Lecture 4: The representation theoretic Fourier transform.

Def. The unnormalized FT ${}^u\hat{f}$ of $f \in C_c^\infty(G/H)$ by

$$
{}^u\hat{f}(P, \xi, \lambda) := \underbrace{\int_{G/H} f(x)\, \pi_{P,\xi,\lambda}(x)\, j(P,\xi,\lambda)\, d\dot{x}}_{\in\, V_P(\xi)^* \otimes C^\infty(K : \xi|_{H_P})}
$$

for $P \in \mathcal{P}_\sigma(A_q)$, $\xi \in \widehat{X}_{P,*,ds}$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$

Example $H = K$, $\sigma = \theta$, $P = P_\emptyset$, $\xi = 1$. Then $V(\xi) = \mathbb{C}$

$j(P,\xi,\lambda)(nak) = a^{-\lambda-\rho_\emptyset} = \mathbb{1}_{P_\emptyset,\lambda}(nak)$, so

$$
\hat{f}(P_\emptyset, \xi, \lambda) = \int_{G/K} f(x)\, \pi_{P_\emptyset,1,\lambda}(x)\, \mathbb{1}_{P_\emptyset,\lambda}\, d\dot{x}
$$

$$
= \pi_{P_\emptyset,1,\lambda}(f)\, \mathbb{1}_{P_\emptyset,\lambda}
$$

Rem The map $f \longmapsto {}^{u}\hat{f}(P,\xi,\lambda)$ intertwines L with $1 \otimes \pi_{P,\xi,\lambda}$. (4.2)

Thm (Plancherel identity) For $f \in C_c^\infty(G/H)$,

$$\|f\|^2_{L^2(X)} = \sum_{P \in \mathbb{P}_\sigma} [W : W_P^*] \underbrace{\sum_{\xi \in ds(P)}}_{= \hat{X}_{P,*,ds}} \int_{i\mathfrak{a}^*_{Pq}} \|{}^{u}\hat{f}(P,\xi,\lambda)\|^2_{HS} \; \underbrace{d\mu_{P,\xi}(\lambda)}_{\text{Pl measure}} .$$

Notation

- $P_1, P_2 \in \mathcal{P}_\sigma(A_q)$ are associated if $\mathfrak{a}_{P_1 q}$ and $\mathfrak{a}_{P_2, q}$ are W-conjugate, notation $P_1 \sim P_2$
- $\mathbb{P}_\sigma$ is a complete set of representatives for $\mathcal{P}_\sigma(A_q)/\sim$.
- $W_P^* = N_W(\mathfrak{a}_P)$
- Recall $\hat{X}_{P,*,ds} = \bigcup_{v \in {}_P\mathcal{W}} (M_P / M_P \cap vHv^{-1})^\wedge_{ds}$

Plancherel, part II.

$$f \mapsto \hat{f} \text{ intertwines } L \text{ with } \bigoplus_{P} \bigoplus_{\xi} \int_{i\mathfrak{a}^*_{Pq}} 1_{V_P(\xi)} \otimes \bar{\pi}_{P,\xi,\lambda} \, d\mu_{P,\xi}(\lambda)$$

Rem $V_P(\xi)$ plays role of multiplicity space.

Standard intertwiners

Suppose $P \in \mathcal{P}_\sigma(A_q)$, $\xi \in \hat{M}_P$, ξ has real inf character

Thm (Knapp–Stein, Vogan–Wallach) $\exists!$ meromorphic family

$$\mathfrak{a}^*_{P\mathbb{C}} \ni \lambda \longmapsto A(\bar{P}, P, \xi, \lambda)$$

of intertw ops $\pi_{P,\xi,\lambda} \longrightarrow \pi_{\bar{P},\xi,\lambda}$ s.t. for $\langle \mathrm{Re}\,\lambda, \alpha \rangle \gg 0$

$(\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}))$ $f \in C^\infty(P : \xi : \lambda)$

$$(A(Q, P, \xi, \lambda) f)(x) = \int_{\bar{N}_P \cap N_Q} f(n x) \, dn \qquad (x \in G).$$

Remark $A(\bar{P}, P, \xi, \lambda)\, A(P, \bar{P}, \xi, \lambda) = \eta(P, \bar{P}, \xi, \lambda) \cdot \mathrm{Id}$

with $\eta(P, \bar{P}, \xi, \cdot) : \mathfrak{a}^*_{PC} \to \mathbb{C}$ meromorphic function.

