P^G H_\sigma}\}$.

Theorem: [Knapp-Stein] Let $W'_\sigma := W'_{\sigma,0}$.

(1) There is a subgroup $R_\sigma \subseteq W_\sigma$ such that $W_\sigma = W'_\sigma \rtimes R_\sigma$.

(2) For each $\chi \in \mathfrak{a}^*$ we have $W'_{\sigma,\chi} = W_{\sigma,\chi} \cap W'_\sigma$.

(3) The $I_{w,\chi}$s can be chosen so that $I_{w,\chi} = \mathrm{id}_{\mathrm{Ind}_P^G H_\sigma}$ for all χ and all $w \in W'_{\sigma,\chi}$.

(4) For each χ the representation $w \mapsto I_{w,\chi}$ of $W_{\sigma,\chi}$ contains every $\rho \in \widehat{W_{\sigma,\chi}/W'_{\sigma,\chi}}$.

We will sketch a proof of (2) and (3).

Knapp-Stein's homotopy argument

Theorem: For $w \in W_{\sigma,\chi}$ we have $I_{w,\chi} = c\,\mathrm{id} \iff I_{w,0} = c\,\mathrm{id}$.

Corollary: $W'_{\sigma,\chi} = W_{\sigma,\chi} \cap W'_\sigma$, and we can normalise the $I_{w,\chi}$ so that $I_{w,\chi} = \mathrm{id}$ for all χ and all $w \in W'_{\sigma,\chi}$. □

Knapp-Stein's homotopy argument

Theorem: For $w \in W_{\sigma,\chi}$ we have $I_{w,\chi} = c\,\mathrm{id} \iff I_{w,0} = c\,\mathrm{id}$.

Proof: W_σ acts linearly on $\mathfrak{a}^*$. So $w \in W_{\sigma,\chi} \Rightarrow w \in W_{\sigma,t\chi}\ \forall t \in \mathbb{R}$.

$w^m = 1 \Rightarrow I_{w,t\chi}^m = \mathrm{id}\ \forall t \in \mathbb{R} \Rightarrow \mathrm{spec}(I_{w,t\chi}) \subseteq \{m^{\mathrm{th}} \text{ roots of } 1\}$

$\mathrm{Ind}_P^G \sigma$ is admissible: each K-isotypical subspace (K a maximal compact subgroup of G) is finite-dimensional. [Harish-Chandra]

$\Rightarrow \mathrm{Ind}_P^G H_\sigma = \overline{\bigcup_{i=1}^\infty H_i}$, $H_i \subseteq H_{i+1}$, $\dim H_i < \infty$, $I_{w,t\chi} H_i = H_i$.

$t \mapsto I_{w,t\chi}$ SOT-cts $\Rightarrow \mathrm{spec}(I_{w,t\chi}\big|_{H_i})$ varies continuously with t

$\Rightarrow$ if $\mathrm{spec}(I_{w,t\chi}) = \{c\}$ for one t, then the same holds for all t. □

Corollary: $W'_{\sigma,\chi} = W_{\sigma,\chi} \cap W'_\sigma$, and we can normalise the $I_{w,\chi}$ so that $I_{w,\chi} = \mathrm{id}$ for all χ and all $w \in W'_{\sigma,\chi}$. □

$C_r^*(G_{\mathbb{R}})$ up to Morita equivalence

G : real reductive group

Recall: • $C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$

• If $W = W' \rtimes R$, and $W'_x = W_x \cap W'$ for each $x \in X$, then $C_0(X, \mathrm{K}(H))^W \underset{\mathrm{M}}{\sim} C_0(X/W') \rtimes R$.

Exercise: if $A_i \underset{\mathrm{M}}{\sim} B_i$ for $i \in I$ then $\bigoplus_i A_i \underset{\mathrm{M}}{\sim} \bigoplus_i B_i$.

Corollary: [Wasserman] For each real reductive group G we have

$$C_r^*(G) \underset{\text{Morita}}{\sim} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

□

Corollary: $K_*(C_r^*(G)) \cong \bigoplus_{[P,\sigma],\, W'_\sigma=\{1\}} \mathbb{Z}$ [see Hang Wang's lectures]

Example : $\mathrm{SL}(2,\mathbb{R})$

$G = \mathrm{SL}(2,\mathbb{R})$ has two conjugacy classes of parabolic subgroups:

$P = G$: $\quad M = G, \quad A = \{1\}$

$\widehat{M}_{L^2} = \{\sigma_n \mid n \in \mathbb{Z} \setminus \{0\}\}$ (discrete series), $\quad W_P = \{1\}, \quad \mathfrak{a}^* = 0$

$\Longrightarrow C_r^*(G)_{[P,\sigma_n]} \cong \mathrm{K}(H_{\sigma_n}) \underset{\mathrm{M}}{\sim} \mathbb{C}$

$P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$: $\quad M = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \ (a > 0)$

$\widehat{M}_{L^2} = \{\mathrm{triv}, \mathrm{sign}\}$, $W_P = \left\{1, w = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right\}$, $\mathfrak{a}^* \cong \mathbb{R}$, $w\chi = -\chi$

$W_{\mathrm{triv}} = W'_{\mathrm{triv}} = W \Longrightarrow C_r^*(G)_{[P,\mathrm{triv}]} \underset{\mathrm{M}}{\sim} C_0([0,\infty))$

$W_{\mathrm{sign}} = R_{\mathrm{sign}} = W \Longrightarrow C_r^*(G)_{[P,\mathrm{sign}]} \underset{\mathrm{M}}{\sim} C_0(\mathbb{R}) \rtimes W$

Conclusion: $C_r^*(G) \underset{\mathrm{M}}{\sim} C_0(\mathbb{Z} \setminus \{0\}) \oplus C_0([0,\infty)) \oplus C_0(\mathbb{R}) \rtimes W.$

How does all of this work for p-adic groups?

What changes over $\mathbb{Q}_p$: $\mathbb{Q}_p$ vs $\mathbb{R}$

Let p be a prime in $\mathbb{N}$.

We will just talk about $\mathbb{Q}_p$, but everything here is also valid for finite extensions of $\mathbb{Q}_p$.

$$\mathbb{Q}_p := \overline{\mathbb{Q}}^{|\ |_p} \quad \text{where} \quad |p^k(a/b)|_p := p^{-k} \quad (p \nmid a, b)$$

Some important differences between $\mathbb{Q}_p$ and $\mathbb{R}$:

- $|a+b|_p \leq \max\{|a|_p, |b|_p\}$ (while in $\mathbb{R}$ often $|a+b| = |a| + |b|$)
- $\mathbb{Z}_p = \overline{\mathbb{Z}}^{|\ |_p}$ is compact in $\mathbb{Q}_p$ (while $\mathbb{Z}$ is discrete in $\mathbb{R}$)
- $\mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times$: discrete $\times$ profinite (while $\mathbb{R}^\times = e^{\mathbb{R}} \times \{\pm 1\}$)
- p-adic groups have lots of compact subgroups

  (while Lie groups have no small subgroups)

What changes over $\mathbb{Q}_p$: discrete series

G : p-adic reductive group (eg $\mathrm{GL}(n, \mathbb{Q}_p)$, $\mathrm{SL}(n, \mathbb{Q}_p)$)

- We don't have $G = \underset{\substack{\text{compactly}\\ \text{generated}}}{M} \times \underset{\text{central}}{A}$

  (think about $\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{Q}_p)$)

- $\widehat{G}_{L^2}$:= irreps whose matrix coefficients are L^2 modulo the centre of G.

- Theorem: $\widehat{G}_{L^2} \neq \emptyset$. In fact, G has irreps whose matrix coefficients are compactly supported modulo the centre.

What changes over $\mathbb{Q}_p$: twisting by characters

- $X_G := \{\chi : G \to \mathrm{U}(\mathbb{C}) \mid \chi(g) = 1 \text{ if } g \in \text{a compact subgroup}\}$
- Example: $X_{\mathrm{GL}(n,\mathbb{Q}_p)} \cong \mathrm{U}(\mathbb{C})$, via $e^{it} \mapsto (g \mapsto |\det(g)|_p^{it})$
- Theorem: X_G is a compact torus (ie $X_G \cong \mathrm{U}(\mathbb{C})^n$).
- X_G acts on $\widehat{G}_{L^2}$ ($\chi : \sigma \mapsto \sigma \otimes \chi$), possibly non-freely
- We get a Fourier transform

$$\pi_{G,\sigma} : G \to C(X_G, \mathrm{K}(H_\sigma)).$$

What changes over $\mathbb{Q}_p$: parabolic induction

Here, not much changes:

- Given a parabolic subgroup $P = LN$, and $\sigma \in \widehat{L}_{L^2}$, we get a family of representations of G: $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ for $\chi \in X_P$.
- These representations can all be realised on the same Hilbert space $\mathrm{Ind}_P^G H_\sigma$.
- For each (P, σ) there is a partial Fourier transform

$$\pi_{P,\sigma} : C_r^*(G) \to C(X_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

- The complete Fourier transform

$$\bigoplus_{[P,\sigma]} \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C(X_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$$

  is injective [Harish-Chandra, Bernstein]

What changes over $\mathbb{Q}_p$: intertwining operators

Here there are more significant changes.

- two sources of intertwiners $\mathrm{Ind}_P^G(\sigma \otimes \chi) \to \mathrm{Ind}_P^G(\sigma \otimes \chi')$:
  - a Weyl-type group W_P
  - the stabiliser of σ in X_P

  Consequence: we define W_σ as a subgroup of $X_P \rtimes W_P$.

- $I_{w_1, w_2 \chi} I_{w_2, \chi} = \gamma_{P,\sigma}(w_1, w_2) I_{w_1 w_2, \chi}$ for some 2-cocycle γ on W_σ, which [as far as I know] cannot always be trivialised.

  Consequence: we need to deal with projective representations of W_σ, and twisted crossed products.

- X_P is a torus, so the fixed-point sets X_P^w need not be connected. So we can't [as far as I know] always arrange that $I_{w,\chi} \in \mathbb{C}\,\mathrm{id} \implies I_{w,\chi} = \mathrm{id}$.

  Consequence: we need to keep track of a projective character $w \mapsto i_{w,\chi}$ of $W'_{\sigma,\chi}$ for each χ.

Fourier transform and Morita equivalence for p-adic groups

G : p-adic reductive group

Theorem: [Plymen; Harish-Chandra] The Fourier transform

$$\bigoplus_{[P,\sigma]} \pi_{P,\sigma} : C_r^*(G) \to \bigoplus_{[P,\sigma]} C(X_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$$

is an isomorphism. □

Theorem: [with Clare] For each (P,σ) the bimodule $C(X_P, \mathrm{Ind}_P^G H_\sigma)$ gives a Morita equivalence between $C_r^*(G)_{(P,\sigma)}$ and the ideal

$$C(X_P, W_\sigma, I) :=$$

$$\left\{ \sum_{w \in W_\sigma} f_w w \in C(X_P) \rtimes_{\gamma_{P,\sigma}} W_\sigma \;\middle|\; \begin{array}{c} \forall \chi \in X_P,\ \forall w' \in W'_{\sigma,\chi},\ \forall w \in W_\sigma : \\ f_{w'w}(\chi) = i_{w',\chi} \gamma_{P,\sigma}(w',w) f_w(\chi) \end{array} \right\}$$

of the twisted crossed product $C(X_P) \rtimes_{\gamma_{P,\sigma}} W_\sigma$. □

When is $C_r^*(G)_{(P,\sigma)} \underset{\mathrm{M}}{\sim} C(X_P/W'_\sigma) \rtimes_\gamma R_\sigma$?

Fix P and σ, and drop them from the notation.

Theorem: [Afgoustidis-Aubert; special cases by Plymen et al]

- If $W = W' \rtimes R$, where $W'_\chi = W_\chi \cap W'$ for all $\chi \in X$, then

$$C_r^*(G)_{(P,\sigma)} \underset{\mathrm{M}}{\sim} C(X/W') \rtimes_\gamma R.$$

- For G a split classical group, one can characterise precisely when the above condition is satisfied. □

Example: $\nexists W'$ with $W'_\chi = W_\chi \cap W'$ for all χ

$$G = \mathrm{Sp}(4, \mathbb{Q}_p) = \left\{ g \in \mathrm{GL}(4, \mathbb{Q}_p) \mid g^t \left[\begin{smallmatrix} & I \\ -I & \end{smallmatrix}\right] g = \left[\begin{smallmatrix} & I \\ -I & \end{smallmatrix}\right] \right\}$$