Pf for generic λ, $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)_K$ is irreducible (& admissible)

$A(\bar{P}, P, \xi, \lambda)\, A(P, \bar{P}, \xi, \lambda)$ intertwines $\pi_{\bar{P}, \xi, \lambda}$ with itself.

Apply Schur's lemma. $\boxtimes$

Lemma $\eta(P, \bar{P}, \xi, \lambda) \geq 0$ for $\lambda \in i\mathfrak{a}_P^*$. Pf. use $A(\bar{P}, P, \xi, \lambda) = A(P, \bar{P}, \xi, -\bar{\lambda})^*$.

Plancherel part III

$$d\mu_{P,\xi}(\lambda) = \eta(\bar{P}, P, \xi, \lambda)^{-1} \cdot \underbrace{d\mu_P(\lambda)}_{\text{Lebesgue measure on } i\mathfrak{a}_P^*}$$

Normalized j

Def. $j^\circ(P, \xi, \lambda) := A(\bar{P}, P, \xi, \lambda)^{-1}\, j(\bar{P}, \xi, \lambda)$.

Def $\hat{f}$ as "$\hat{f}$" but with j° in place of j.

Cor For $f \in C_c^\infty(G)$,

$$\hat{f}(P,\xi,\lambda) = A(\bar{P}, P, \xi, \lambda)^{-1}\, {}^u\hat{f}(\bar{P},\xi,\lambda).$$

Cor $\|{}^u\hat{f}(P,\xi,\lambda)\|^2 = \eta(P,\bar{P},\xi,\lambda)\, \|\hat{f}(P,\xi,\lambda)\|^2 \quad (\lambda \in i\mathfrak{a}_P^*)$

Thm (normalized version of Plancherel): for $f \in C_c^\infty(G/H)$,

$$\|f\|^2_{L^2(G/H)} = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^*] \sum_{\xi \in ds(P)} \int_{i\mathfrak{a}_P^*} \|\hat{f}(P,\xi,\lambda)\|^2 \, d\mu_P(\lambda)$$

Lebesgue measure

Strategy of proof: restrict to left K-finite f.

Fix $\delta \in \hat{K}$ and assume $f \in C_c^\infty(G/H)[\delta]$.

$$C_c^\infty(G/H)[\delta] \simeq V_\delta \otimes \mathrm{Hom}_K(V_\delta, C_c^\infty(G/H)) \simeq$$

$$\underset{\tau = 1 \otimes \delta^\vee}{\left([V_\delta \otimes V_\delta^*]\right.} \otimes \underset{L}{\left. C_c^\infty(G/H)\right)^K} \simeq C^\infty(\tau : G/H)$$

$$(V_\tau = V_\delta \otimes V_\delta^*)$$

For (τ, V_τ) f.d. unitary rep of K, define

$$C_c^\infty(\tau : G/H) := \{ f \in C_c^\infty(G/H, V_\tau) \mid f(kx) = \tau(k) f(x) \;\; \forall x \in G/H,\, k \in K \}.$$

Accordingly $\hat{f}(P,\xi,\lambda) \in V_P(\xi)^* \otimes C^\infty(\xi : K_P\backslash K : \tau)$

Lemma For $\xi \in \hat{X}_{P,*,ds}$, $v \in {}_P\mathcal{W}$,

$$V_p(\xi,v)^* \otimes C^\infty(\xi : K_P\backslash K : \tau) \simeq L^2_\xi(\tau_P : \overbrace{M_P/M_P\cap vHv^{-1}}^{X_{P,v}})$$ (isometry)

Proof $$\text{LHS} = \Big[\underset{1}{V_P(\xi,v)^*} \otimes \overbrace{\underset{R}{C^\infty\mathrm{Ind}_{K_P}^{K}(\xi|_{K_P})} \otimes \underset{\tau}{V_\tau}}\Big]^K$$

Frobenius Reciprocity $$= \Big[\underbrace{V_P(\xi,v)^* \otimes \mathcal{H}_\xi}_{\underset{L^2_\xi(X_{P,v})}{\overset{\wr}{\simeq}}} \otimes V_{\tau_P},\Big]^{K_P}$$

$$= \text{RHS} \quad \text{(isometrically)} \quad \square$$

Here $V_p(\xi,v)^*$ has been equipped with the unique inner product which makes $\wr\!\!\!\simeq$ an isometry. Accordingly $V_p(\xi) \simeq \bigoplus_{v\in {}_P\mathcal{W}} V_p(\xi,v)$ is equipped with the direct sum inner product.