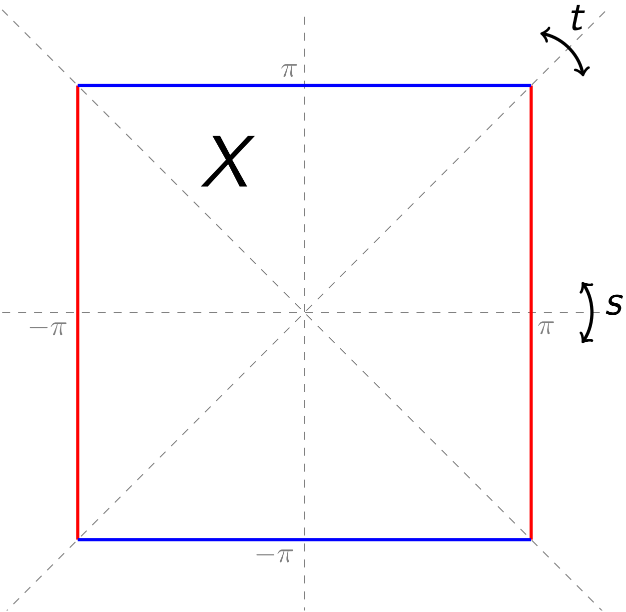
$$L = \begin{bmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{bmatrix}, \quad N = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & c & d \\ & & 1 & 0 \\ & & -a & 1 \end{bmatrix}, \quad \sigma = \mathrm{triv} \in \widehat{L}$$

$$\cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$$

$$X_P \cong \mathbb{R}^2 / 2\pi\mathbb{Z}^2 \qquad (x, y) : (p^n, p^m) \mapsto e^{i(xn+ym)}$$

$$W_\sigma = \langle s, t \rangle, \quad s(x, y) = (x, -y), \quad t(x, y) = (y, x)$$

Example: $\nexists W'$ with $W'_\chi = W_\chi \cap W'$ for all χ



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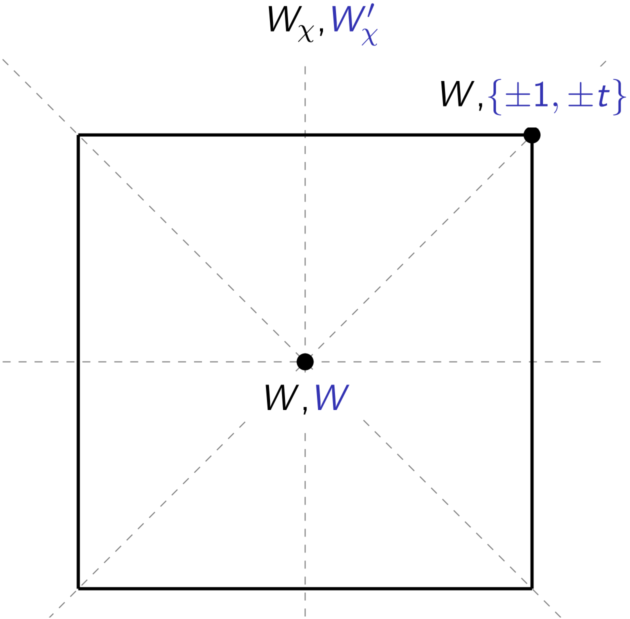


Computing $K_*(C_r^*(G))$ [joint work with Pierre Clare]

K_* is Morita-invariant and commutes with $\bigoplus$

$\Longrightarrow$ enough to compute $K_*(C(X_P, W_\sigma, I))$ for each (P, σ).

Fix P and σ, and drop them from the notation.

X (a compact torus) has a W-invariant CW-structure; choose one.

The Knapp-Stein homotopy argument implies:

- W_χ, W'_χ, and i_χ depend only on the open cell $z \ni \chi$
- if $z' \in \partial z$ then
  - $W_z \subseteq W_{z'}$
  - $W'_z = W_z \cap W'_{z'}$
  - $i_z = i_{z'}\big|_{W'_z}$

Computing $K_*(C_r^*(G))$ [joint work with Pierre Clare]

Reminder: $z' \in \partial z \Rightarrow W_z \subseteq W_{z'}$, $W'_z = W_z \cap W'_{z'}$, and $i_z = i_{z'}\big|_{W'_z}$

For each open cell z define

$$\mathcal{R}_z := \operatorname{Rep}_{\gamma,\overline{i_z}}(W_z) := \mathbb{Z}\left\{\rho \in \widehat{W_z}^{\gamma} \;\middle|\; \overline{i_z} \subseteq \rho\big|_{W'_z}\right\}.$$

For each $w \in W$ we have a map $\mathrm{Ad}_w : \mathcal{R}_z \to \mathcal{R}_{wz}$.

For each $z' \in \partial z$ we have a map restrict : $\mathcal{R}_{z'} \to \mathcal{R}_z$.

$\mathcal{R}$ is an equivariant cohomological coefficient system [Bredon]

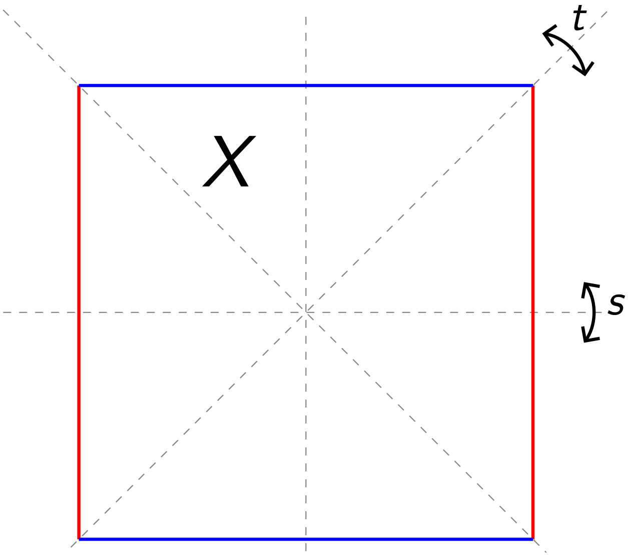
$\rightsquigarrow$ equivariant cohomology groups $H^*_W(X;\mathcal{R})$ [example soon]

Theorem: $K_*(C(X,W,I)) \cong H^*_W(X;\mathcal{R})$, up to a filtration.

Proof: filtration of X by skeleta $\Longrightarrow$ filtration of $C(X,W,I) \Longrightarrow$ spectral sequence converging to $K_*(C(X,W,I))$ with $E^\infty = E^2 = H^*_W(X;\mathcal{R})$ [Atiyah-Hirzebruch; Bredon; Schochet] □