Take the direct sum over the finitely many $\xi \in \hat{X}_{P,*,ds}$ for which $\xi|_{K_P}$ and $\tau|_{K_P}$ have a K_P-type in common, we get

$$\bigoplus_{\xi} \overline{V_P(\xi)} \otimes C^\infty(\xi : K_P\backslash K : \tau) \simeq \bigoplus_{v} L^2_d(\tau : X_{P,v})$$

Here we use that for every $\delta \in \widehat{K_P}$ there exist only finitely many $\xi \in \left(M_P/M_P \cap vHv^{-1}\right)^\wedge_{ds}$ s.t. $\mathcal{H}_\xi[\delta] \neq 0$. However, since $\dim M_P < \dim G$ we can still hope to prove this result by induction on $\dim G$.

Notice that, for $\varphi \otimes \eta \in C^\infty(\xi : K_P \backslash K : \tau) \otimes \overline{V_P(\xi)}$,

$$\langle \hat{f}(P, \xi, \lambda), \varphi \otimes \eta \rangle =$$

$$\int_{G/H} f(x) \langle \pi_{P,\xi,\lambda}(x) j^\circ(P,\xi,\lambda|\eta), \varphi \rangle \, d\dot{x} =$$

$$\int_{G/H} \langle f(x), E^\circ_-(P, \psi_{\varphi\otimes\eta}, \lambda)(x) \rangle \, d\dot{x}$$ ← to be defined below.

Def $\mathcal{A}_{2,P,\tau} := \bigoplus_{v \in {}_P\mathcal{W}} L^2_d(\tau : X_{P,v})$, and for $\psi \in \mathcal{A}_{2,P,\tau}$,

Def $E^\circ(P, \psi, \lambda) \in C^\infty(\tau : G/H)$, linear in $\psi \in \mathcal{A}_{2,P,\tau}$

by $$E^\circ(P, \psi_{\varphi\otimes\eta}, \lambda)(x) = \overline{\langle \pi_{P,\xi,\lambda}(x) j^\circ(P,\xi,\lambda|\eta), \varphi \rangle}$$

$(\varphi \in C^\infty(\xi_P : K_P \backslash K : \tau),\ \eta \in V_P(\xi, v))$.

(4.9)

Def $\mathcal{F}_P : C_c^\infty(\tau : G/H) \longrightarrow \mathcal{M}(\mathfrak{a}^*_{P\mathbb{C}}) \otimes \mathcal{A}_{2,P,\tau}$

$$\langle \mathcal{F}_P f(\lambda), \psi \rangle := \int_{G/H} \langle f(x), \overset{\circ}{E}(P, \psi, -\bar{\lambda})(x) \rangle \, d\dot{x}$$

Cor For $f \in C_c^\infty(\tau : G/H)$:

$$\langle \sum_\xi \hat{f}(P, \xi, \lambda), T \rangle = \langle \mathcal{F}_P f(\lambda), \psi_T \rangle$$

Since $T \longmapsto \psi_T$, $\bigoplus_\xi V_P(\xi) \otimes C^\infty(\xi_P : K_P \backslash K : \tau) \longrightarrow \mathcal{A}_{2,P,\tau}$

is an isometry, it now follows that:

$$\mathcal{F}_P f(\lambda) = \psi\left(\sum_\xi \hat{f}(P : \xi : \lambda)\right)$$

$$\|\mathcal{F}_P f(\lambda)\|^2 = \sum_\xi \|\hat{f}(P : \xi : \lambda)\|^2.$$

4.10

Thm (Spherical version of Plancherel).

For $f \in C_c^\infty(\tau : G/H)$:

$$\|f\|^2_{L^2(\tau : G/H)} = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^*] \int_{i\mathfrak{a}_P^*} \| \mathcal{F}_P f(\lambda) \|^2 \underbrace{d\mu_P}_{\text{Lebesgue}} (\lambda).$$

In the next lecture, aspects of the proof of this theorem, which implies the earlier version of Plancherel.