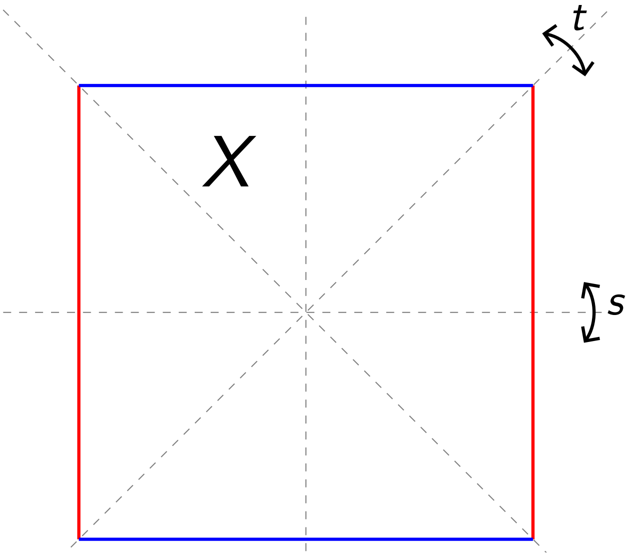
The Sp(4) example again

$$W = \{\pm 1, \pm s, \pm t, \pm st\}$$



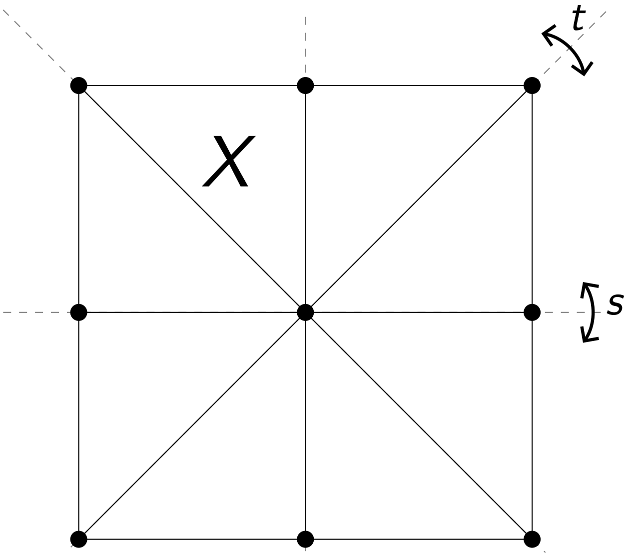
The Sp(4) example again

Choose a W-CW-structure



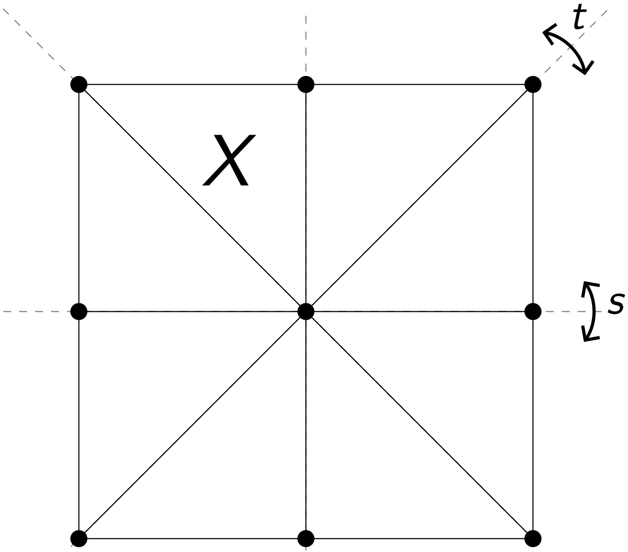
The Sp(4) example again

Choose a W-CW-structure



The Sp(4) example again

$$C^m_W(X;\mathcal{R}) = \left(\bigoplus_{z\in X^m} \mathcal{R}_z\right)^W$$



The Sp(4) example again

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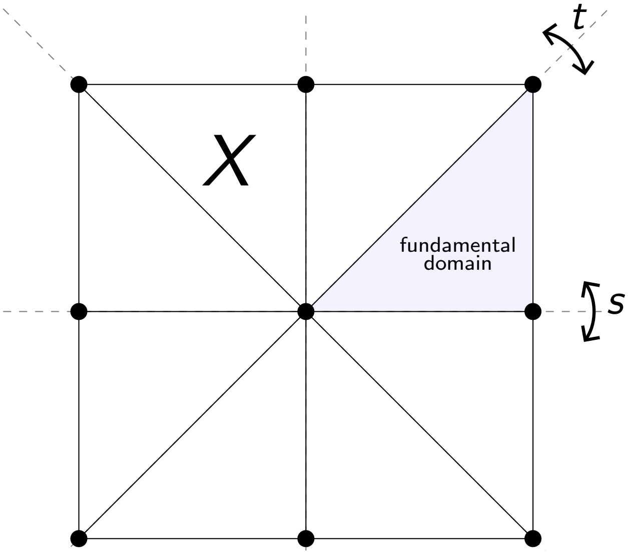
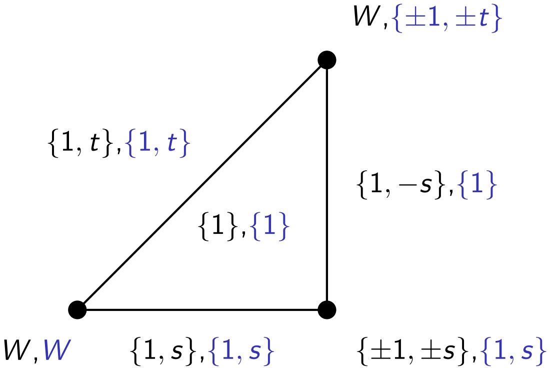


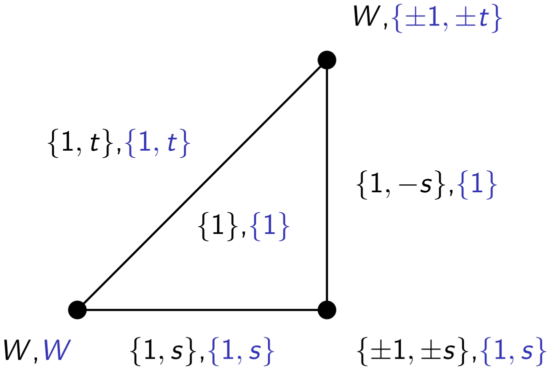
Intertwiners and scalar intertwiners [Keys]

W_z, W'_z:



Intertwiners and scalar intertwiners [Keys]

Here $\gamma, i \equiv 1$, so $\mathcal{R}_z = \mathrm{Rep}(W_z/W_z')$



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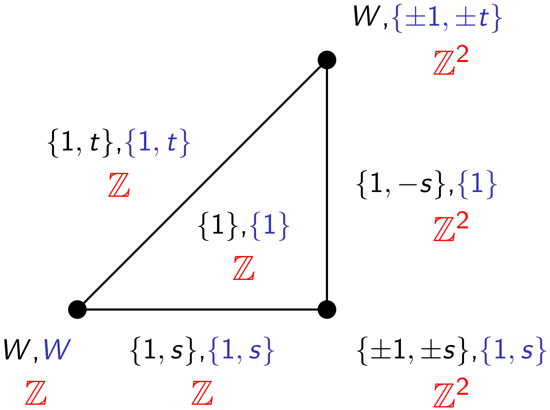
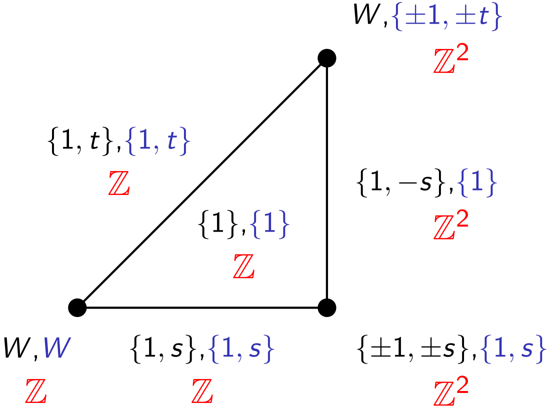


Intertwiners and scalar intertwiners [Keys]

$z' \in \partial z$: $d : \mathcal{R}_{z'} \to \mathcal{R}_z$, $\pm$ restriction

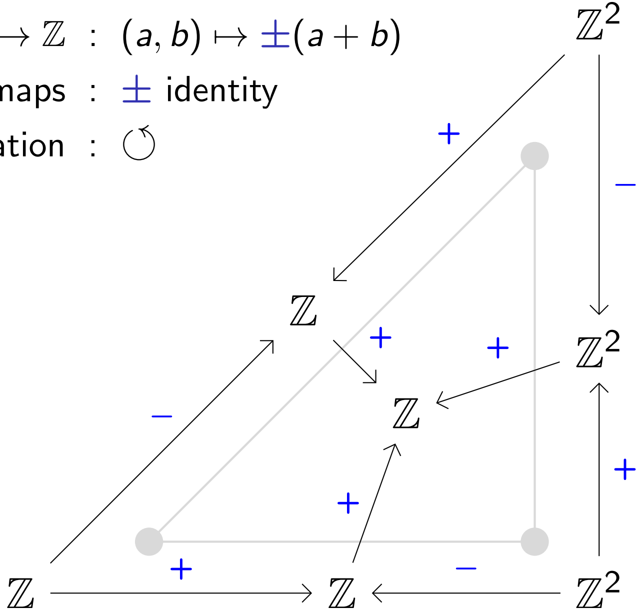


Restriction maps

$\mathbb{Z}^2 \to \mathbb{Z}$: $(a, b) \mapsto \pm(a + b)$

other maps : $\pm$ identity

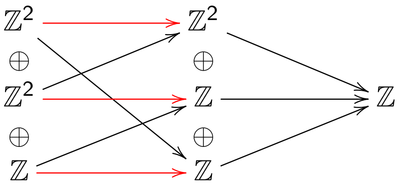
orientation : $\circlearrowleft$



Computing $H^*_W(X;\mathcal{R})$

$\mathbb{Z}^2 \xrightarrow{(a,b)\mapsto\pm(a+b)} \mathbb{Z}$; other maps are $\pm$ identity; $\xrightarrow{+}$, $\xrightarrow{-}$

degree 0 degree 1 degree 2



$H^0 \cong \mathbb{Z}^2$ $H^1 = 0$ $H^2 = 0$

Conclusion: The direct-summand of $K_* C^*_r(\mathrm{Sp}(4,\mathbb{Q}_p))$ associated to the trivial representation of a minimal parabolic subgroup has $K_0 \cong \mathbb{Z}^2$ and $K_1 = 0$. □

Thanks for having me!